Introduction to Computational Topology

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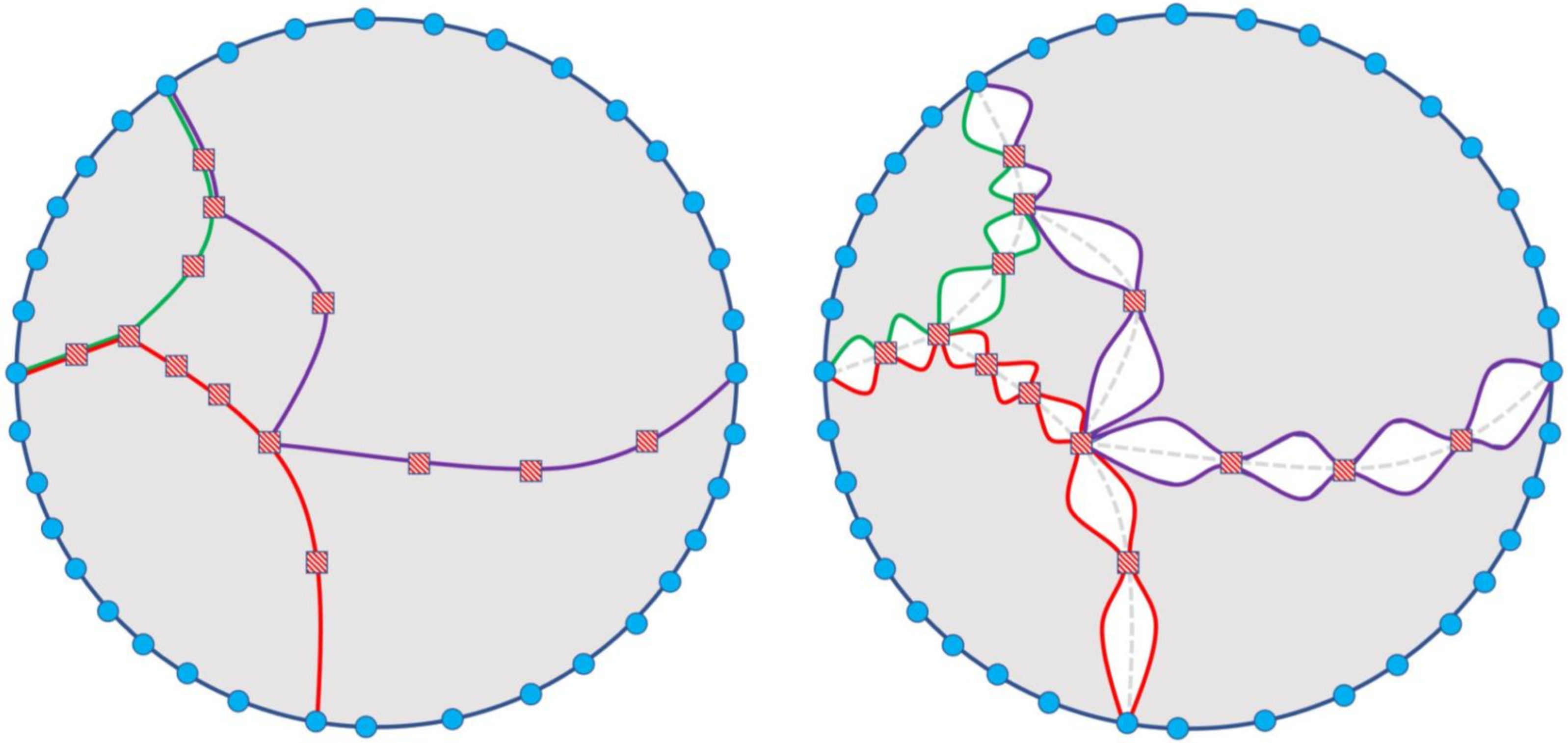
Lecture 17, November 11, 2021

Planar ε-Emulators

ε-Emulator

- Graph H is an ε-emulator of G with respect to terminals T if
  - $e^{-\varepsilon}\, dist_G(x, y) \leq dist_H(x, y) \leq e^{\varepsilon}\, dist_G(x, y)$ for all pairs of terminals x, y



