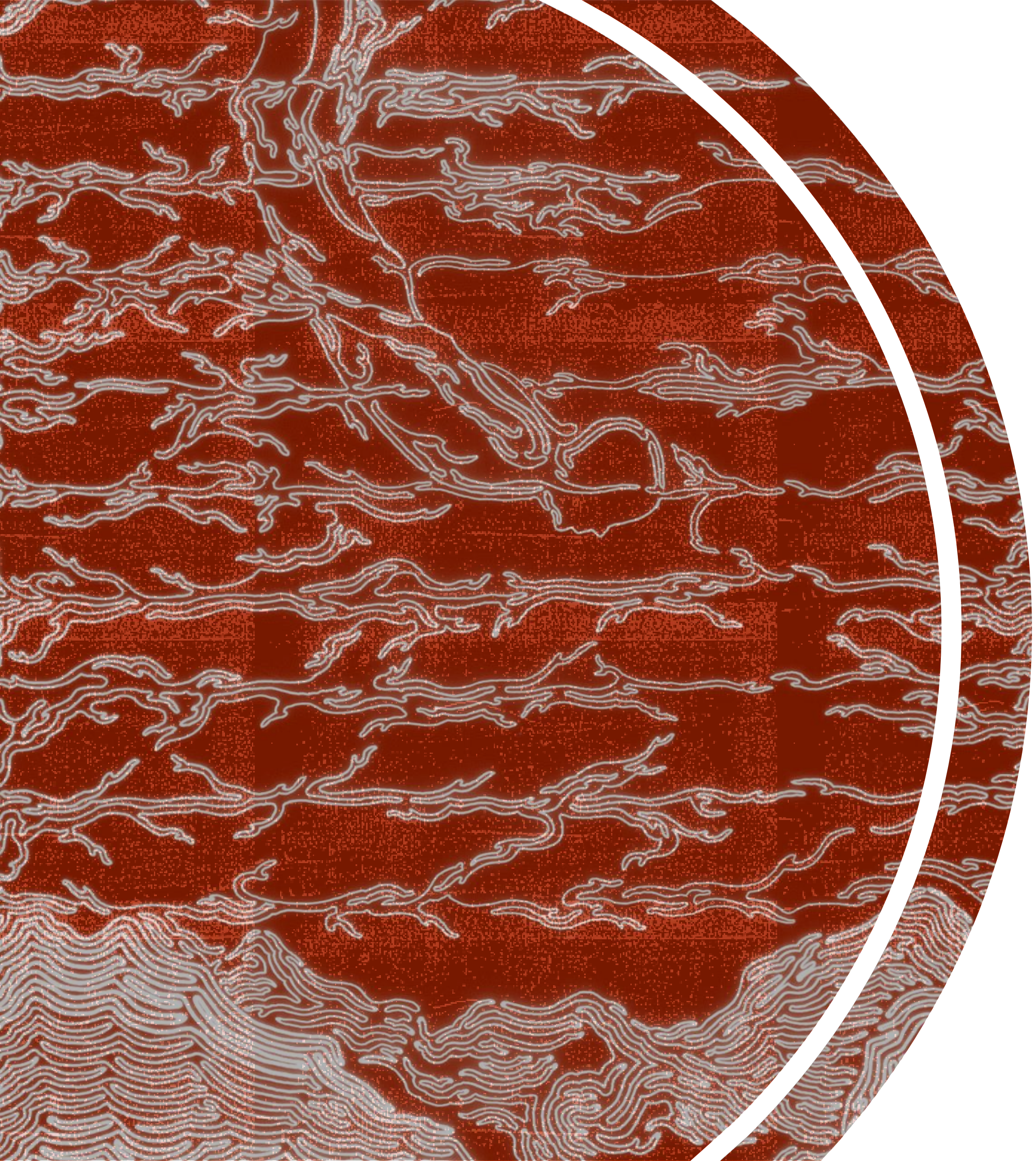


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 18, November 16, 2021

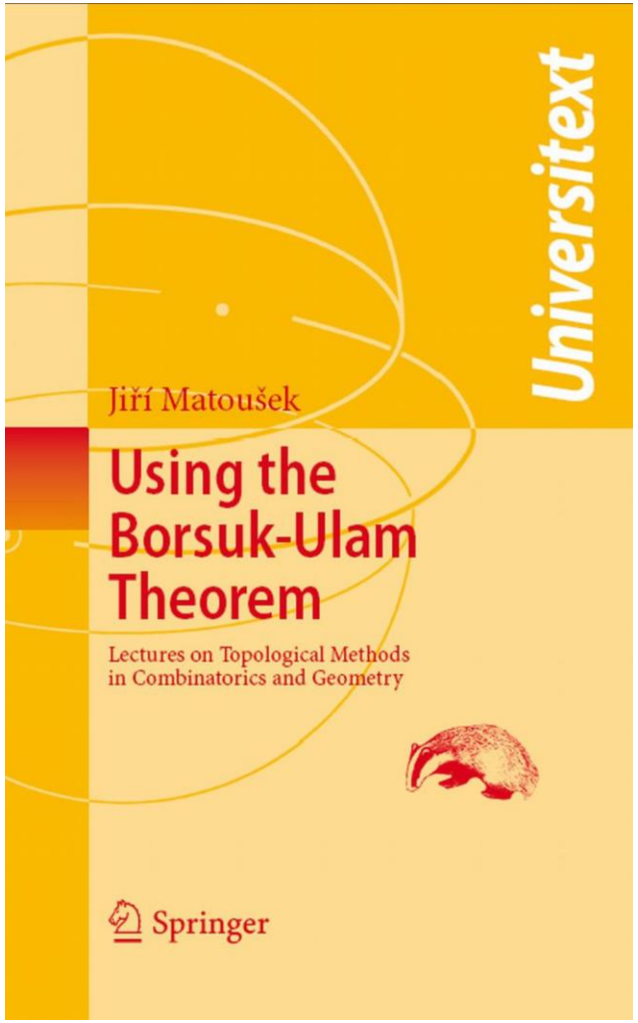
Administrivia

- Final project presentation: 11/24 (Wed)
  - Signup sheet on slack
- All coursework due on 11/26 (Fri)

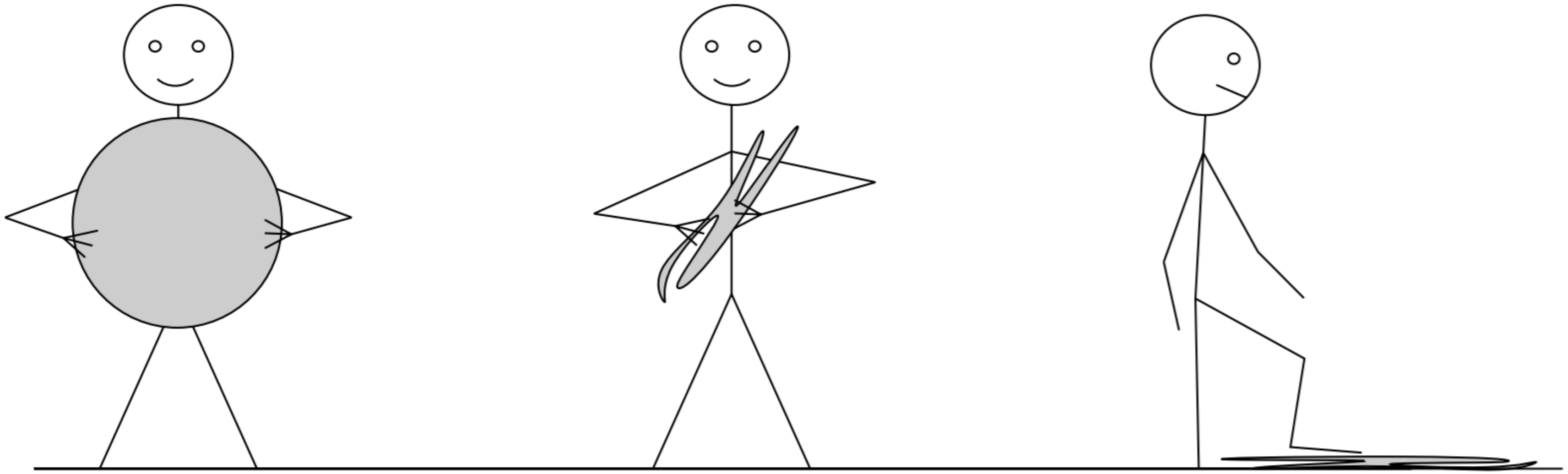
Applications of Fixed-Point Theorems

Using the Borsuk–Ulam Theorem

- Topological methods in combinatorics and geometry



Jiří Matoušek



Borsuk-Ulam Theorem [Borsuk 1933]

Every map $f: S^n \to R^n$ has a point x where
$f(x) = f(-x)$

$g(x) := f(x) - f(-x)$

Equivalent Formulations

$\forall x \ -f(x) = f(-x)$

- Every map $f: S^n \to \mathbb{R}^n$ that is antipodal has a point x where $f(x) = 0$

