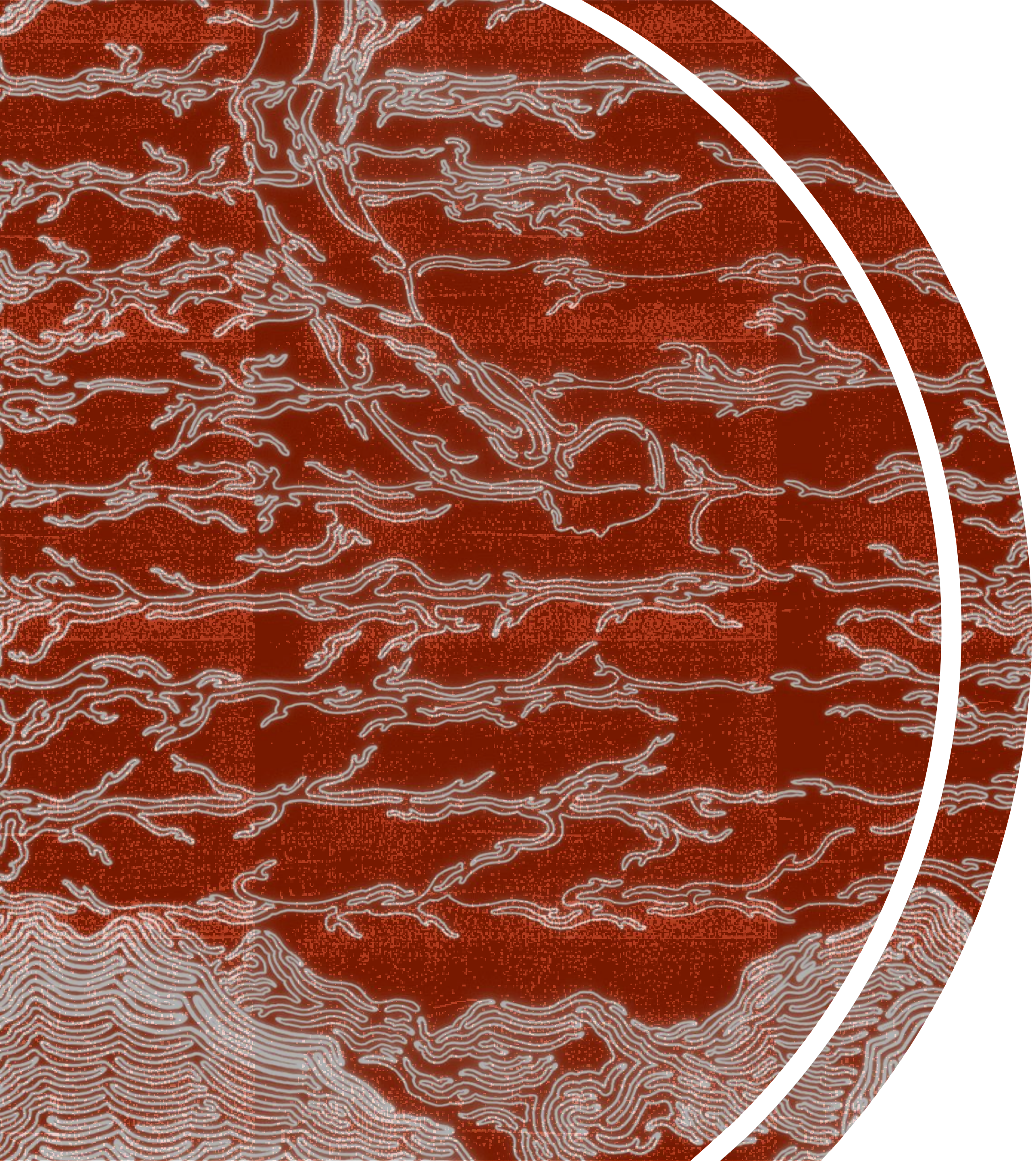
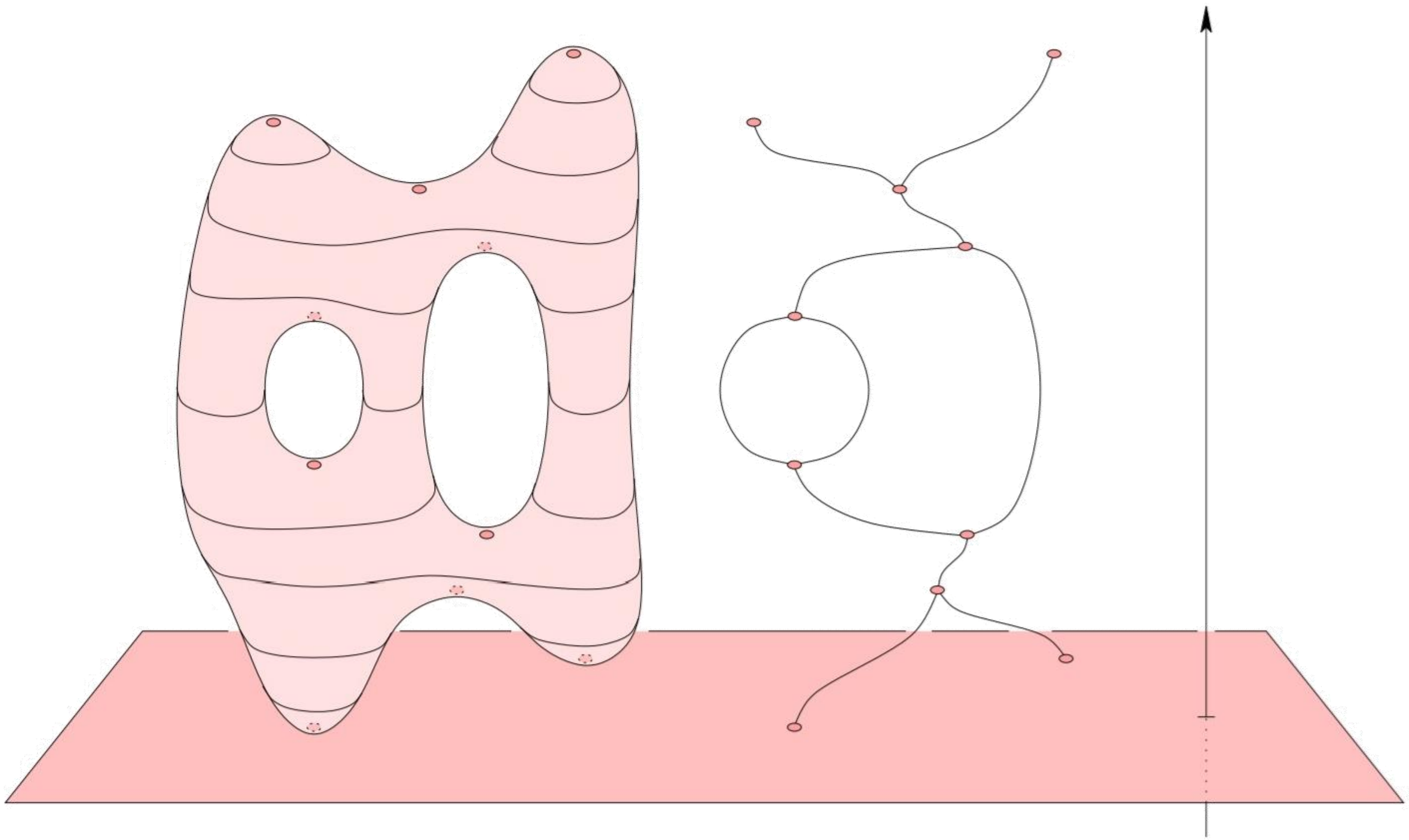