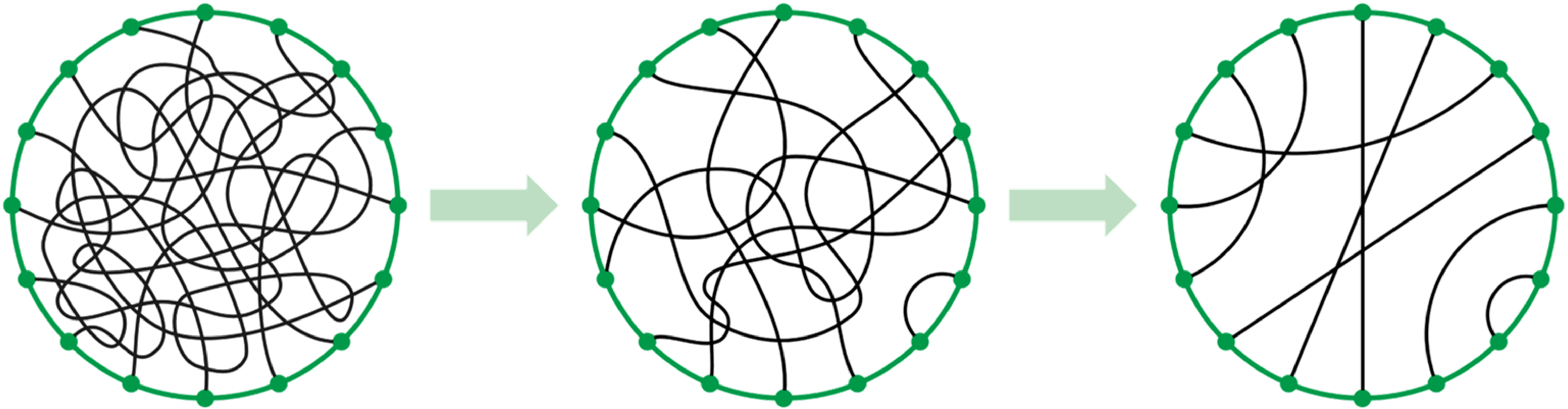
Planar ε-Emulators

[Chang-Krauthgamer-Tan 2022]

Every planar piece with k bdry vertices has a planar ε-emulator of size $O(k \log^{O(1)} k / \varepsilon^{O(1)})$, which can be computed in $O(n \log^* n / \varepsilon^{O(1)})$ time

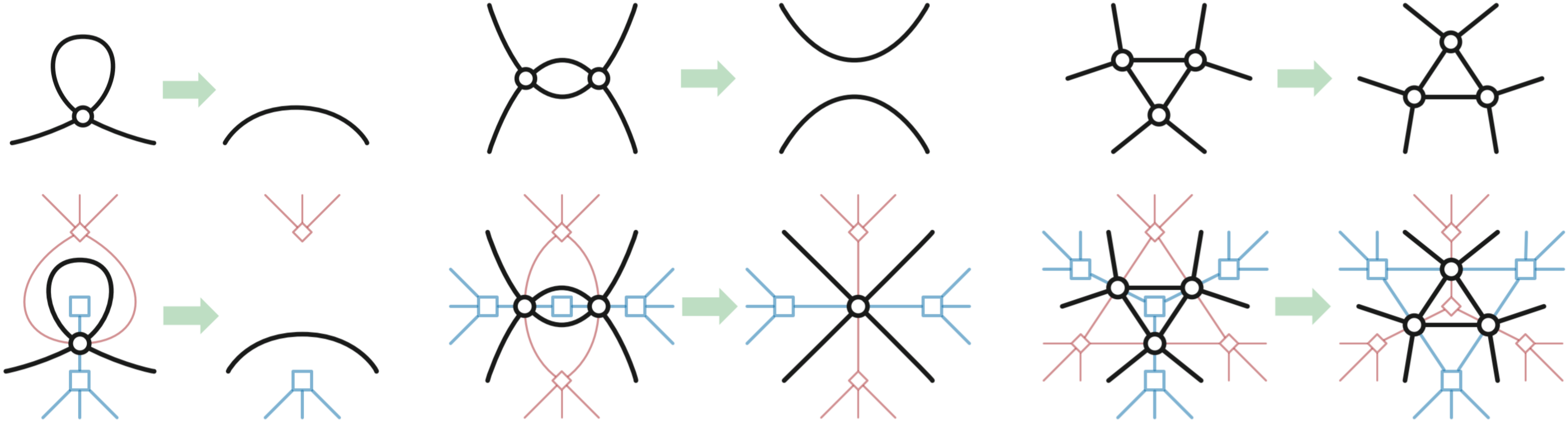
View from Behind the Scene

- During grad school I was thinking about how to tighten tangles.



