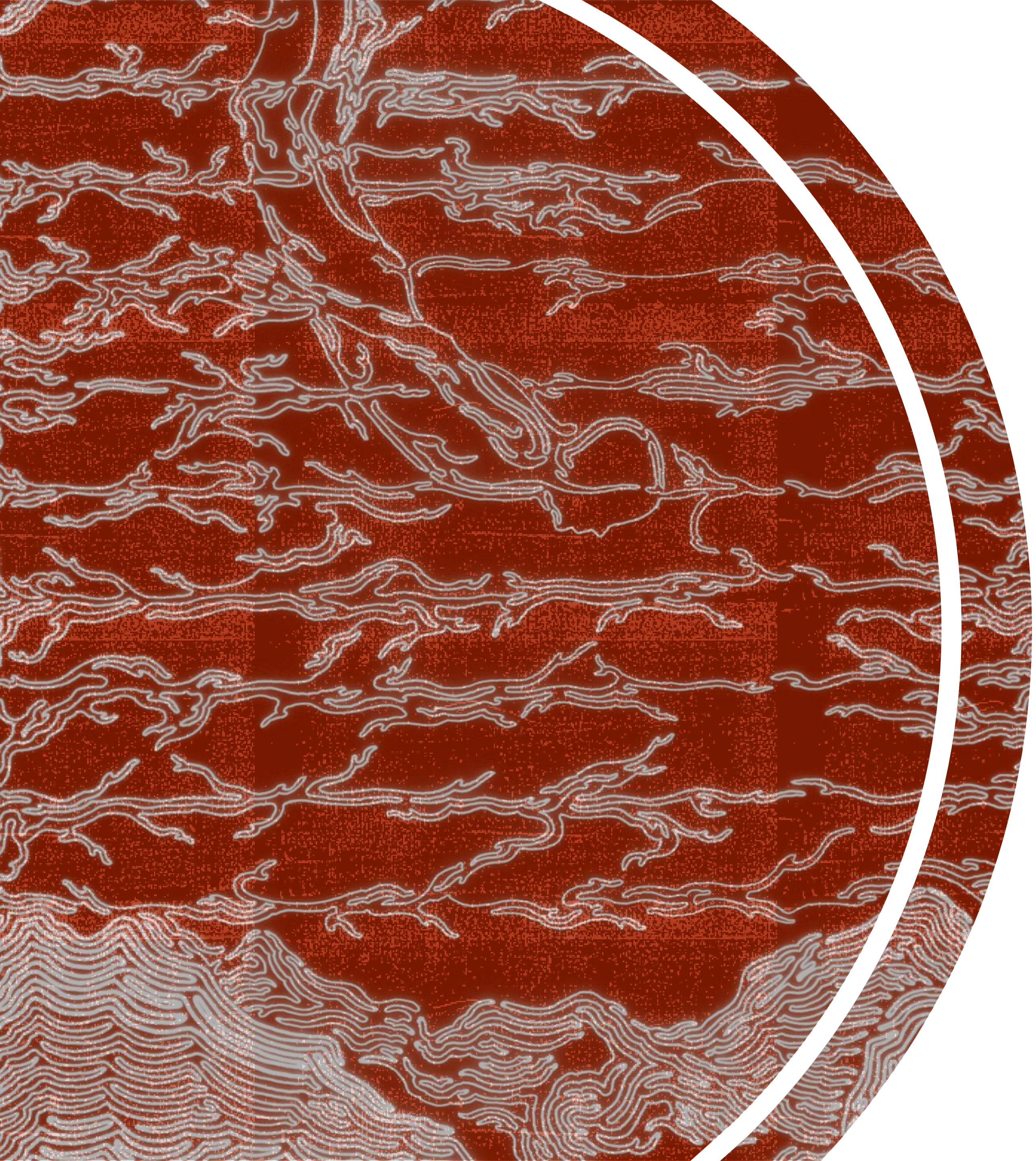

Introduction to
Computational Topology

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Lecture 16, November 4, 2021

Discrete Morse Theory

Today's Goal

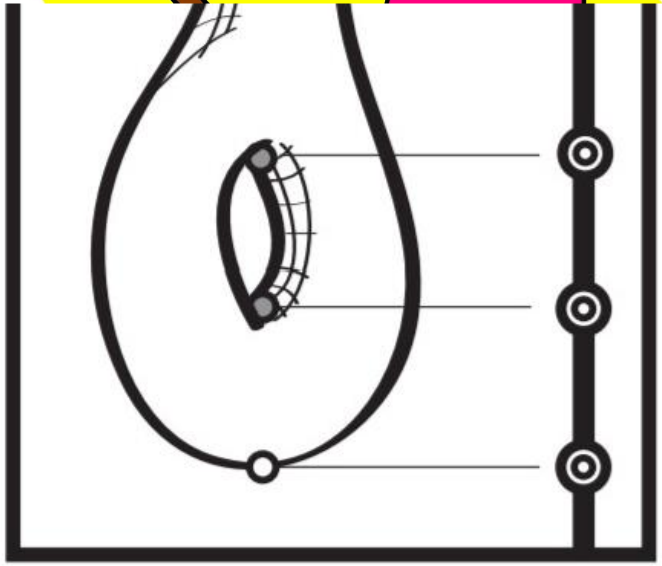
- Introduce a discrete version of the Morse theory that works for complexes

Definitions

- **Height function h**: M -> R

- **Sub-level set $M_{\leq a}$**: $h^{-1}(-\infty, a] = \{x : h(x) \leq a\}$

- **Critical points**: where the topology changes

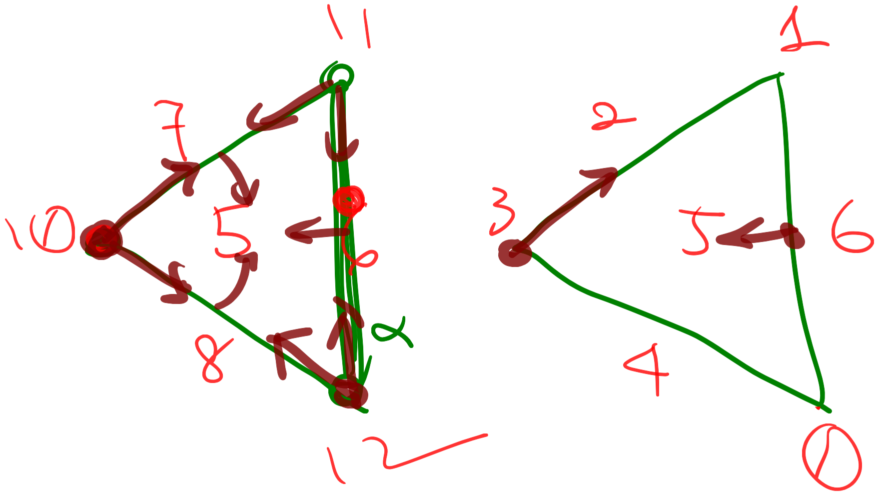


Definitions

- Intuition: Morse function h is not important, only gradient field ∇h

- Discrete gradient Given $f: K \rightarrow R$,
  - For k-simple α and (k+1)-simplex β: $f(\alpha) \geq f(\beta)$

Draw arrow from α to β if $f(\alpha) \geq f(\beta)$ & $\alpha \leq \beta$.