by BU. $f(x^*) = f(-x^*) = -f(x^*) \Rightarrow f(x^*) = 0$

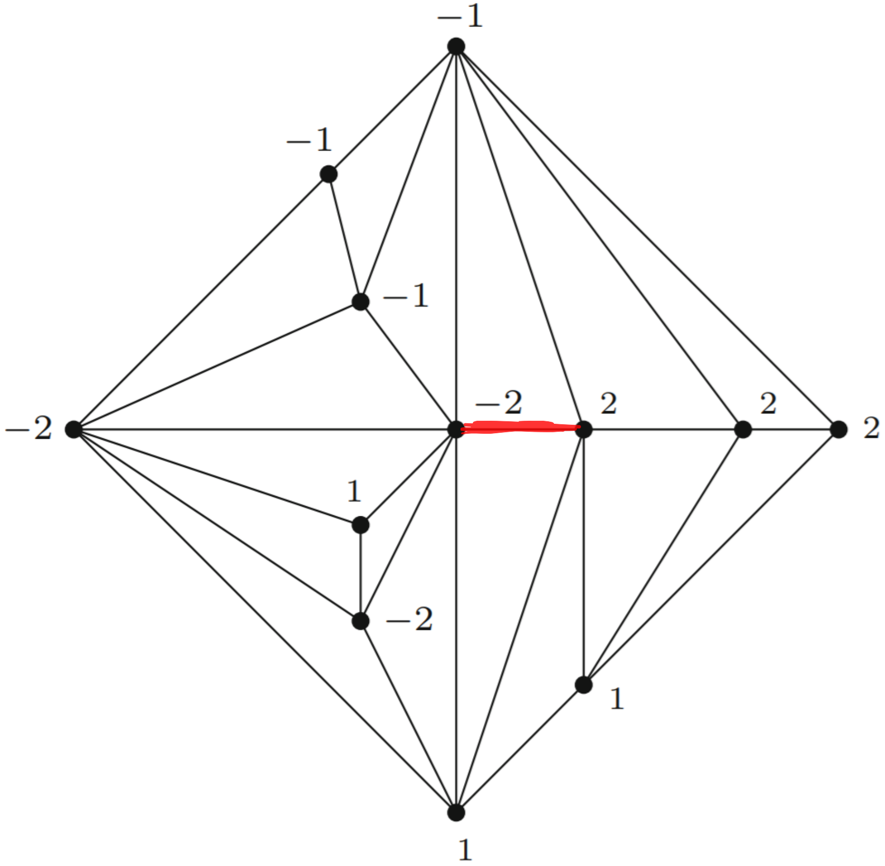
- There is no antipodal map $g: S^n \to S^{n-1} \hookrightarrow D^n$

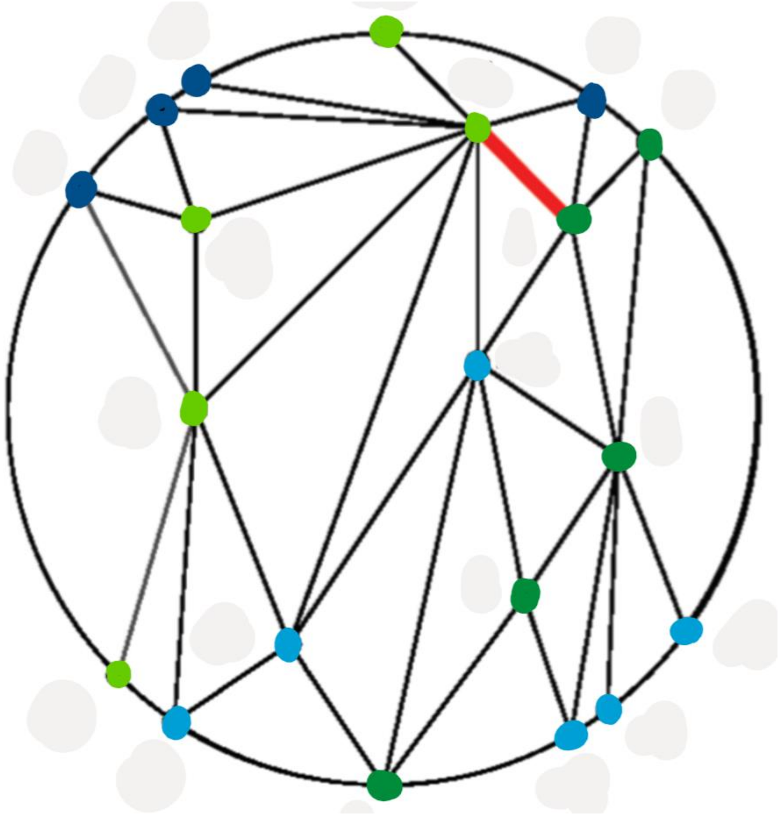
$h(x) := \frac{f(x)}{\|f(x)\|}$

$\hat{g}: S^n \to D^n$. $\hat{g}$ has no zero pts $\to$ ✗

- There is no map $h: D^n \to S^{n-1}$ that is antipodal on ∂D^n

$S^n \to D^n \xrightarrow[\text{rst. 3}]{} S^{n-1}$





Tucker's Lemma

[Tucker 1946] [Lefschetz 1949]

Every [±d]-labeled triangulation of D^d antipodal on bdry contains an edge with complementary labels

Lusternik-Schnirelmann Theorem

[Lusternik-Schnirel'man 1930]

If S^n is covered by n+1 closed sets $U_1, \ldots, U_{n+1}$,
at least one set U_i contains a pair of antipodal points

Proof of Lusternik-Schnirelmann Theorem.