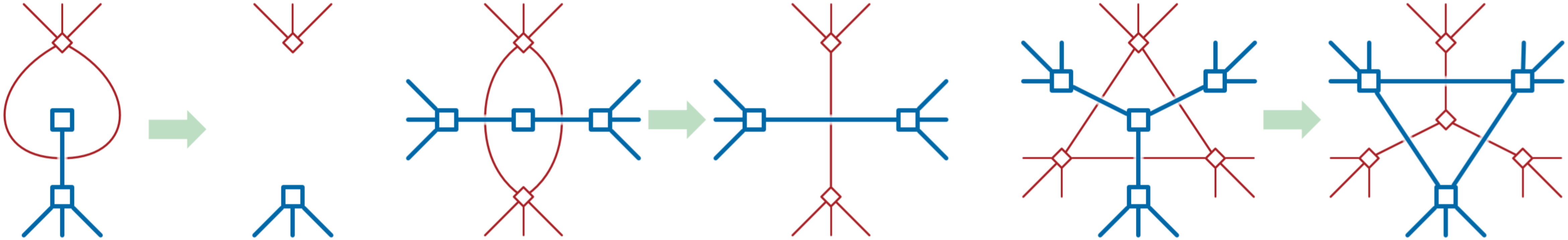
View from Behind the Scene

- You can also tighten tangles using electrical moves



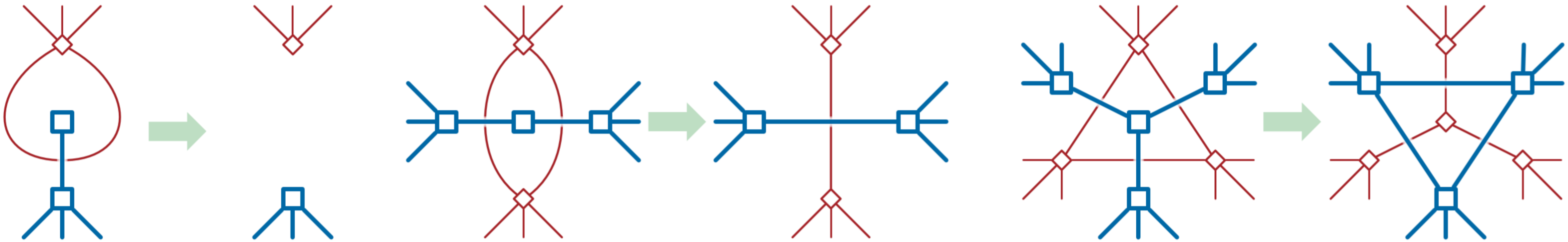
View from Behind the Scene

- Reconstruction problem:
  - Given voltage-current measurements, reconstruct resistor network
    [CdV-Gitler-Vertigan 1996] [Curtis-Ingerman-Mooers-Morrow 1998]
  - Given distance measurements, reconstruct weighted planar graph [Chang-Ophelders 2020]



View from Behind the Scene

- Electrical transformations preserve distances
- $\Omega(n^2)$ lower bound [Krauthgamer-Zondiner 2012] [Cossarini 2019] [Chang-Ophelders 2020]
  - Electrical moves ~ Homotopy moves