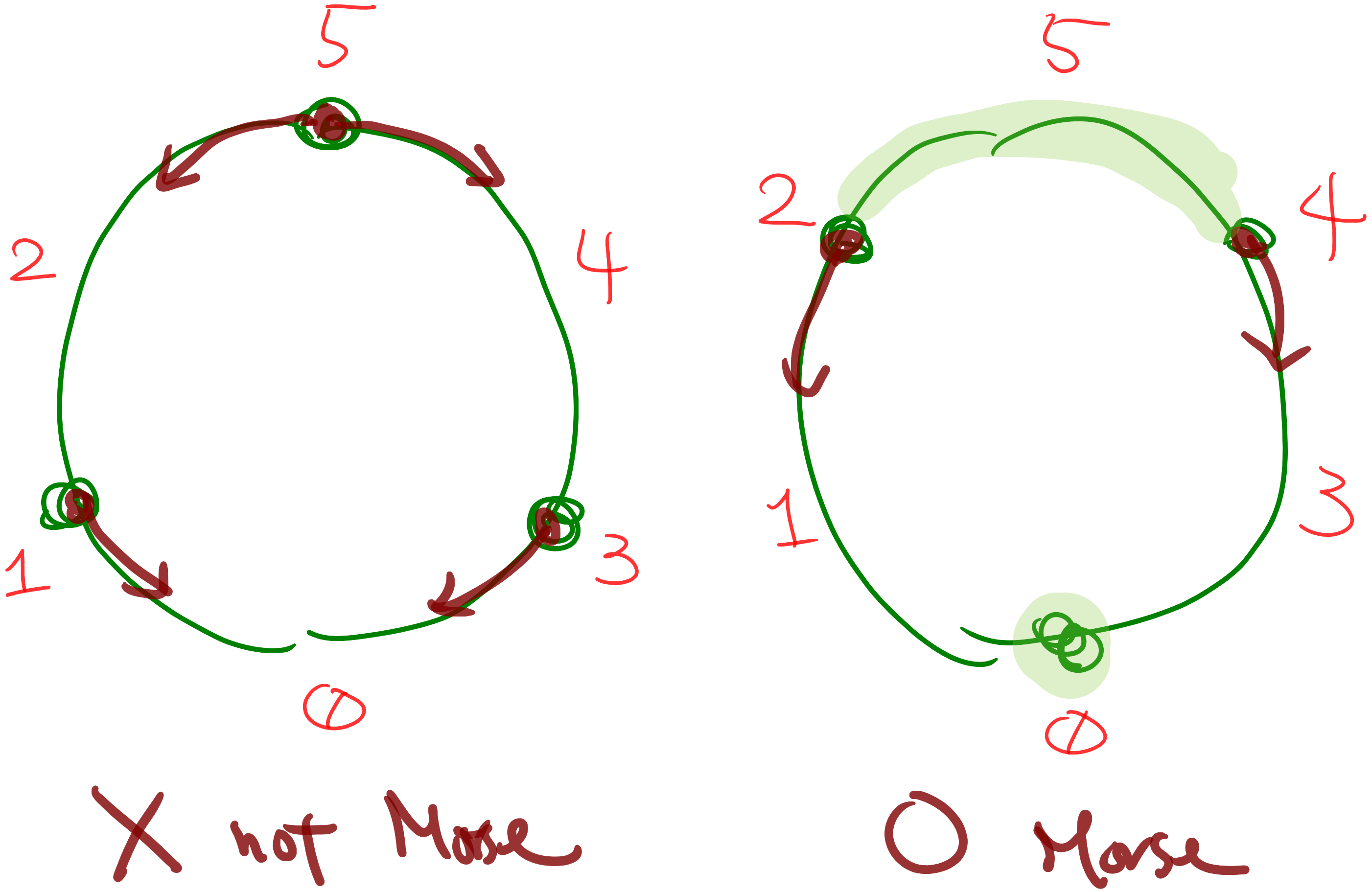
- Discrete Morse function
  - All discrete gradients are unique (if exist)

Definitions

- Critical cell
  - Cell with no discrete gradient

- Sub-level set $K_{\leq C}$

$$\{\beta : \beta \text{ in those } \alpha \text{ that } f(\alpha) \leq C\}$$



Example

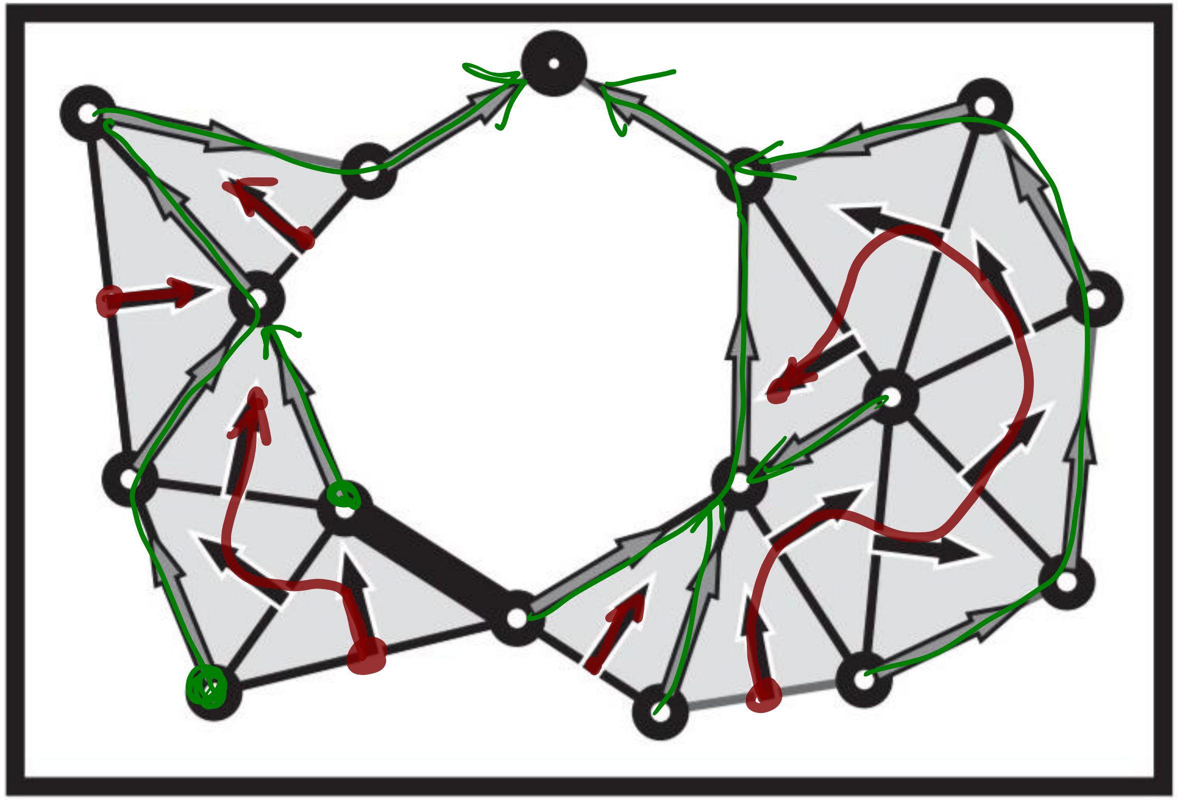
- Which one is Morse?