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 15, November 2, 2021

Morse Theory



Reeb Graph

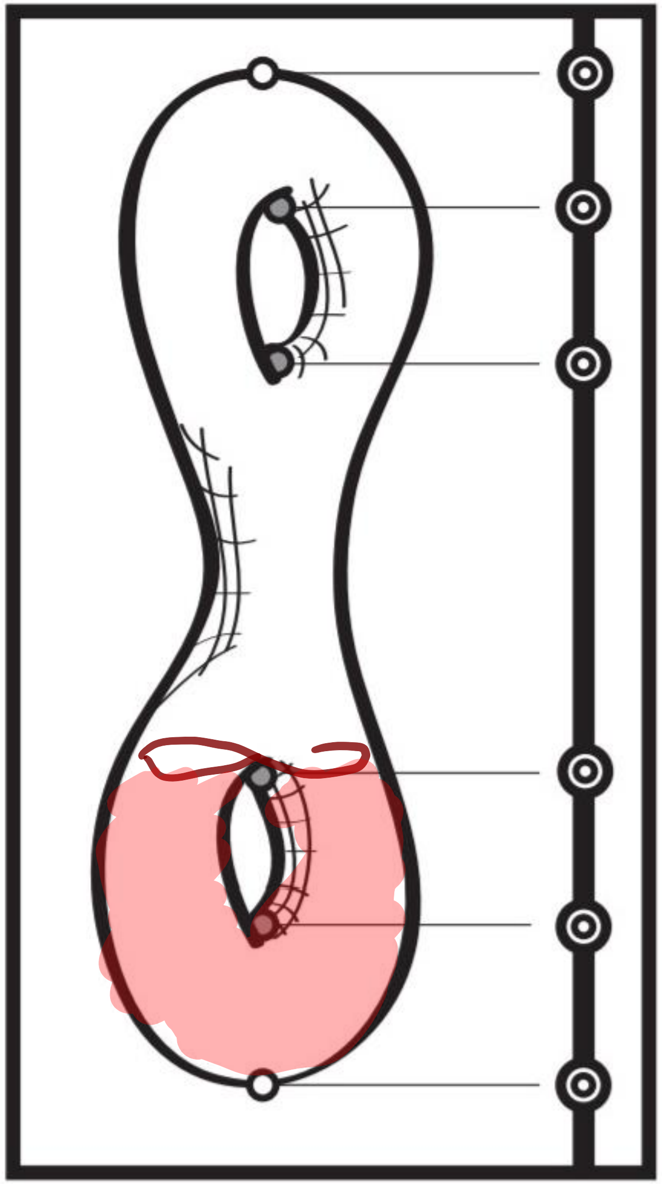
- $\beta_1(\text{Reeb}(M)) \leq \beta_1(M)$

Morse Theory

- Topology is still useful when the surface is just a terrain!

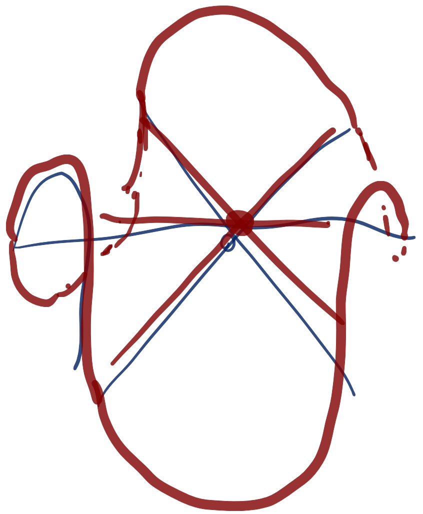
Definitions

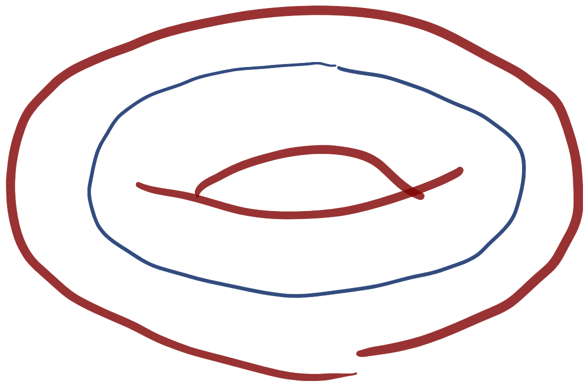
- Height function h: M -> R
- Sub-level set $M_{\leq a}$: $h^{-1}(-\infty, a] = \{x : h(x) \leq a\}$
- Critical points: where the topology changes