Toolbox for Planar Distance Problems

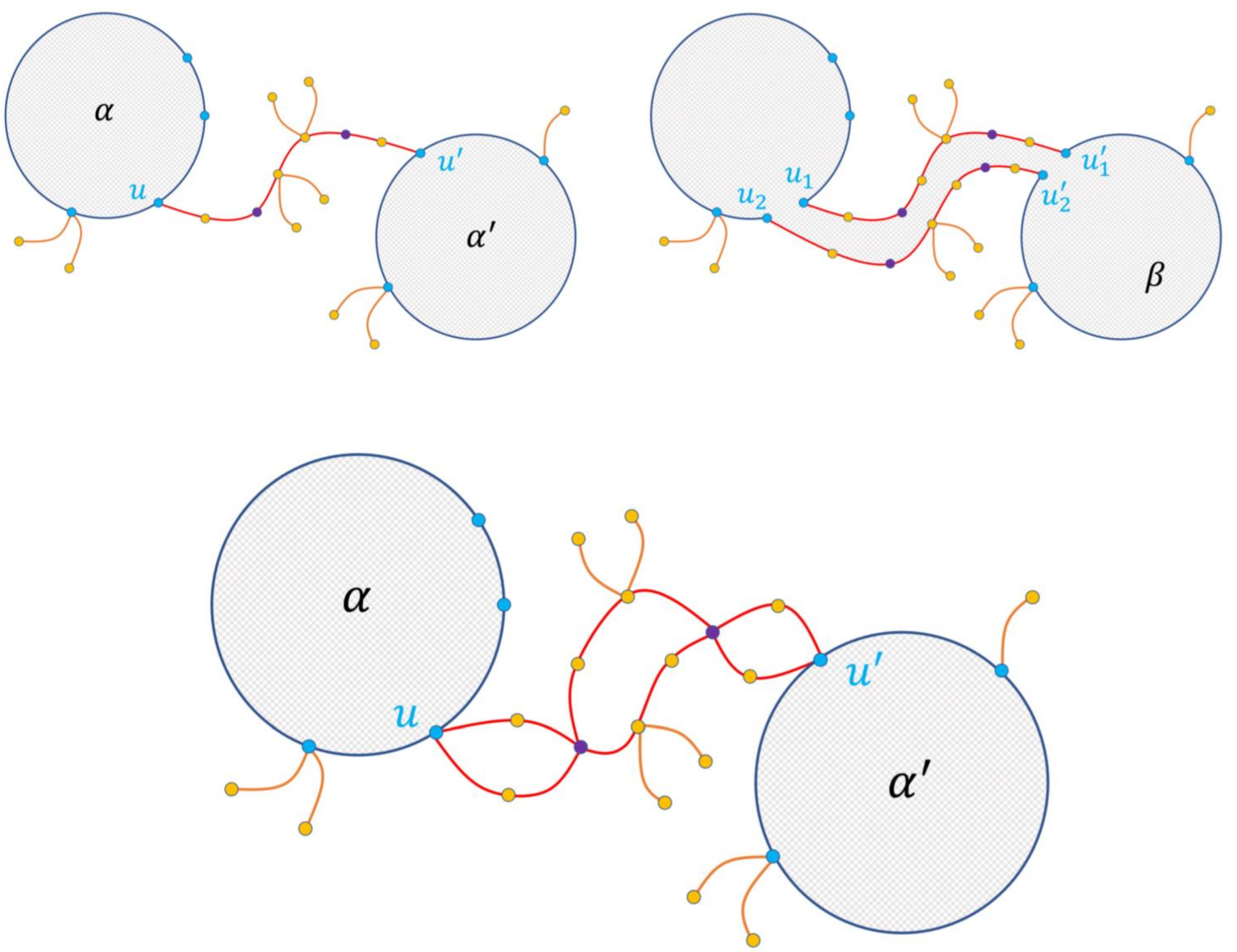
- Multiple-source shortest paths [Klein 2005] [Cabello-Chambers-Erickson 2013]
- Cycle separator decomposition/r-division [Frederickson 1989] [Klein-Mozes-Sommer 2012]
- Monge heap/dense distance graph [SMAWK 1987] [Fakcharoenphol-Rao 2001]
- FR-Dijkstra [Fakcharoenphol-Rao 2001]

DURING STOC 2021 (JUNE 23)

- ZIHAN: [Chang-Ophelders 2020] is nice, but what if terminals not on bdry?
  - $O(k^4)$ and $\Omega(n^2)$ [still open]
- But hey, [Cheung-Goranci-Henzinger 2016] shows $\tilde{O}(k^2/\varepsilon^2)$ if allowing distortion ε

July, 2021

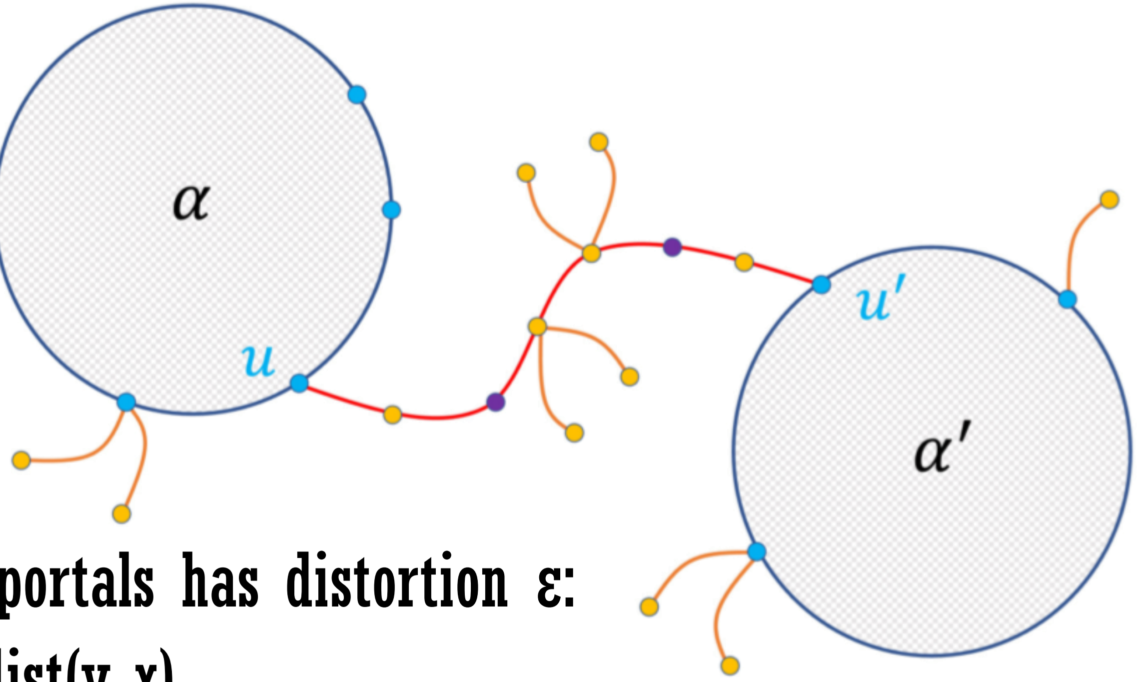
- One-hole planar piece has ε-emulator by modifying [Chang-Ophelders 2020]
- Cut O(1)-hole pieces into one-hole pieces; portals on the cut-open path