$f: S^n \to \mathbb{R}^n$ $\qquad \mathcal{U}_1, \ldots, \mathcal{U}_n, \mathcal{U}_{n+1}$

$x \mapsto \left(d(x, \mathcal{U}_1), \ldots, d(x, \mathcal{U}_n) \right)$

$\exists x^* \text{ s.t. } f(x^*) = f(-x^*)$

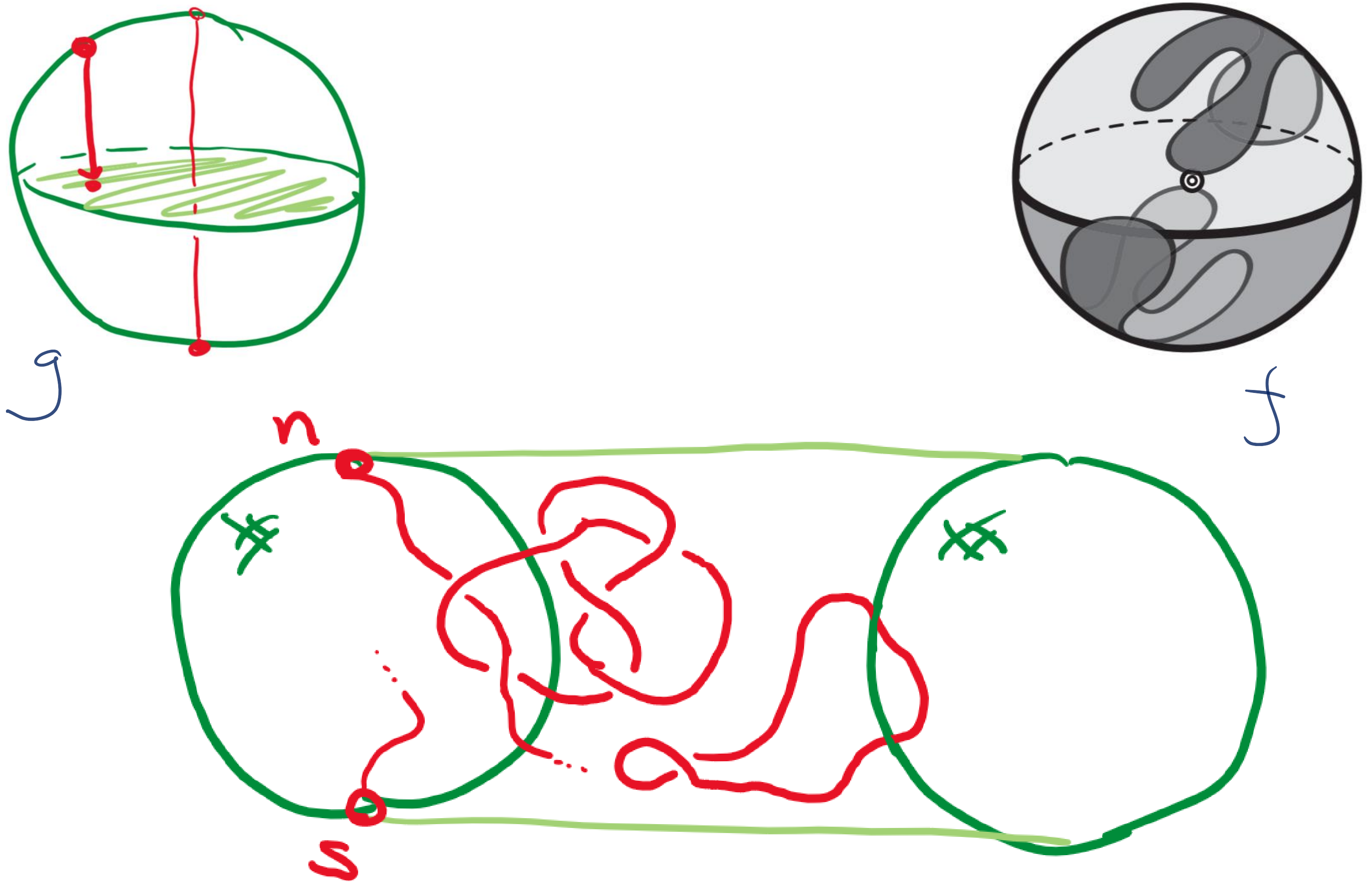
$$\left(d(x^*, \mathcal{U}_1), \ldots, d(x^*, \mathcal{U}_n) \right)$$

$$\shortparallel$$

$$\left(d(-x^*, \mathcal{U}_1), \ldots, d(-x^*, \mathcal{U}_n) \right)$$

- either some $\exists \mathcal{U}_i$: $d(x^*, \mathcal{U}_i) = d(-x^*, \mathcal{U}_i) = 0$.
- or both $x^*, -x^*$ are in $\mathcal{U}_{n+1}$.