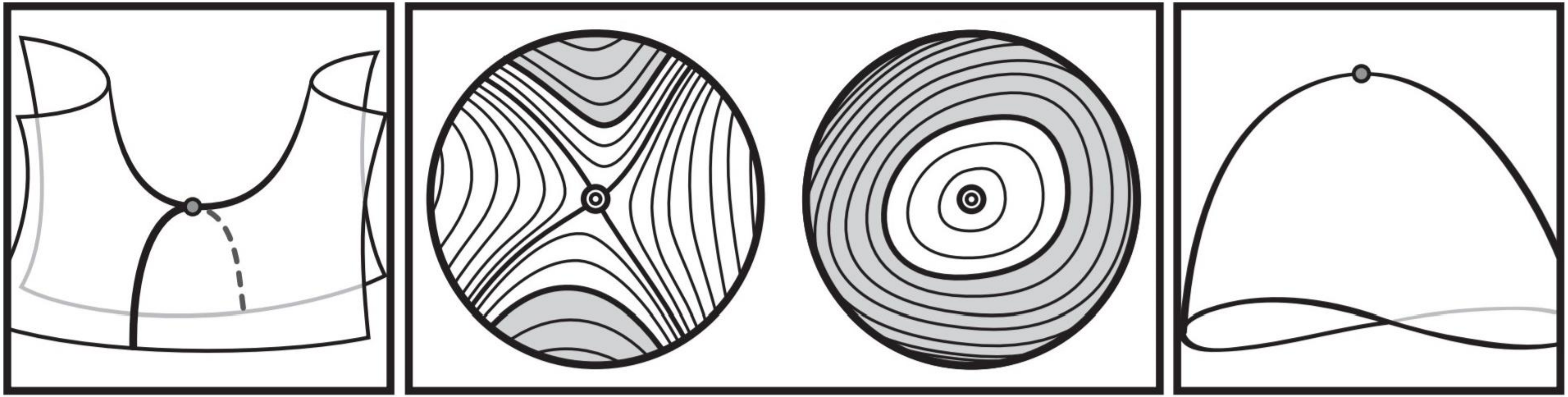


Morse Function

- All critical points are non-degenerate and have distinct function values







Morse Lemma

[Morse 1934]

Given Morse function h and critical point p, locally U(p) looks like $f(x) = f(p) - x_1^2 \dots - x_s^2 + x_{s+1}^2 \dots + x_d^2$

Example

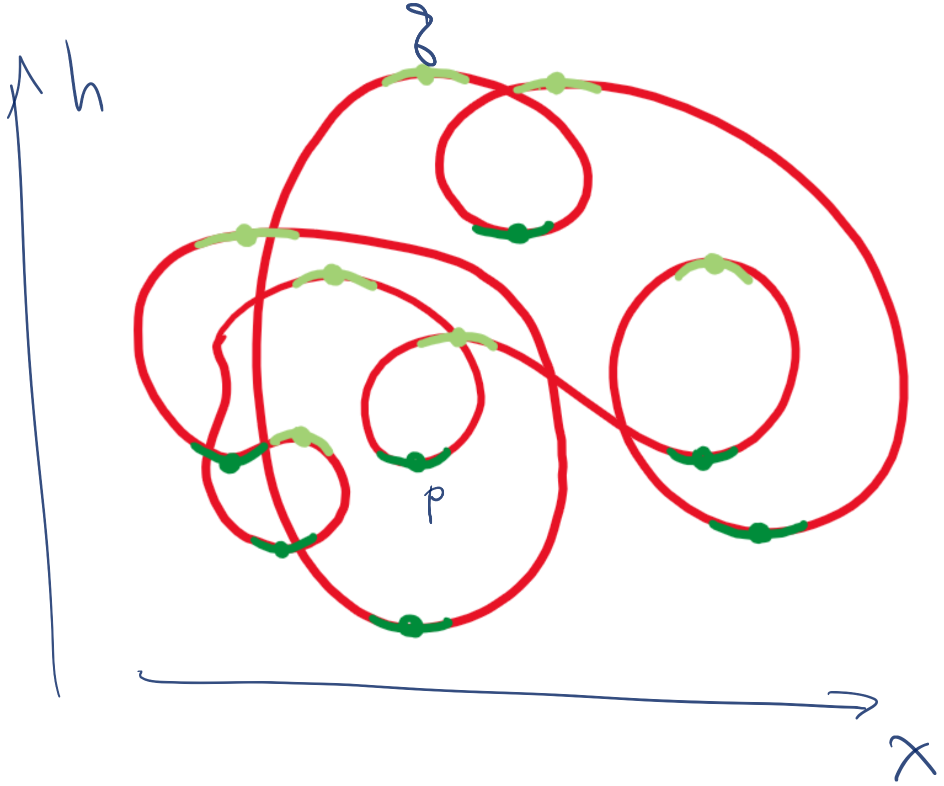
- Rotation number redux
- Morse index $\mu(p)$: number of negative quadratic terms