Cut&Glue

- Cut open the shortest path between two holes

Portals



- ε-cover of v on P
  - Portals on P such taking detours through portals has distortion ε:

$$\mathrm{dist}(v, p) + \mathrm{dist}(p, x) \leq (1+\varepsilon)\ \mathrm{dist}(v, x)$$

- ε-cover of size O(1/ε) exist [Thorup 2004]
  - O(k/ε) portals to remove one hole
  - Each takes O(n log n) time

July, 2021

- One-hole planar piece has ε-emulator by modifying [Chang-Ophelders 2020]
- Cut O(1)-hole pieces into one-hole pieces; portals on the cut-open path

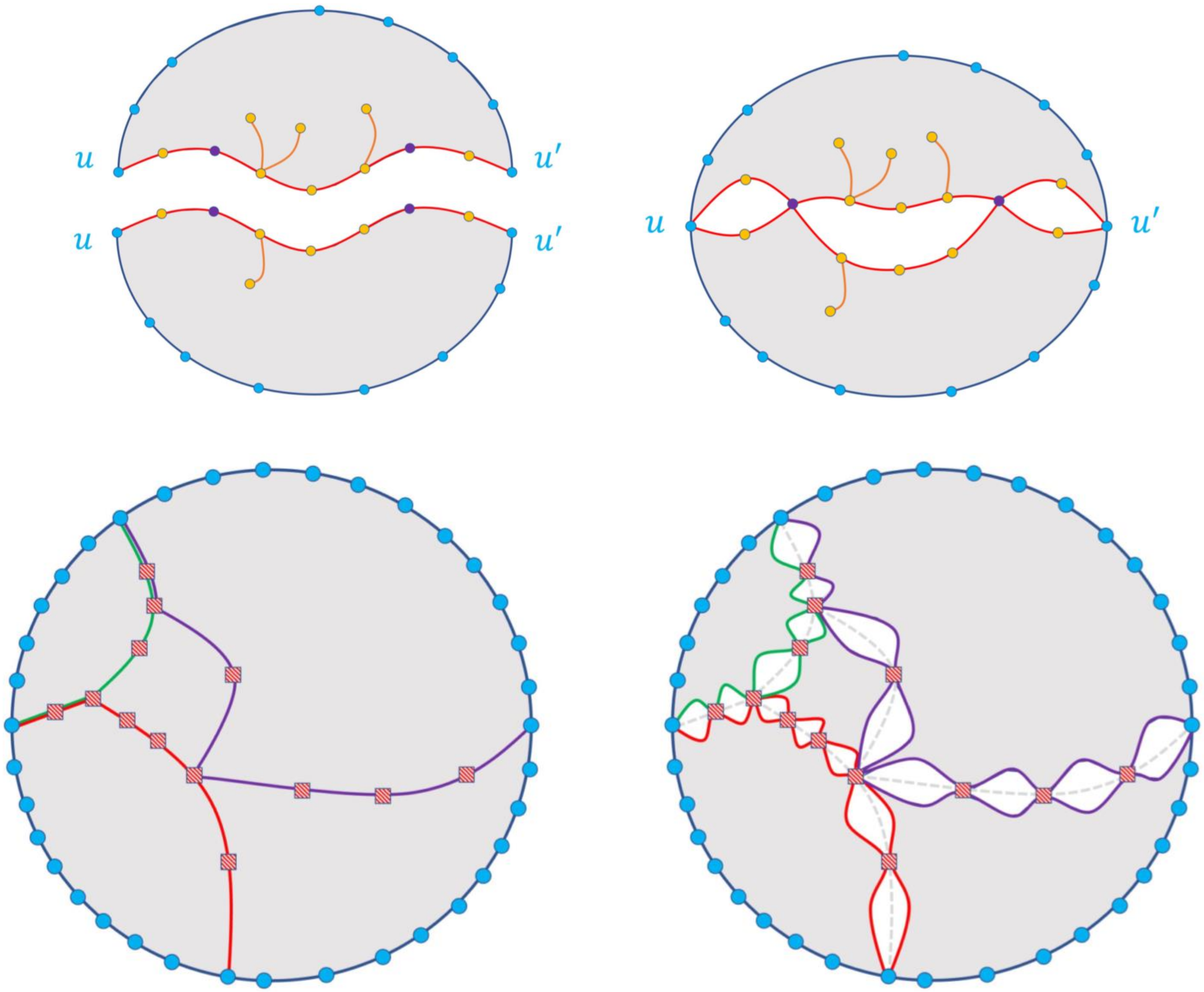
- July 28: Giving TRG talk "Planar emulators for planar graphs"
  - tl;dr Planar graphs are soft and squishy; come and see why.