Discrete Flowlines

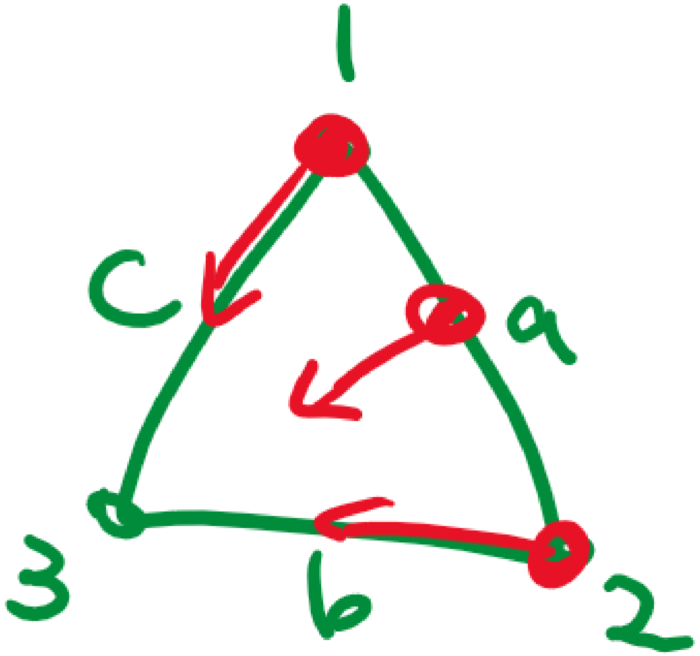
- Pairing of neighboring k- and (k+1)-simplex are canceled

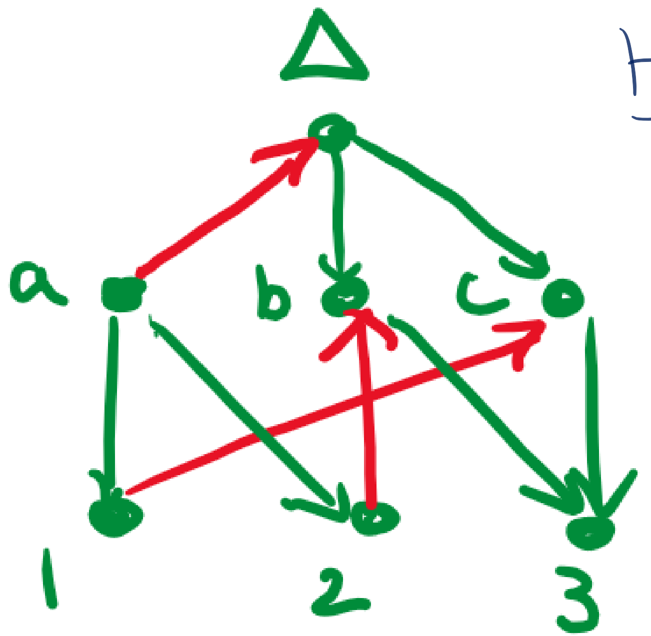
Disc. Flow Lines :

$(\alpha_0 \prec \beta_0) \succ (\alpha_1 \prec \beta_1) \succ \cdots \succ (\alpha_k \prec \beta_k)$ in K



PROPOSITION. A vector field is the gradient field of a discrete Morse function iff it is acyclic.





Hasse diagram:

A cycle in flow line

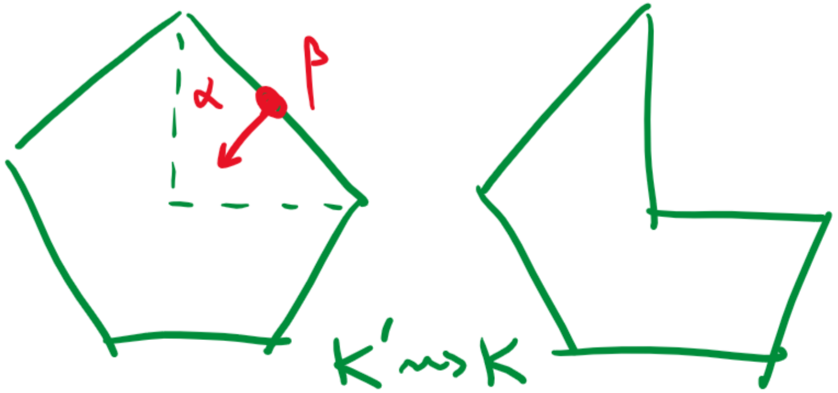
⇓

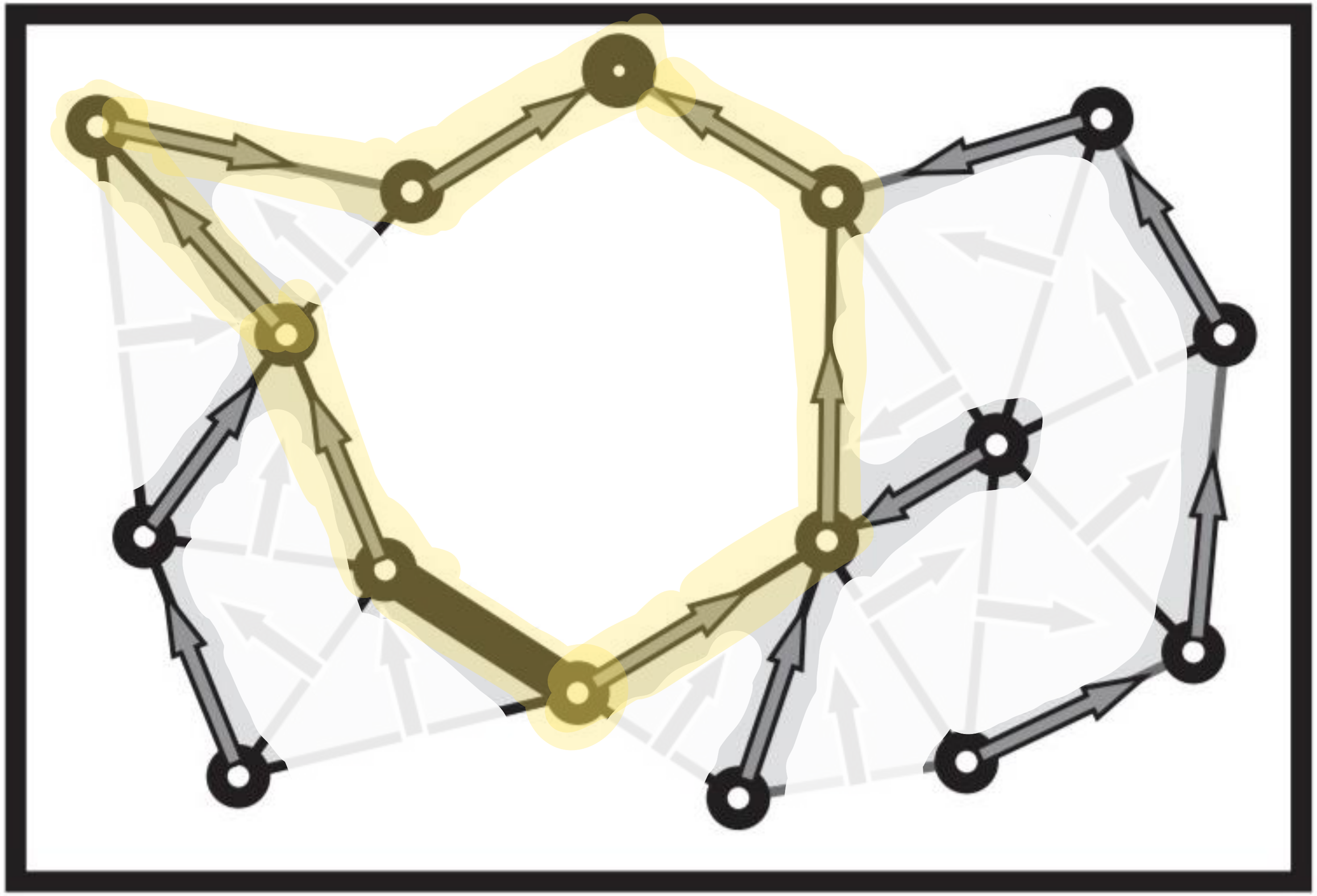
A cycle in diagram.