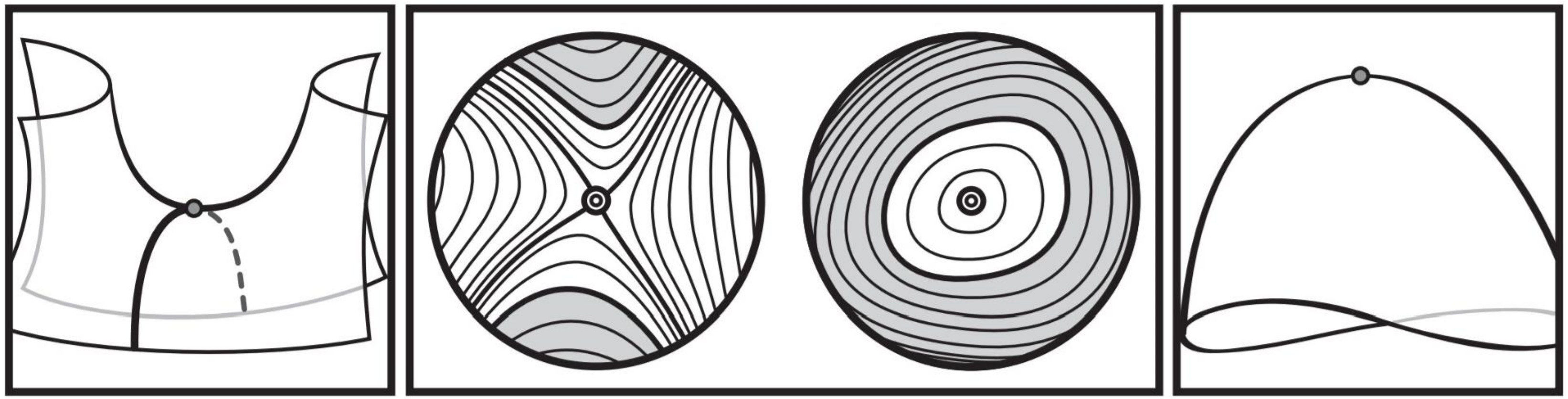
$h(x) = h(p) + x^2$

index = 0

$h(x) = h(q) - x^2$

index = 1

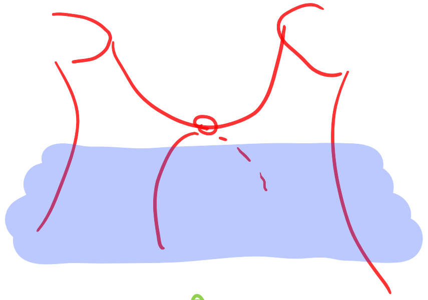




- $h(x) = h(p) + x_1^2 + x_2^2$

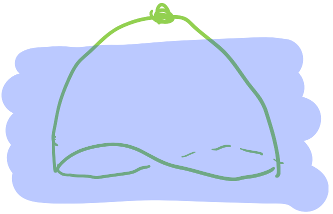
index Ø

- $h(x) = h(p) - x_1^2 + x_2^2$



index 1

- $h(x) = h(p) - x_1^2 - x_2^2$



index 2

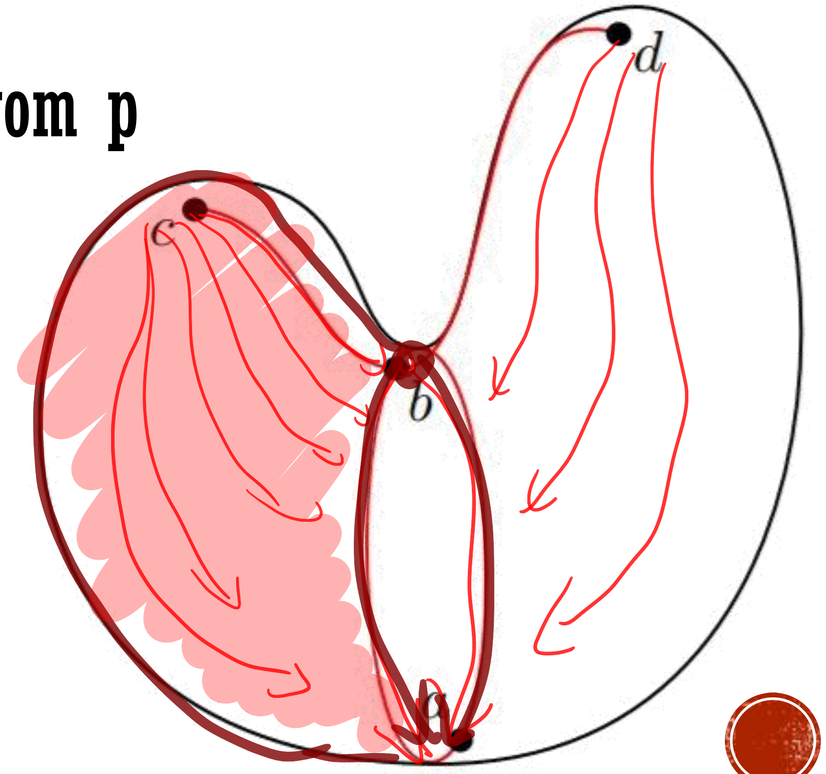
Example

- $U = D^n$
- $L = D^{n-(\mu-1)} \times S^{\mu-1}$

Flowlines

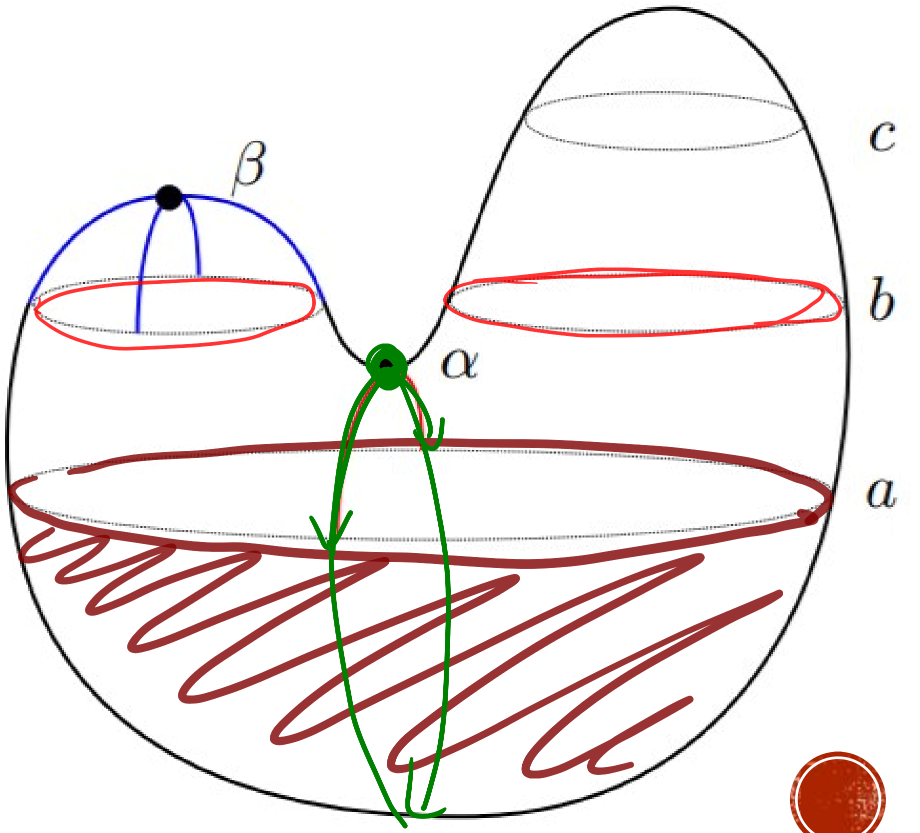
- Gradient field ∇h defines flowlines between critical points
  - M decomposes into flowlines
- Descending manifold M↓(p): flowlines originated from p

- **Proposition.** M↓(p) has dimension μ(p)