July 29, 2021

- Hmm that doesn’t work.

August, 2021

- Aug 1: Wait we can cut open along shortest path and portal it. Why not portal all the way through?



Cut&Glue

- Cut open the shortest path among "balanced" terminal pairs

August, 2021

- Aug 1: Wait we can cut open along shortest path and portal it. Why not portal all the way through?
- Aug 2-11: Working hard
- Applications
  - MSSP, min-cut, diameter… you name it
- Aug 12: Spread is a problem; but we have spread reduction

Intermission

Spread

- Spread Φ
  - Ratio between max and min distance between terminal pairs

Too many Portals

- Instead of O(1)-holes, now we have O(log n) levels
  - $O(k/\varepsilon)$ portals from ε-cover is too much!

- Can take at most $O(k/\log^2 k)$ portals
  - Portals at exponentially-increasing intervals from both ends of P
  - distortion $\log \Phi/(k/\log^2 k)$

- But usually we have spread reduction!

September, 2021

- Sep 2: Trying to convince Robi and Zihan that this is fine
  - Nope, that won't work.

Uncontrollable Spread

- Distortion $\log \Phi/(k/\log^2 k)$
  - When $\Phi \leq \exp(k^{0.9})$, $\sim k^{-0.1}$ distortion

- The spread is changing during D&C as we add portals as terminals

September, 2021

- Sep 2: Trying to convince Robi and Zihan that this is fine
  - Nope, that won't work.

- Zihan: Tricolor sets based on short/medium/long ranges, here's why…
  - Trying hard to make [Chang-Ophelders 2020] useful
- Sep 23
  - Zihan: Hey I fixed it, but the spread is not so good

When Spread is Large

- Hierarchical clustering of terminals
  - Form level-i cluster if within distance $\sim k^{2i}$

- Draw cluster tree
  - Cluster is expanding if parent cluster is at least $\exp(k^{-0.7})$-factor bigger
  - At most $k^{0.7}$ levels if all clusters are expanding
  - Spread at least $\exp(k^{0.9})$, thus some cluster is non-expanding