Properties

- $K_{\leq b} \simeq K_{\leq a}$ if no critical points in (a, b]
- $K_{\leq b} \simeq K_{\leq a} \cup$ {k-handle} if (a, b] has k-dim critical point p

- Collapse (discrete homotopy)
  - If $K' = K \cup \{\alpha, \beta\}$ where β is the face to only α, then K' can be collapsed to K



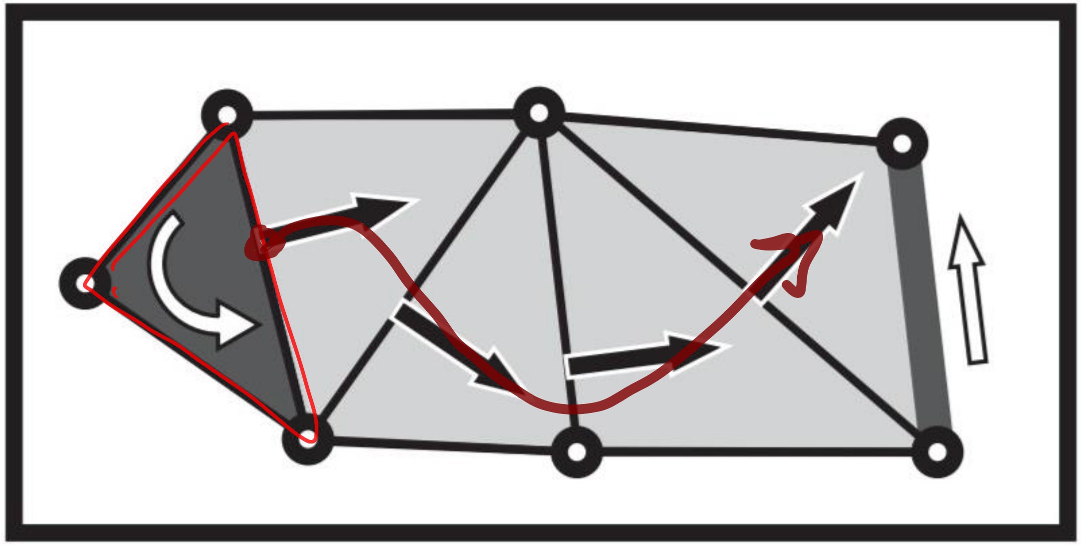


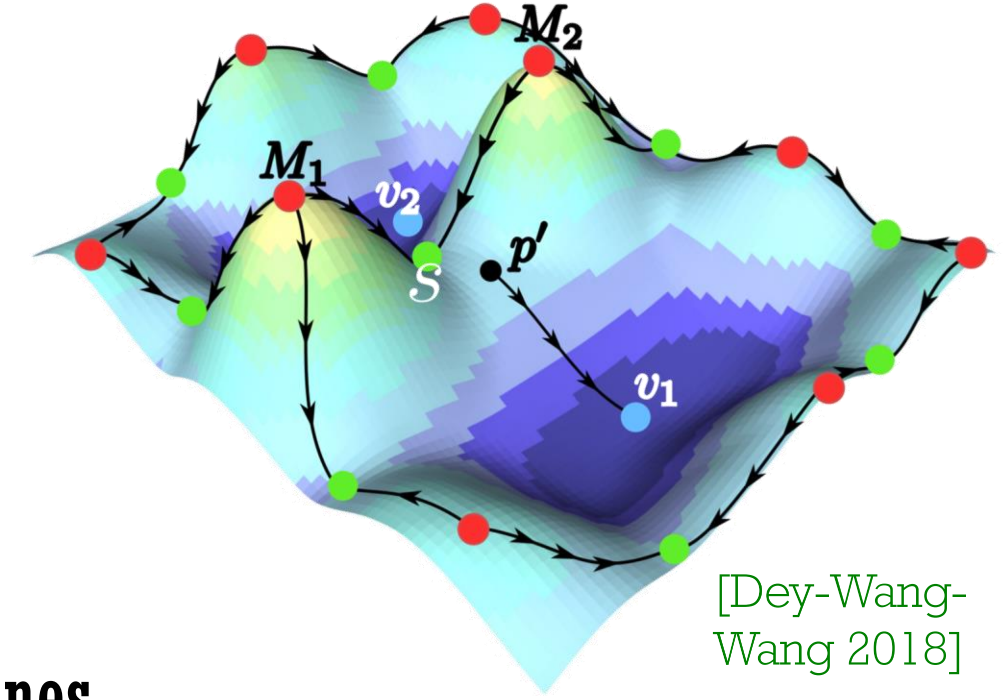
Example

- Collapsing the flowlines

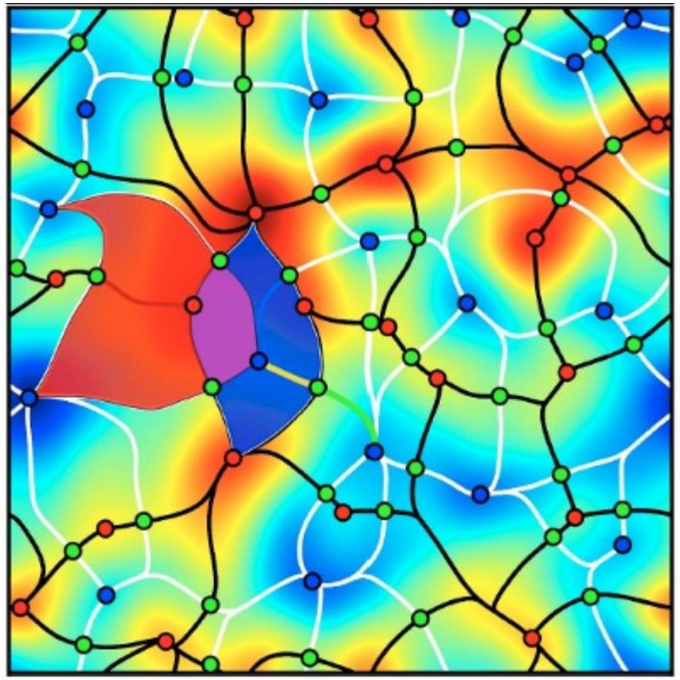
Discrete Morse Complex

- MC_k: ⟨k-dim critical cells⟩
- Boundary map ∂_k:
  all (k-1)-dim critical cells reachable by flowlines