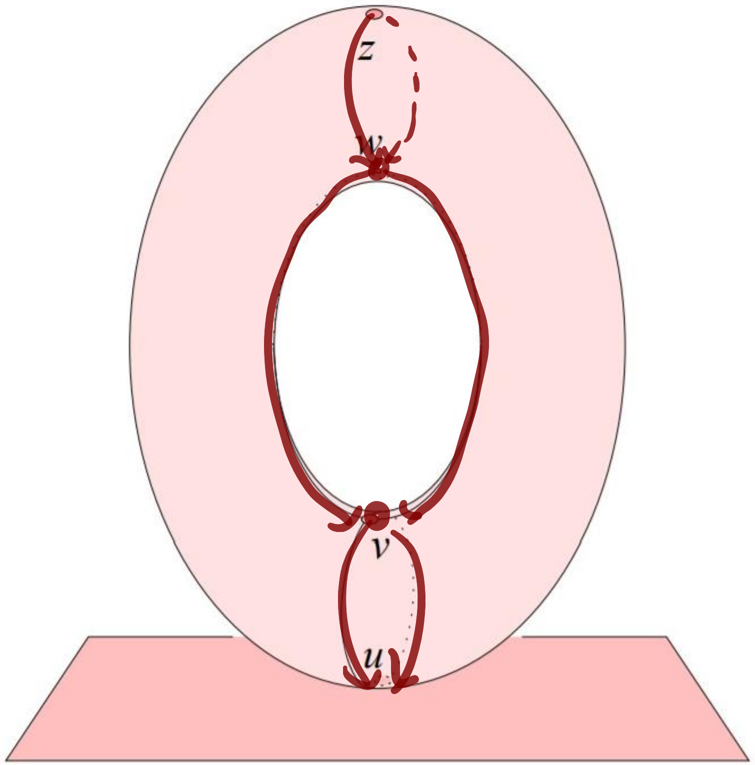
Properties

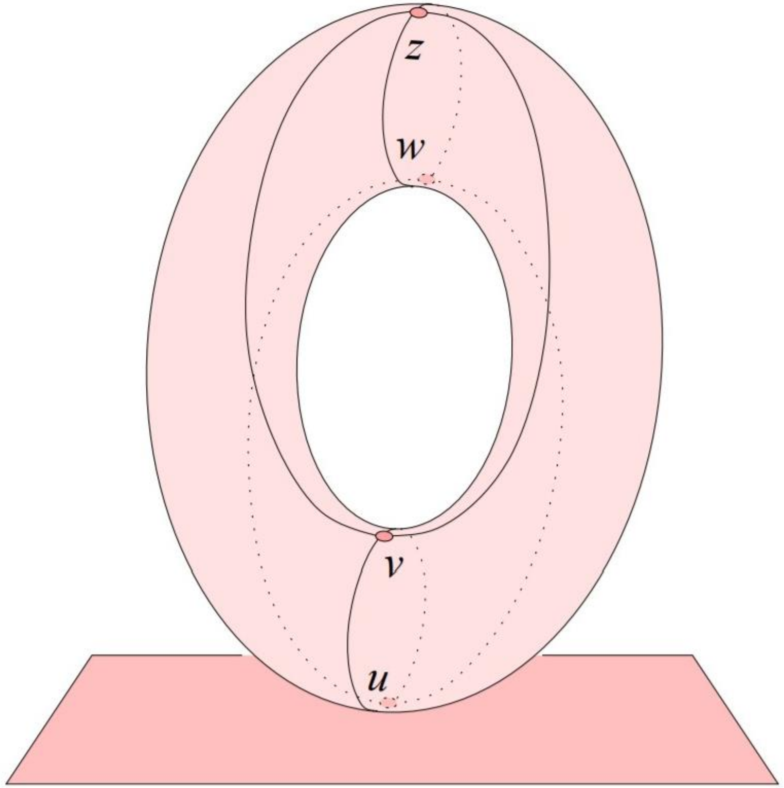
- Descending manifold $M_{\downarrow}(p)$: flowlines originated from p
- Proposition.
  - $M_b \simeq M_a$ if no critical points in $h^{-1}[a, b]$
  - $M_{\leq b} \simeq M_{\leq a} \cup M_{\downarrow}(p)$ if $h^{-1}[a, b]$ has critical point p



Morse-Smale Function

- All flowlines go from k-dim critical pts to (k-1)-dim critical pts





Intermission

Food for Thought.

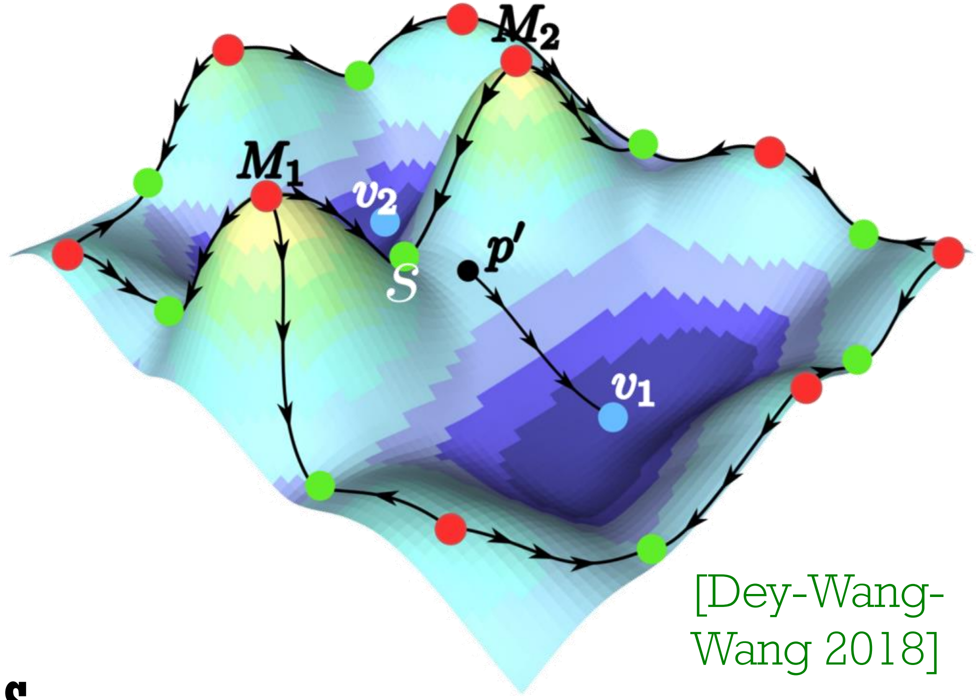
Flowlines going one-dimension lower.

What are we trying to do?