Proving Borsuk-Ulam

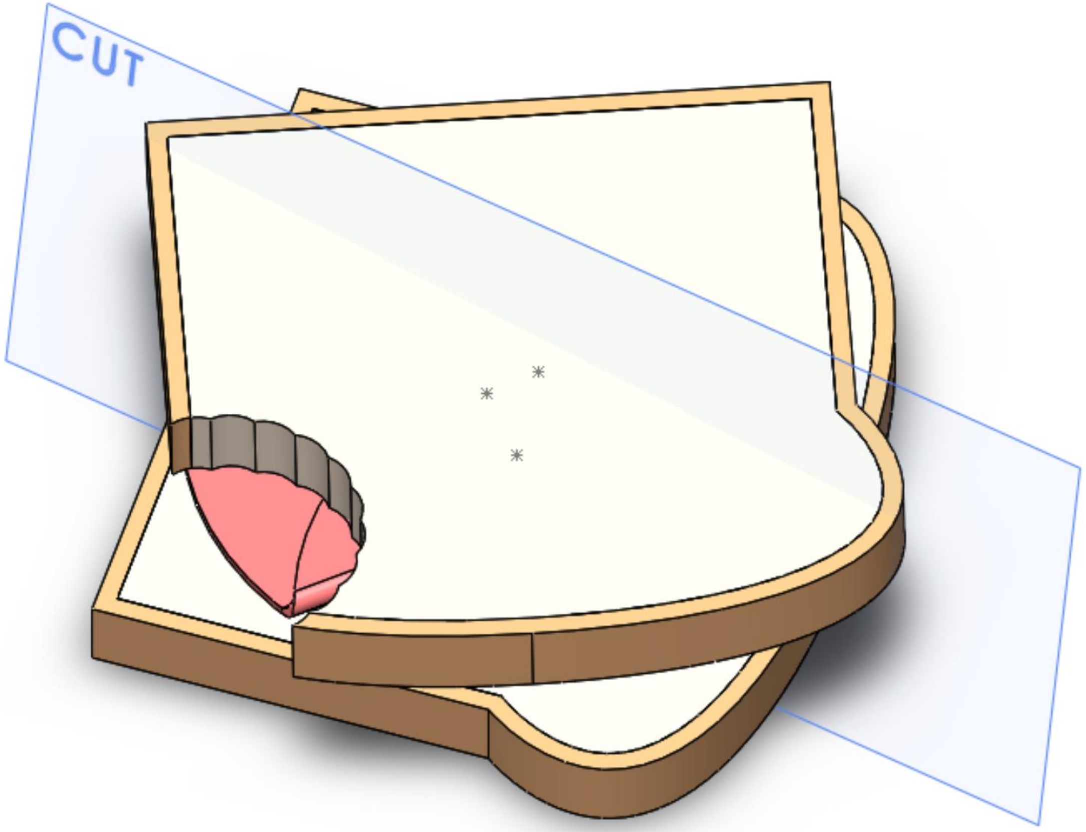


$$F(x,t) := (1-t)\,g(x) + t\cdot f(x)$$

Fake Proof of Borsuk-Ulam

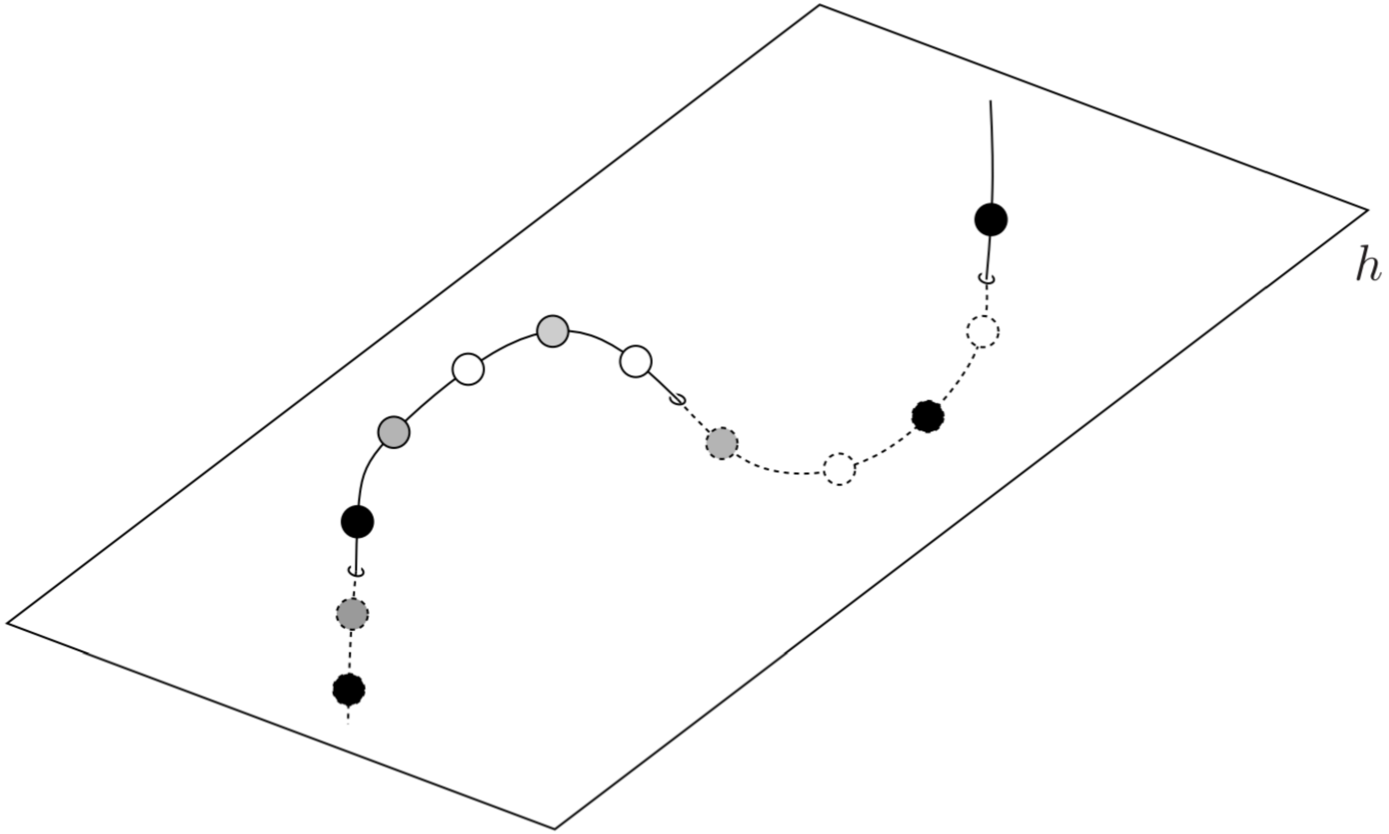
- Every antipodal map f: $S^n \to R^n$ has a point x where $f(x) = 0$

Intermission



Ham Sandwich Theorem [Banach 1938] [Stone-Tukey 1942]