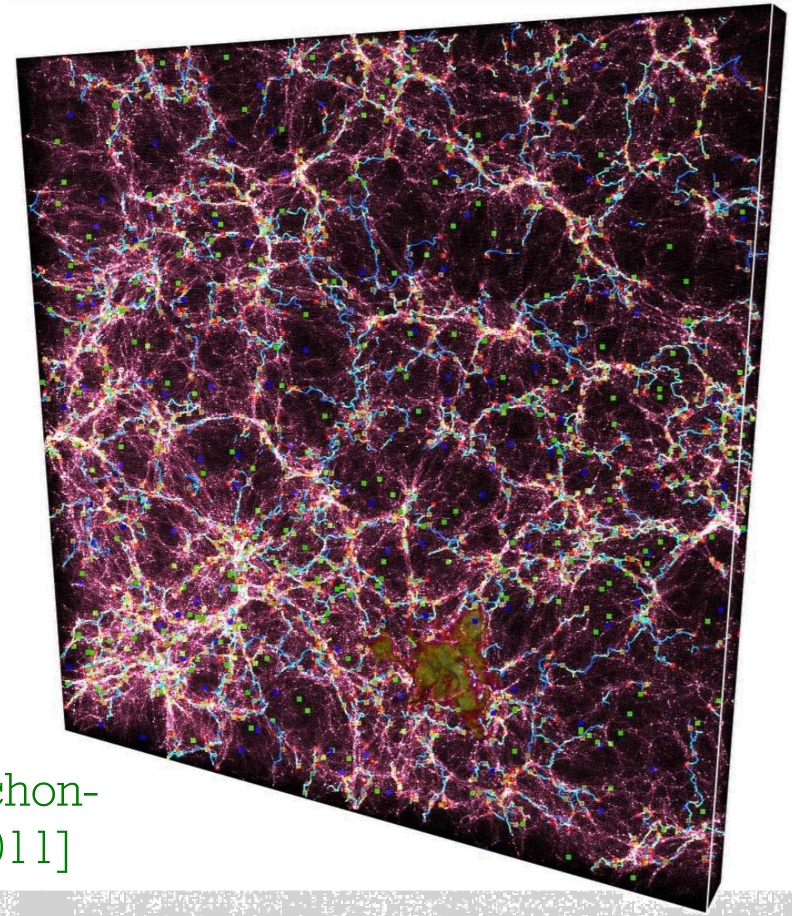




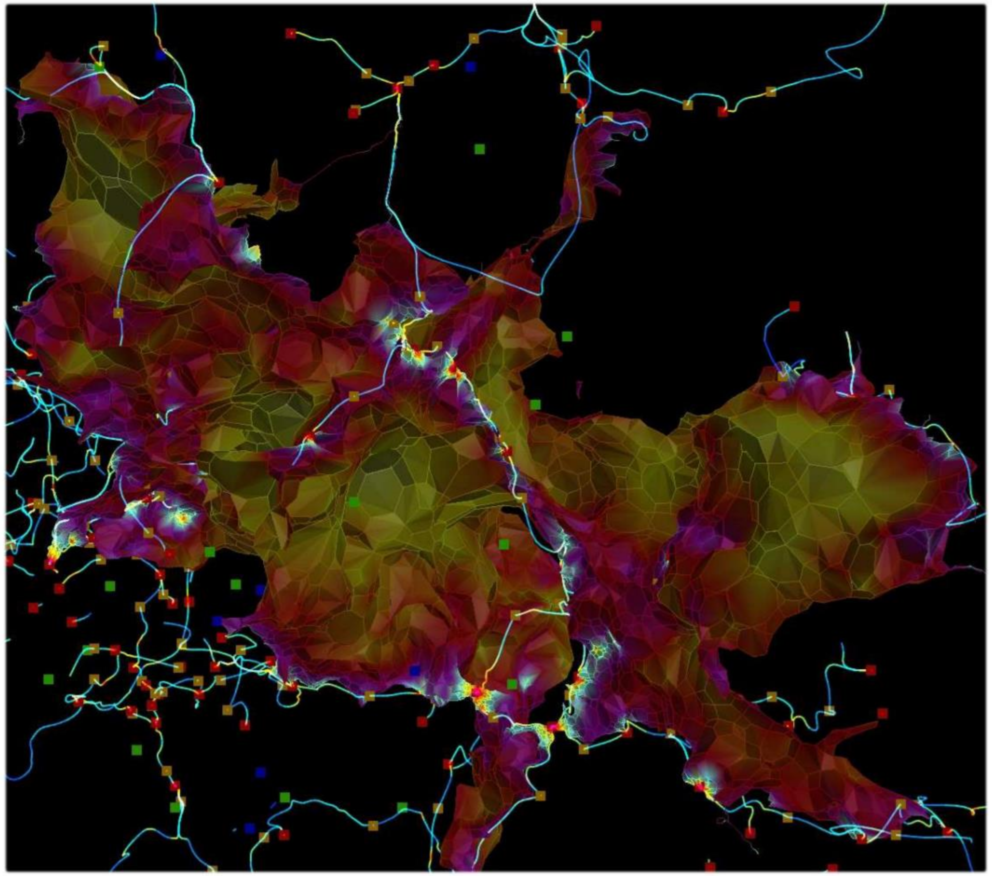
[Dey-Wang-Wang 2018]



[Sousbie 2011]



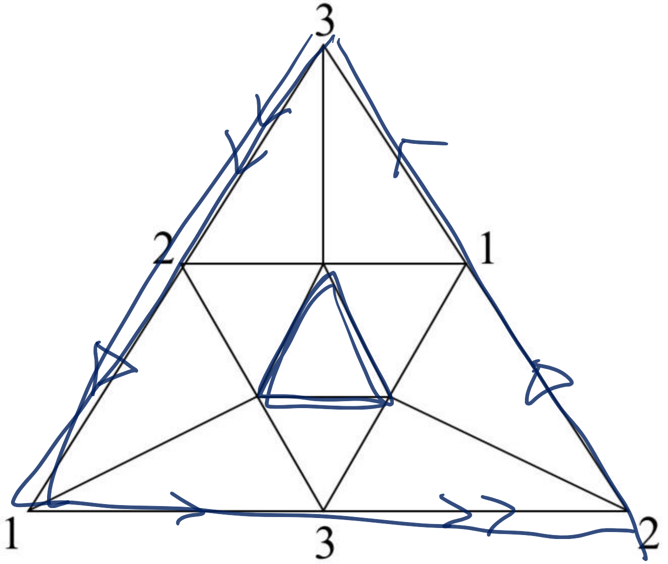
[Sousbie-Pichon-Kawahara 2011]

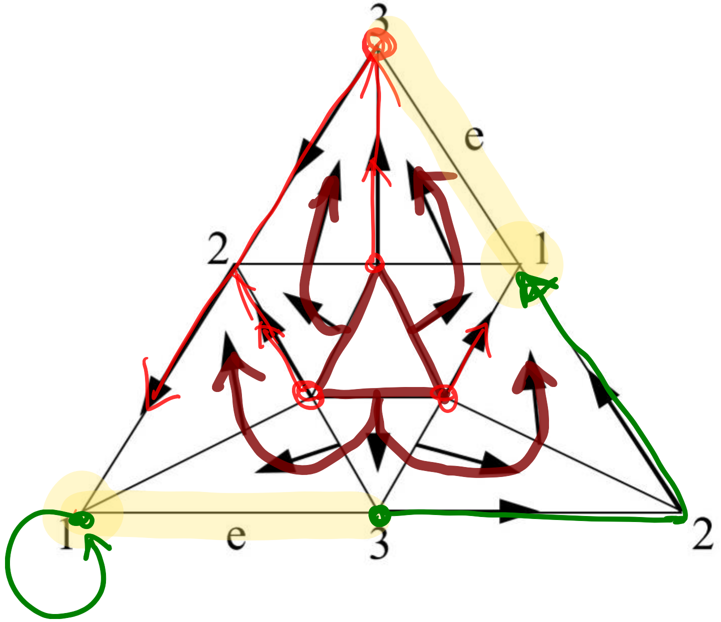


Morse Homology Theorem [Forman 1998]