When Spread is Large

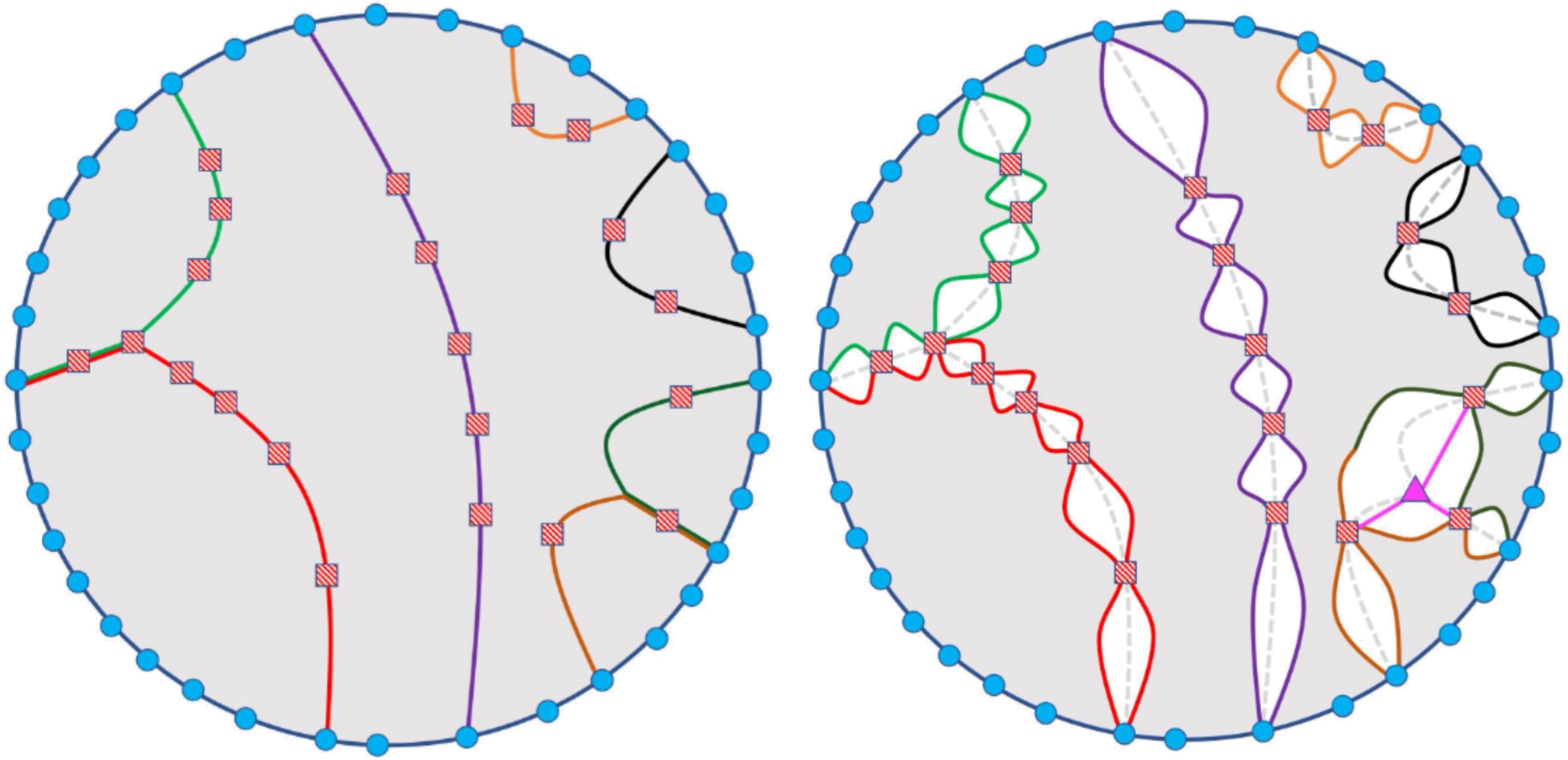
- If the non-expanding cluster C is balanced (between k/5 and 4k/5):
  - Cut along the "flower" formed by terminals in C
  - Portal from parent cluster C' using ε_k-covers for $\varepsilon_k = k^{-0.1}$
- Distance between C and K-C' are far away

When Spread is Large

- If all non-expanding clusters are not balanced:
  - One of such clusters C is huge (of size at least 4k/5)

- Find all non-expanding clusters of maximal level
  - All such clusters are within $\sim k^{0.7}$ levels from C
  - Cut along the "flowers" formed by all terminals in all max level clusters

Lemma. C_1 and C_2 two disjoint clusters.
Then terminal pairs from C_1 and C_2 are non-crossing.