A ham sandwich has a straight cut that divides the ham and two breads evenly



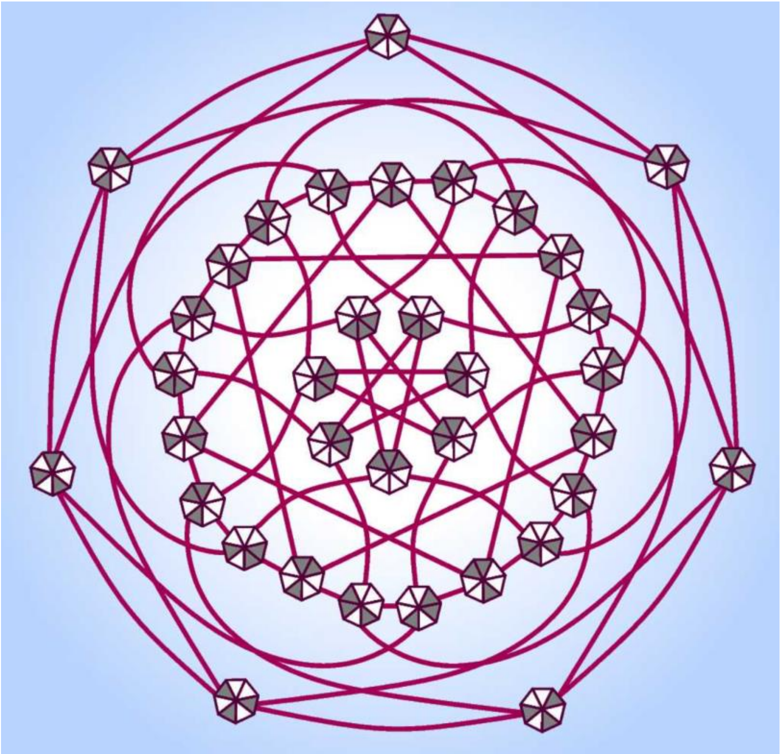
Necklace Splitting Theorem [Alon-West 1986]

Every (open) necklace with d colors of jewels can be divided between two thieves using at most d cuts