$MH_n(K) \simeq H_n(K)$ for any <u>complex</u> K

Exercise



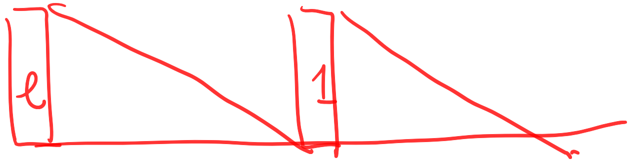


$MC_0 = \langle [1] \rangle$ $MC_1 = \langle [e] \rangle$

$\partial[1] = 0$ $\partial[e] = 1 + 1 = 0 \ /\mathbb{Z}_2.$

1 $\emptyset$

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$



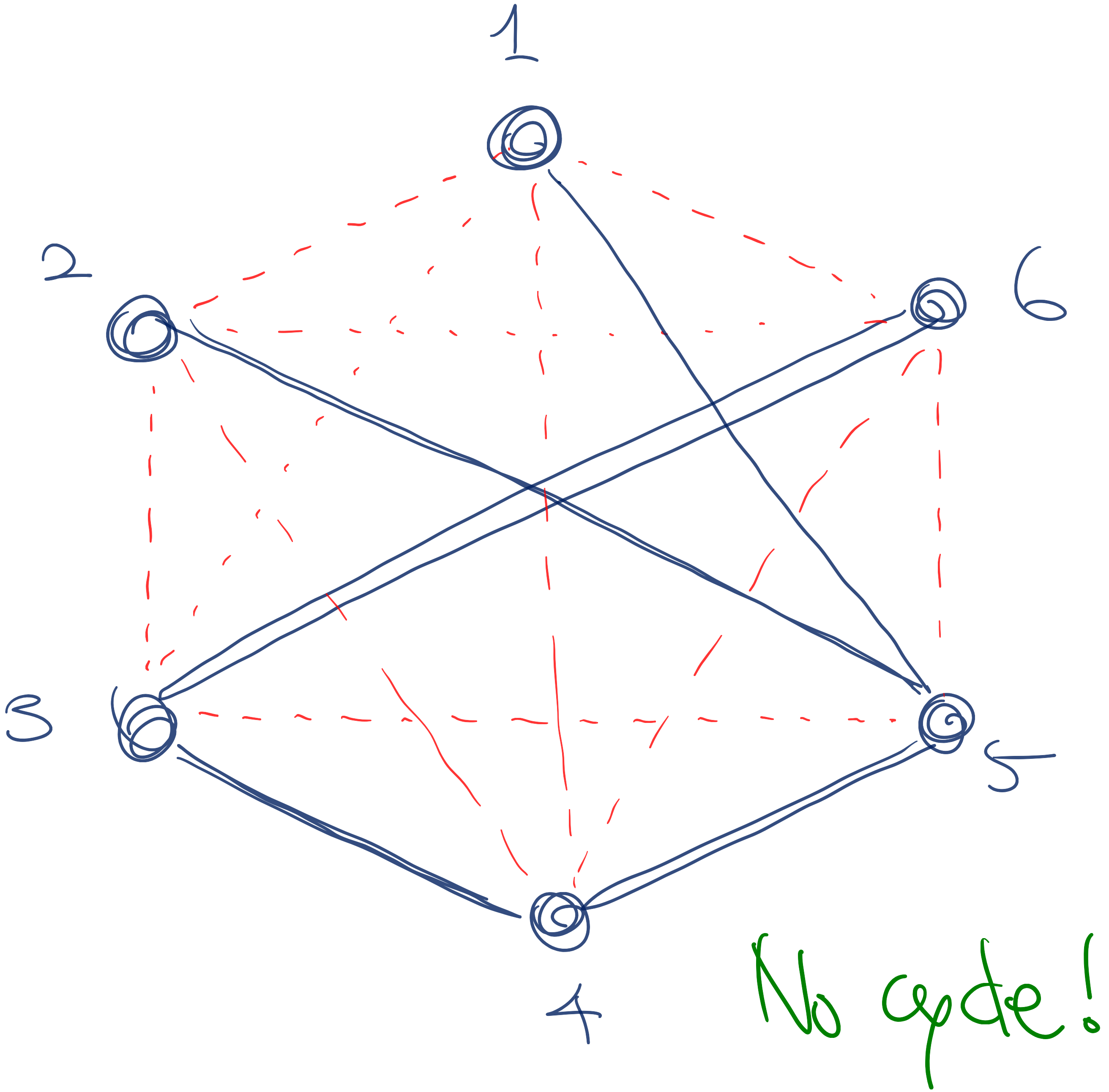
Intermission

Food for Thought.

Forget about homology.

We can use it to simplify complexes!

Evasiveness (Why Lowerbound is Hard)



Motivating Puzzle

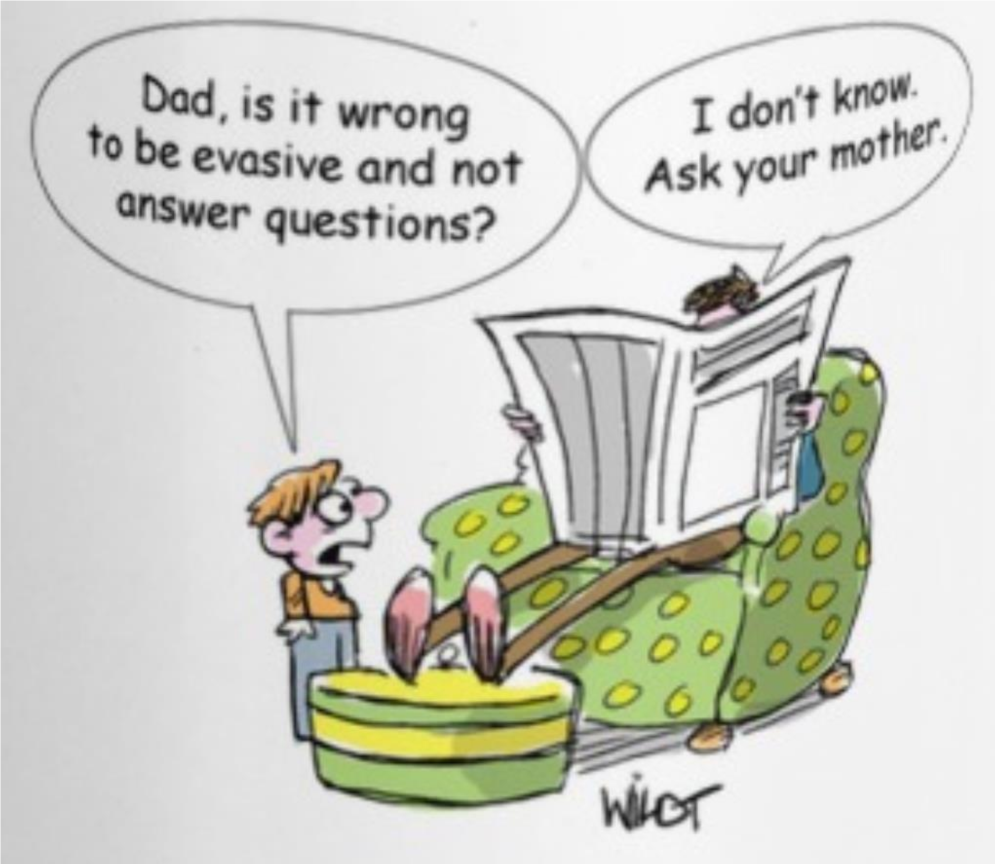
- Question allowed:
  "Is edge (i,j) in G?"
- Goal:
  Does G have a cycle?