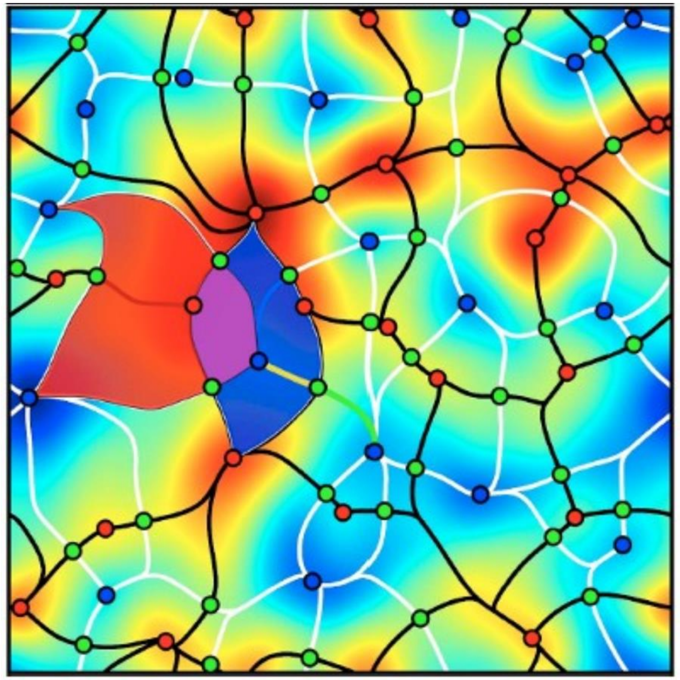
Morse Homology

Morse Complex

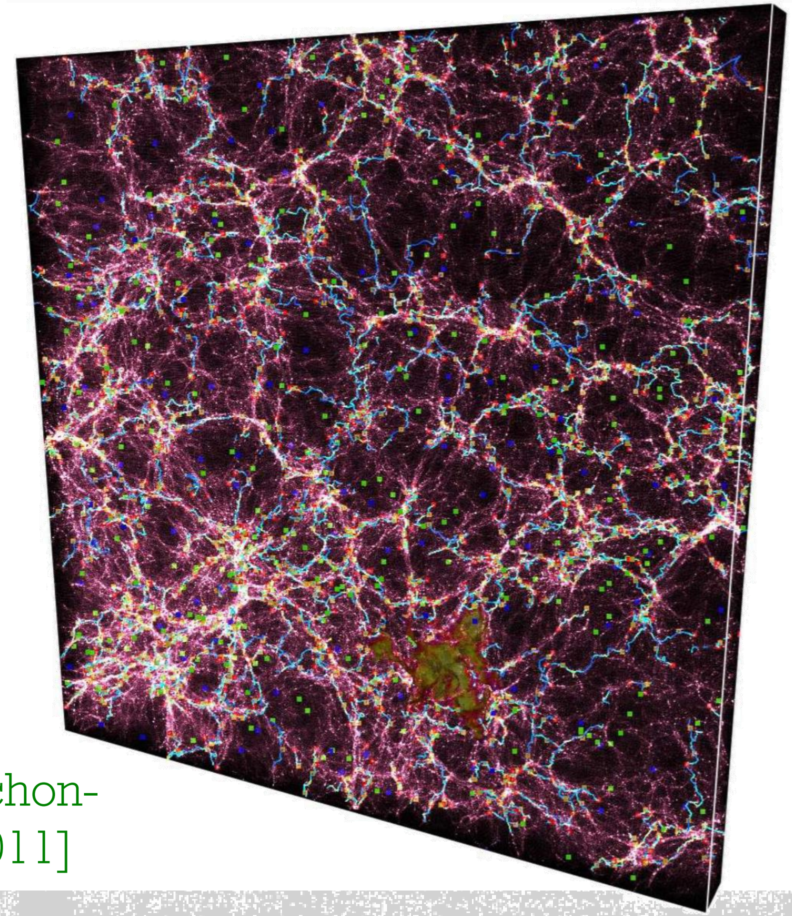
- k-chain-complex MC_k: ⟨~~k-dim~~ index k critical pts⟩
- Boundary map ∂_k:
  all ~~(k-1)-dim~~ index k-1 critical pts reachable by flowlines



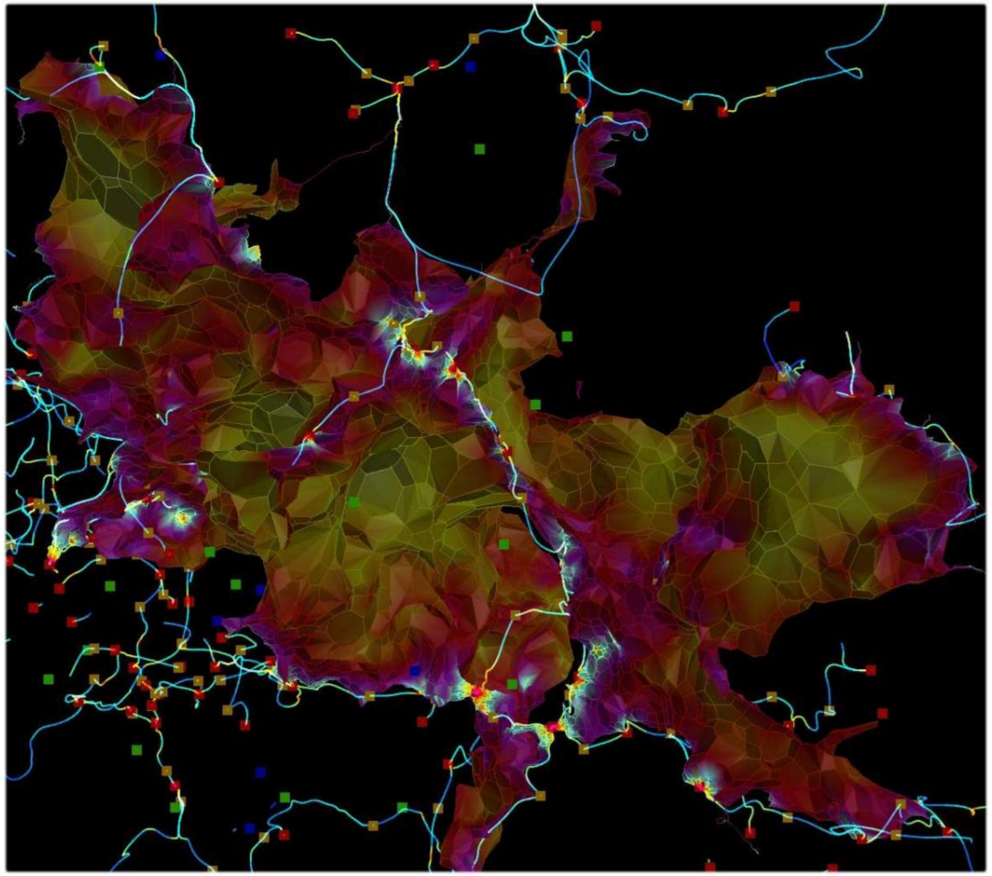
[Dey-Wang-Wang 2018]