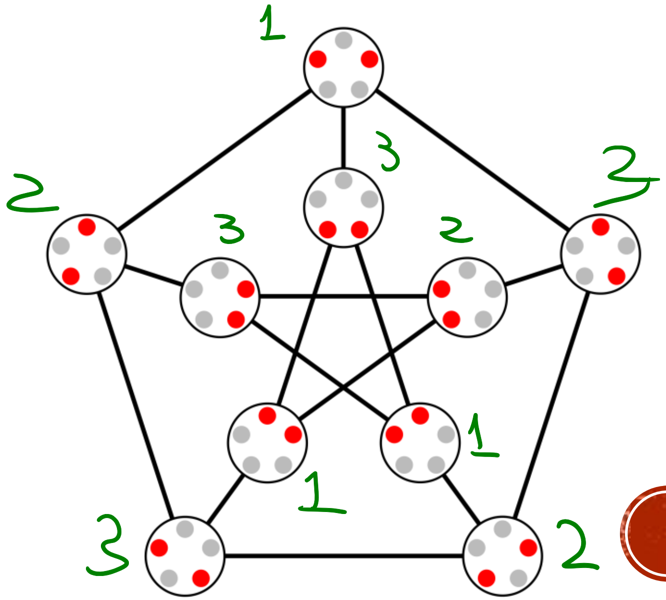
Lovász-Kneser Theorem

Kneser Conjecture

- Kneser graph $KG_{n,k}$
  - Vertices: k-subsets of [n]
  - Edges: (U, V) adjacent if U and V are disjoint



- **Kneser conjecture.**
  $\chi(KG_{n,k}) = n - 2k + 2 \quad \text{for } n \geq 2k - 1$



Note

Kneser's Conjecture, Chromatic Number, and Homotopy

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Communicated by the Editors

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If the simplicial complex formed by the neighborhoods of points of a graph is $(k - 2)$-connected then the graph is not k-colorable. As a corollary Kneser's conjecture is proved, asserting that if all n-subsets of a $(2n - k)$-element set are divided into $k + 1$ classes, one of the classes contains two disjoint n-subsets.

Lovász-Kneser Theorem

[Lovász 1978] [Bárány 1978] [Greene 2002]

$$\chi(KG_{n,k}) = n-2k+2 \quad \text{for } n \geq 2k-1$$

Upper bound.

- $\chi(U) = \min\{\ \min U,\ \ n-2k+2\ \}$

2k-1 elements,

$\chi(u) = \chi(v) = i$

$u \in [n-2k+2, \dots, n]$

$i \in u \qquad i \in v$

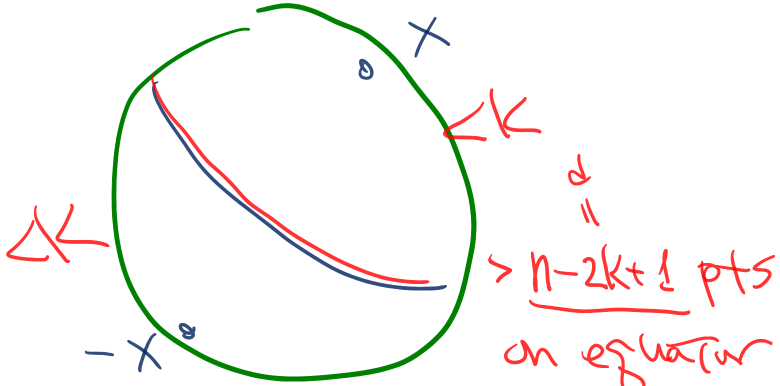
$u \cap v \neq \emptyset \quad \times$

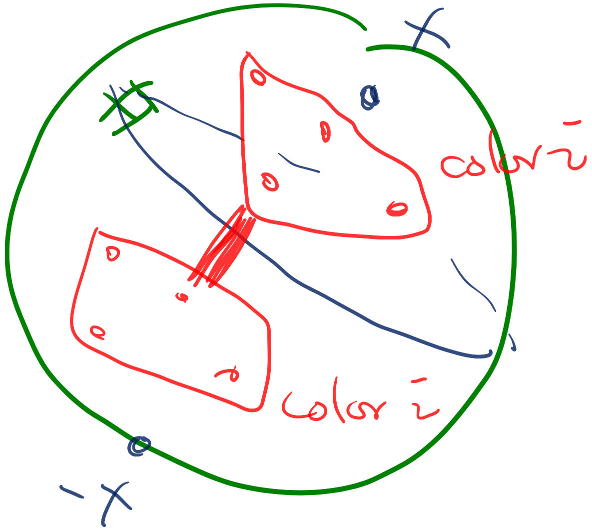
Lower bound. [Lovász 1978] [Greene 2002]

- Put n points on S^d in general position X for $d = n-2k+1$
- Fix a proper d-coloring of $KG_{n,k}$
- Define sets $A_1, \ldots, A_d, A_{d+1}$ on S^d
  - Point x in A_i if open hemisphere centered at x contains a k-tuple of X with color i
  - $A_{d+1} = S^d \setminus (A_1, \ldots, A_d)$