Formulation

- Let $g(x_1, \ldots, x_E)$ be Boolean function
- Property T
  - g(X) = 0 iff graph X has property T
- Monotone property
  - If graph X has T, subgraph Y of X must be in T

- Determine if graph G satisfies T

How many Qs do you need?



Evasiveness Conjecture

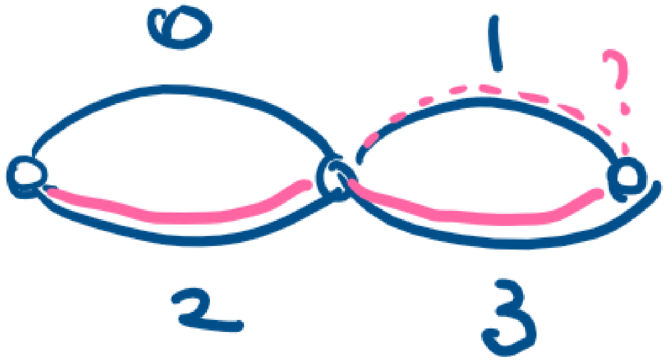
[Aanderaa-Rosenberg 1973]

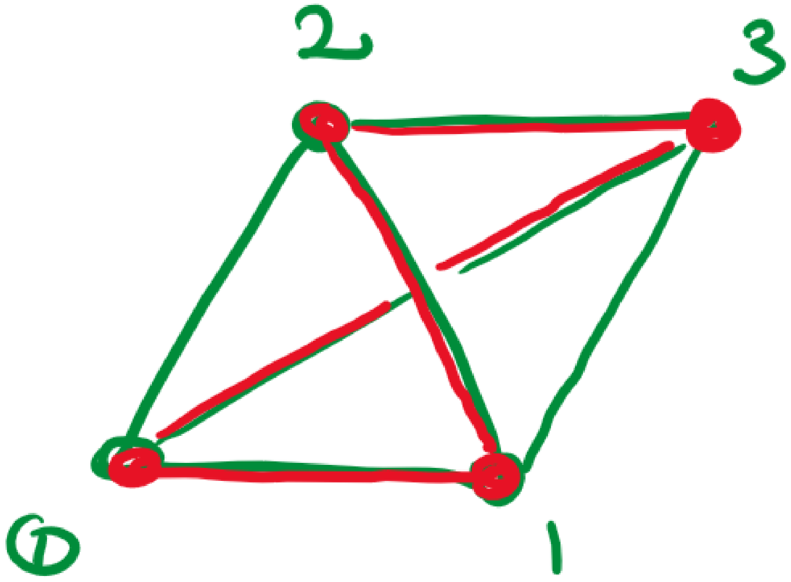
If property T is monotone, nontrivial, and symmetric,
then T is evasive, i.e. requires $\binom{n}{2}$ questions

Topological Approach

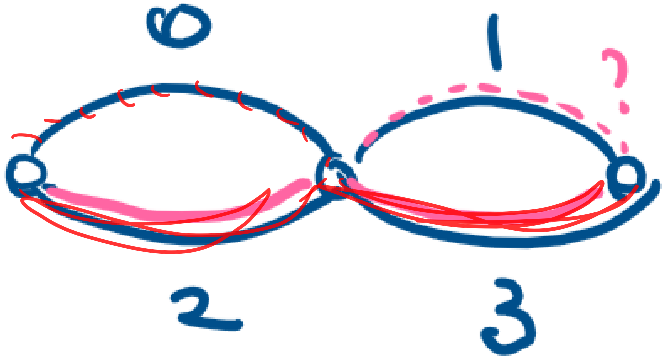
[Kahn-Saks-Sturtevant 1984]