[Sousbie 2011]



[Sousbie-Pichon-Kawahara 2011]



Morse Homology Theorem

[Thom 1949] [Milnor 1963] [Smale 1967]

$$MH_n(M)\otimes\mathbb{Z}_2 \cong H_n(M)\otimes\mathbb{Z}_2$$

(independent to the choose of height function h)

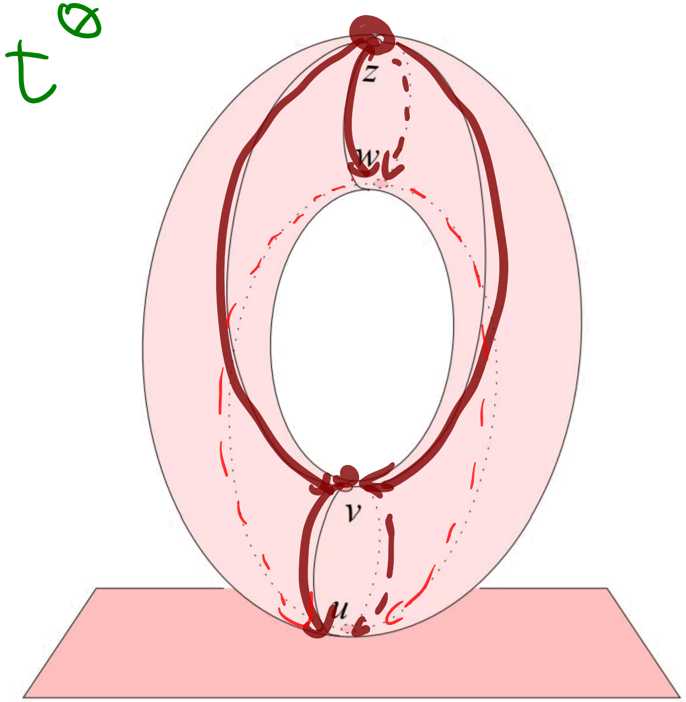
Example

$C_0 = \langle u \rangle$

$C_1 = \langle v . w \rangle$

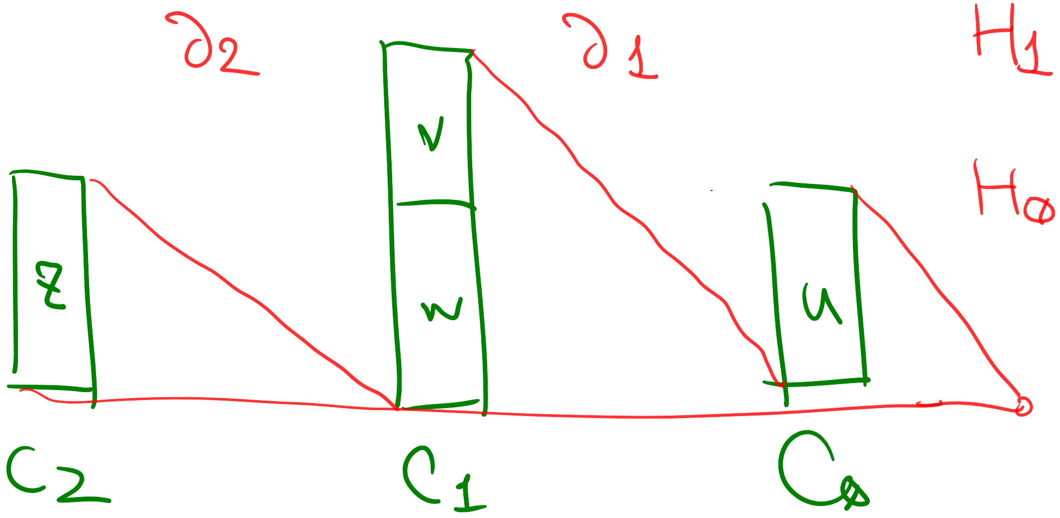
$C_2 = \langle z \rangle$

$t^2 + 2t^1 + t^0$



$\partial v = 2u = \partial w = \emptyset$

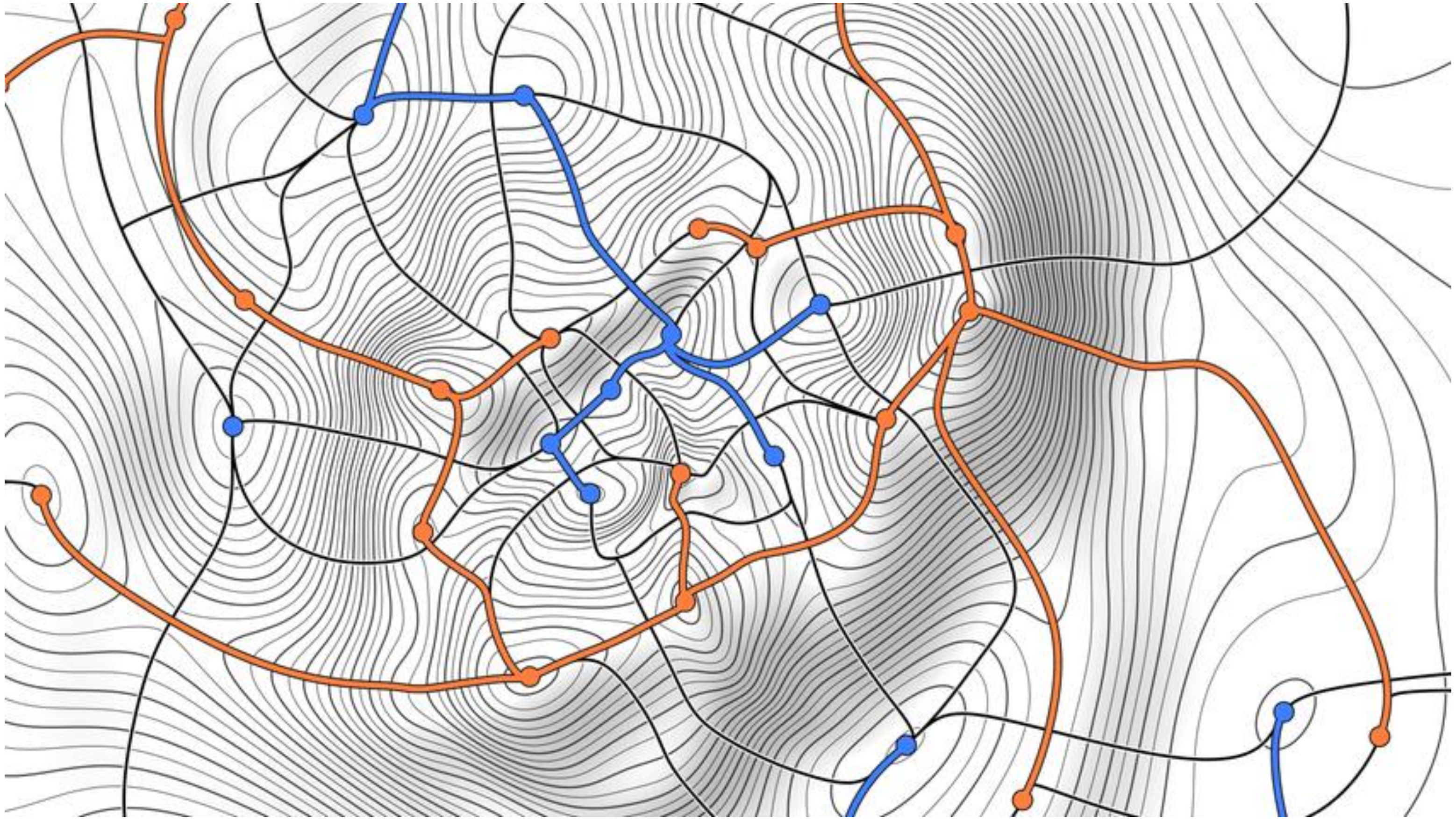
$\partial z = 2w + 2v = \emptyset$



$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \mathbb{Z}_2^2$

$H_0 = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{Z}_2$

$H_2 = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \mathbb{Z}_2$



Morse Inequalities

$$\Sigma_p t^{\mu(p)} = \Sigma_k \beta_k \cdot t^k + (1+t) \cdot Q(t)$$