$i \in [1..d]$

- by LS thm, some A_i contains $x, -x$
- A_{d+1} contains $x, -x$:





GALE'S LEMMA. $2k+d$ points X can be put on S^d such that every open hemisphere contains at least k points from X

[Gale 1956]

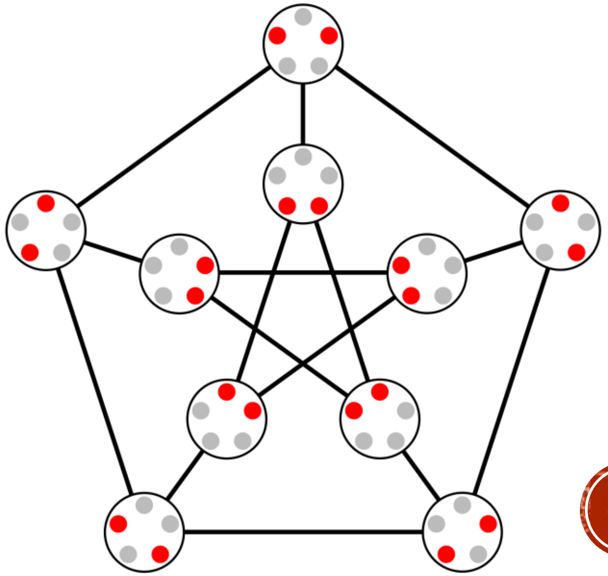
Lower bound. [Bárány 1978]

- Put n points on S^d in general position X for $d = n-2k$ by Gale's Lemma
- Fix a proper $(d+1)$-coloring of $KG_{n,k}$
- Define sets $A_1, \ldots, A_d, A_{d+1}$ on S^d
  - Point x in A_i if open hemisphere centered at x contains a k-tuple of X with color i
  - ~~$A_{d+1} = S^d \setminus (A_1, \ldots, A_d)$~~

Strengthening

- Schrijver graph $SG_{n,k}$
  - Vertices: independent k-subsets of [n]
  - Edges: (U, V) adjacent if U and V are disjoint

- **Schrijver's Theorem.**
  $\chi(SG_{n,k}) = n - 2k + 2 \quad \text{for } n \geq 2k$



GALE'S LEMMA. 2k+d points X can be put on S^d such that every open hemisphere contains k independent points in X

[Gale 1956]

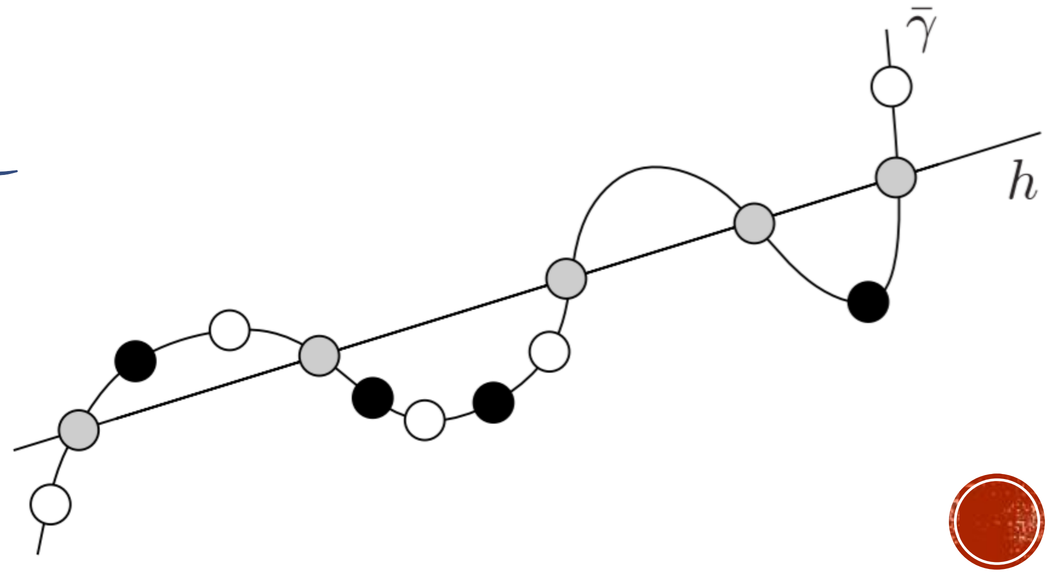
$m(t) = (1, t, t^2, \ldots, t^d)$

$m(1), \ldots, m(2k+d) = w_1 \ldots w_{2k+d}$,

$v_i = (-1)^i \cdot w_i$

We have 2k+d pts, d on hyperplane

$\frac{2k}{2}$ pts will be above h



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Topology is Everywhere

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