- Construct complex K_P
  - Add cell σ if σ satisfies P

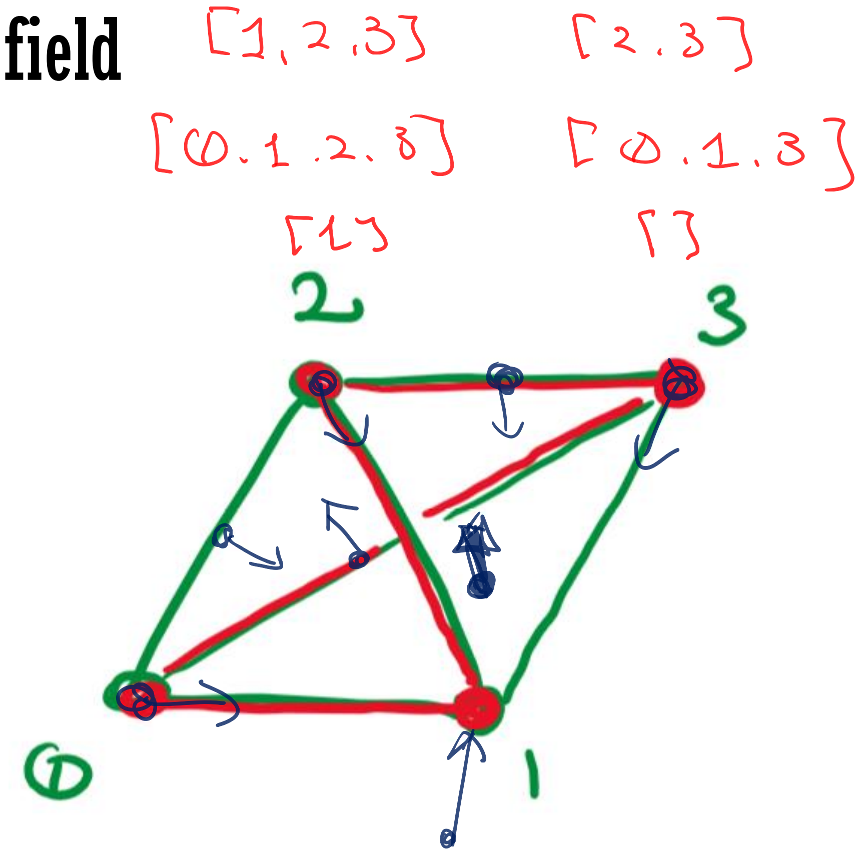


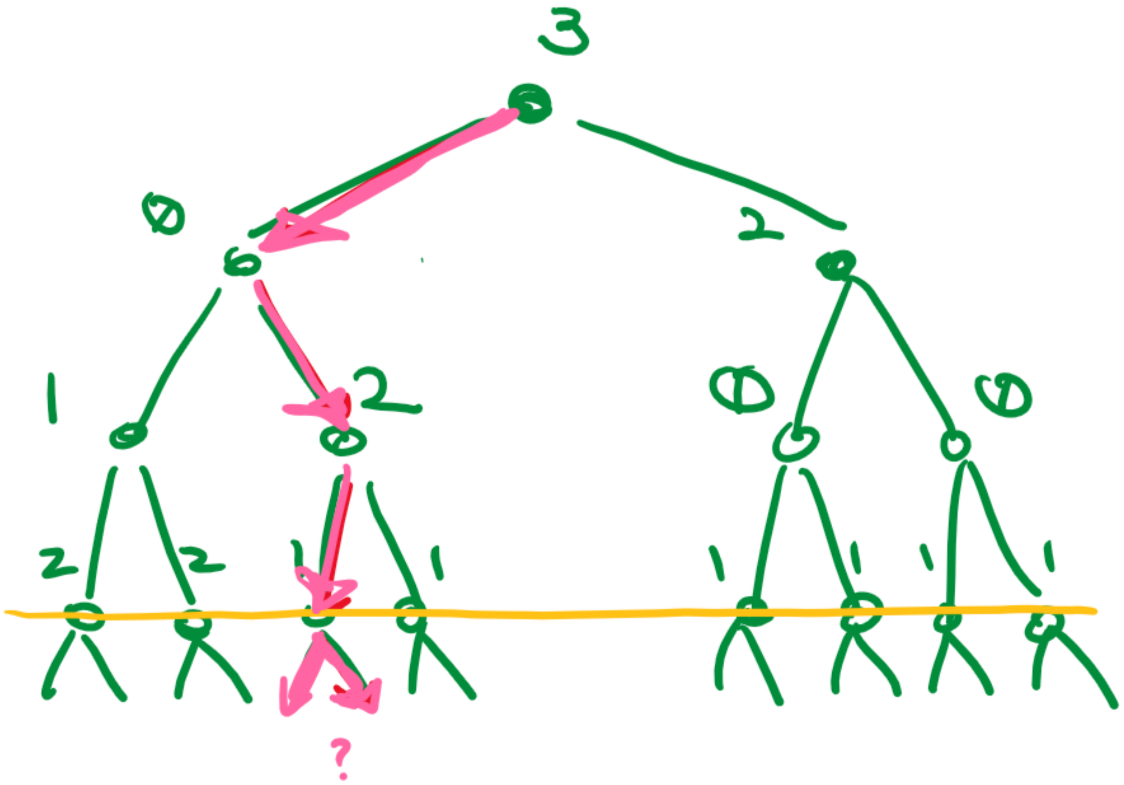


Observation

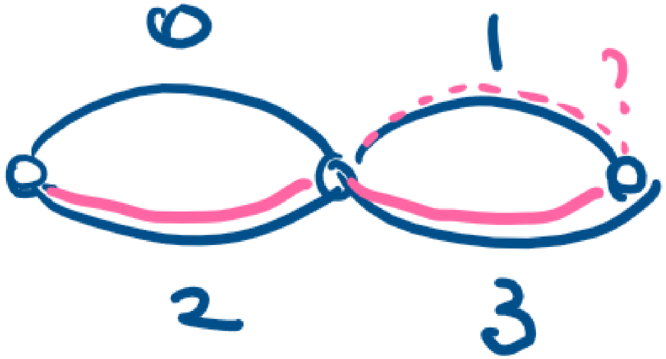


- Guessing algorithm induces discrete gradient field

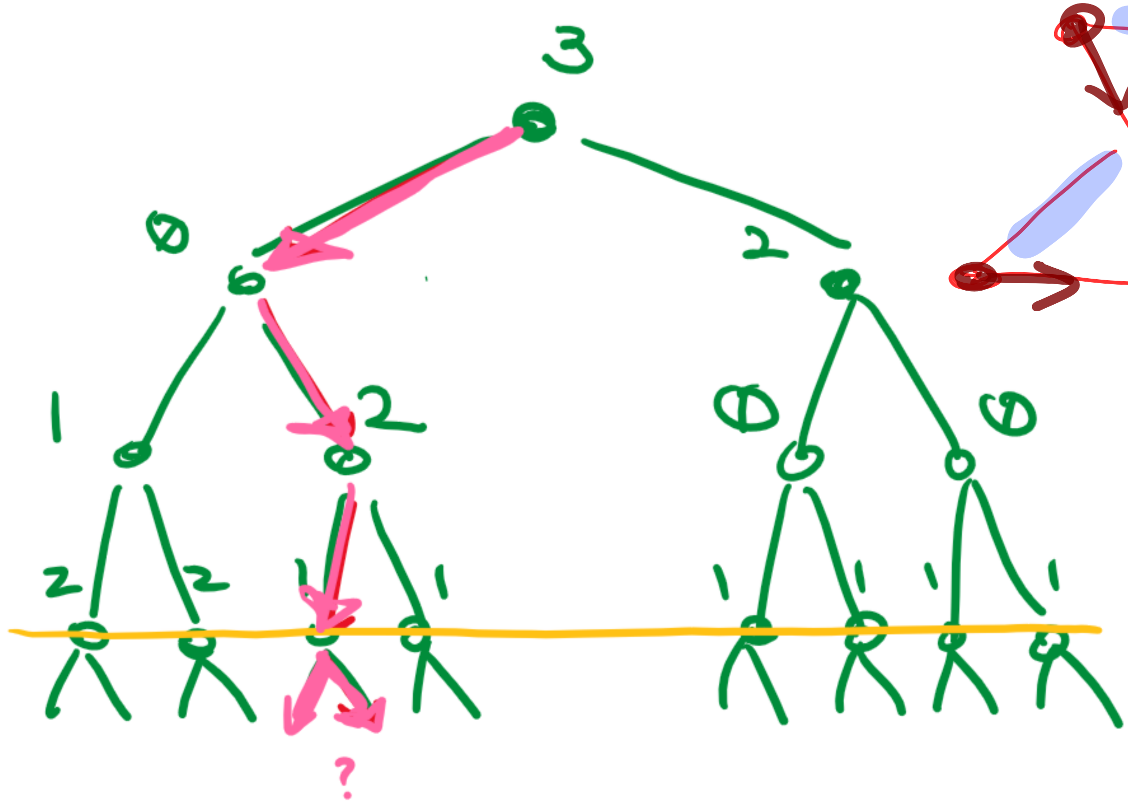


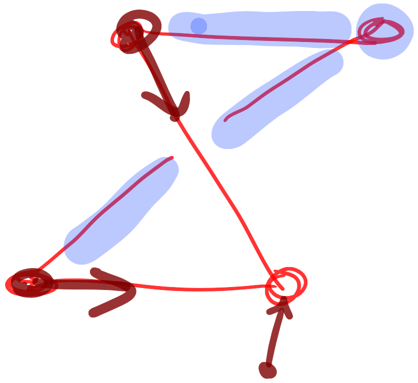


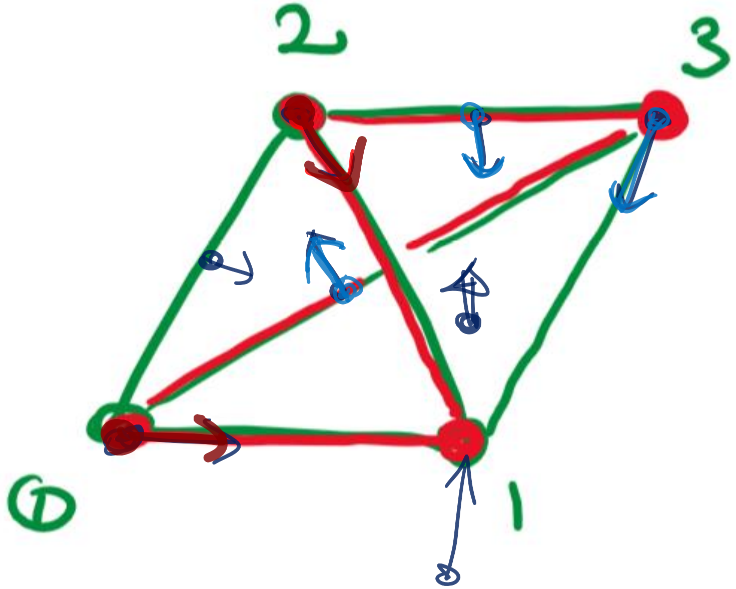
Observation



- Critical cells in K_p correspond to graph pairs that require the last question







Counting Evaders [Forman 2000]

#Evaders under any algorithm is at least

$2 \Sigma_k \dim H_k(K_P)$

Proving Lowerbound by Showing K_p Non-Trivial

Next Time.

Almost end of the term. We'll see!