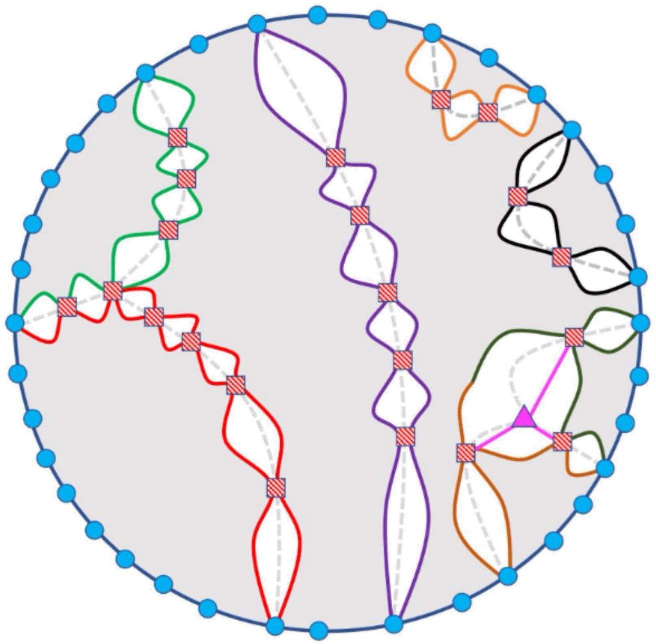
PLANAR ε-EMULATORS

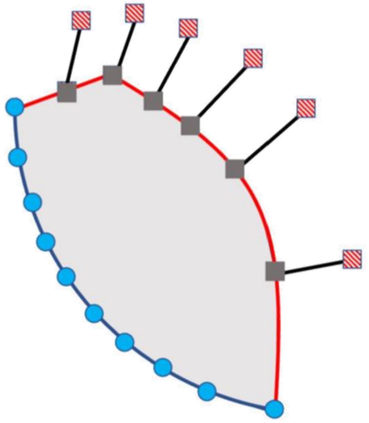
[Chang-Krauthgamer-Tan 2022]

Every planar piece with k bdry vertices has a planar ε-emulator of size $O(k \log^{O(1)} k / \varepsilon^{O(1)})$, which can be computed in $O(n \log^* n / \varepsilon^{O(1)})$ time

OCTOBER, 2021

- Grinding through the details
  - Oh no, too many deg-3 vertices during D&C
  - Wait, the distances are shrinking!?
  - Well the sub-instances are not a disk any more
  - OH MY GOSH WE DON'T HAVE TIME
  - The applications doesn't work #$!&?
  - abort abort ABORT
  - Hey we can bootstrap the running time



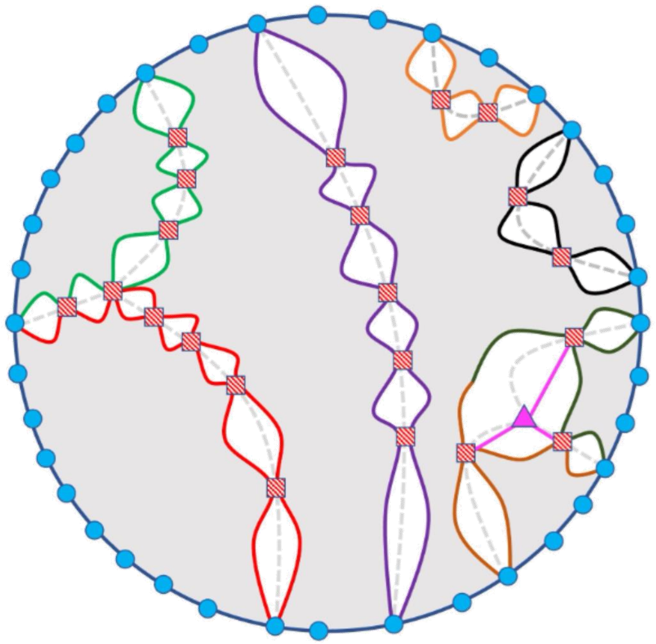


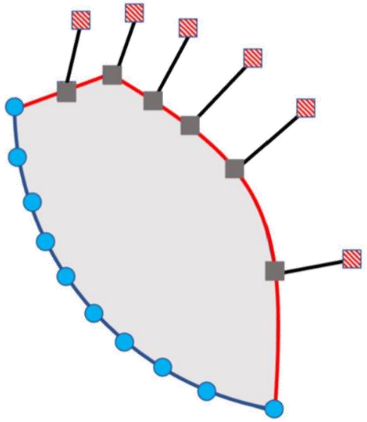
Bootstrap Lemma. Planar ε-emulator can be computed in $O_{\varepsilon}(\, n \log\log n\,)$ time

Bootstrap Lemma. Planar ε-emulator can be computed in $O_\varepsilon(n \log\log\log n)$ time

October, 2021

- Grinding through the details
  - Oh no, too many deg-3 vertices during D&C
  - Wait, the distances are shrinking!?
  - Well the sub-instances are not a disk any more
  - OH MY GOSH WE DON'T HAVE TIME
  - The applications doesn't work #$!&?
  - abort abort ABORT
  - Hey we can bootstrap the running time
  - The neg-weighted shortest path application still doesn't work, oh well





November, 2021

- Nov 4: STOC submission
- Nov 11: Present the result in class
  - Hey, there are typos here and there
  - Wait how does this work again?
  - This slide is self-referencing now

Real Research is Messy

Next Time.

Some more applications to fixed-point theorems.