#k-dim critical pts $\geq \beta_k$

COROLLARY. $\chi(X) = \Sigma_n (-1)^n \cdot \dim H_n(X)$

$$\chi(X) = \sum_i (-1)^i m_i = \sum_i (-1)^i \beta_i$$

$\dim H_i(X)$

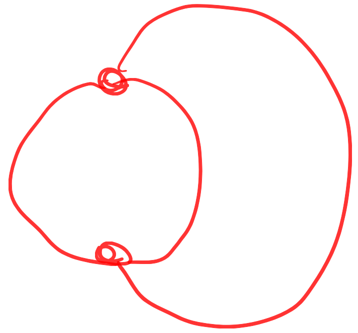
$$\#\text{saddles} \leq s + t - 2 :$$

$$m_1 \leq m_2 + m_0 - 2$$

$$2 \leq m_0 - m_1 + m_2$$

COROLLARY. $\beta_1(\text{Reeb}(M)) = \beta_1(M)$ if $M=\Sigma(g,0)$

$V :=$ #vertices on Reeb graph after contracting deg 1 & 2 vertices



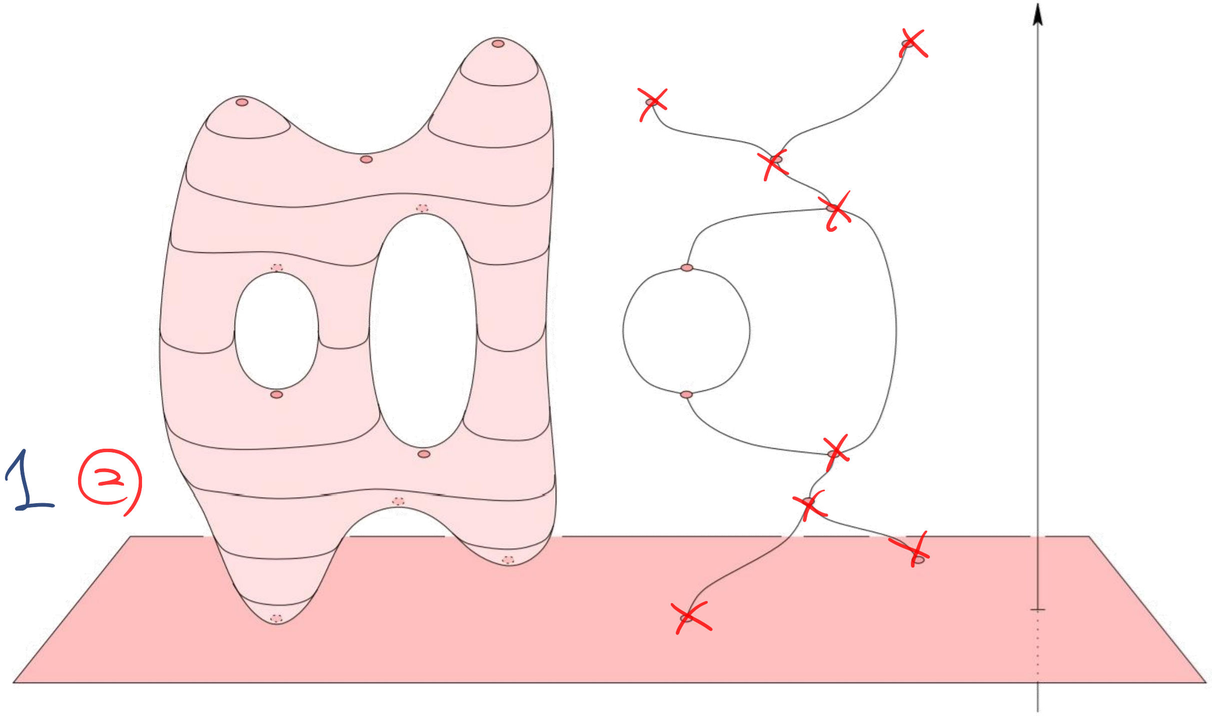
$V = m_1 - (m_0 + m_2)$ ①

$(\sum \deg v)/2$

$\beta_1 = E - V + 1 = \frac{3V}{2} - V + 1 = V/2 + 1$ ②

$\chi = 2g - 2 = m_0 - m_1 + m_2 \overset{①}{=} -V$

$\overset{②}{=} 2(\beta_1 - 1) \Rightarrow g = \beta_1$



Water-rising puts Topology in Geometry

Next Time.

More applications!

What to do when the space is not a surface?