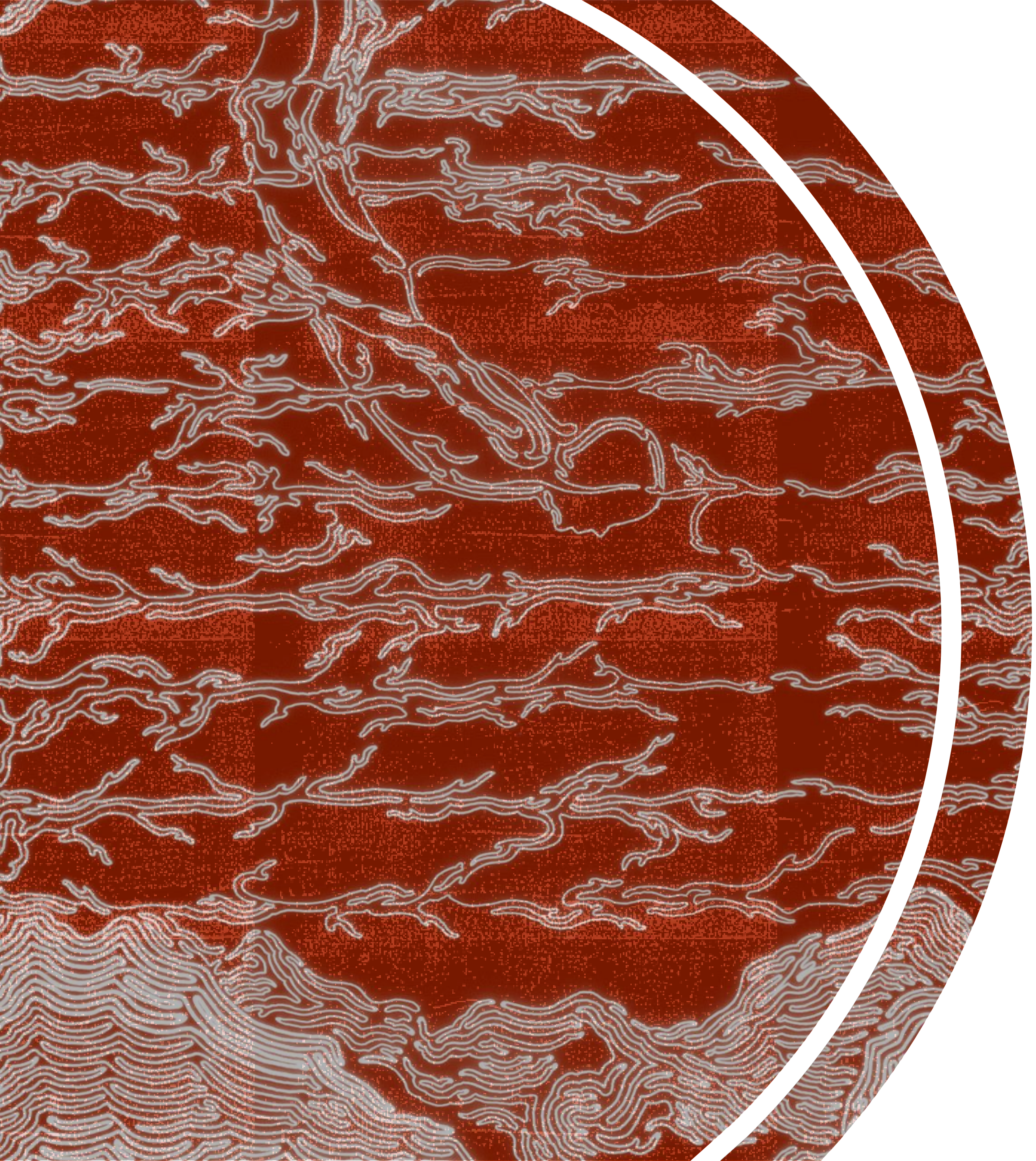


INTRODUCTION TO

COMPUTATIONAL TOPOLOGY

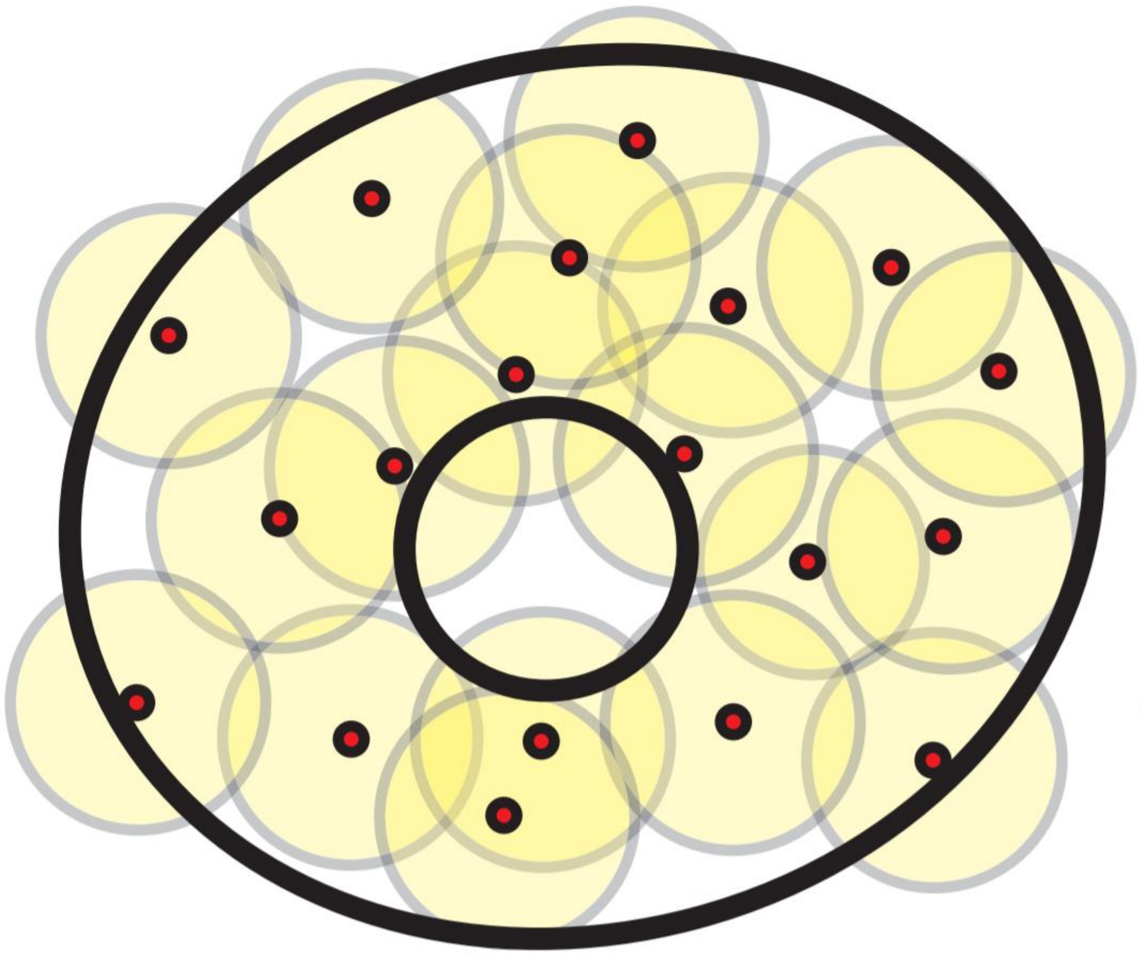
Hsien-Chih Chang

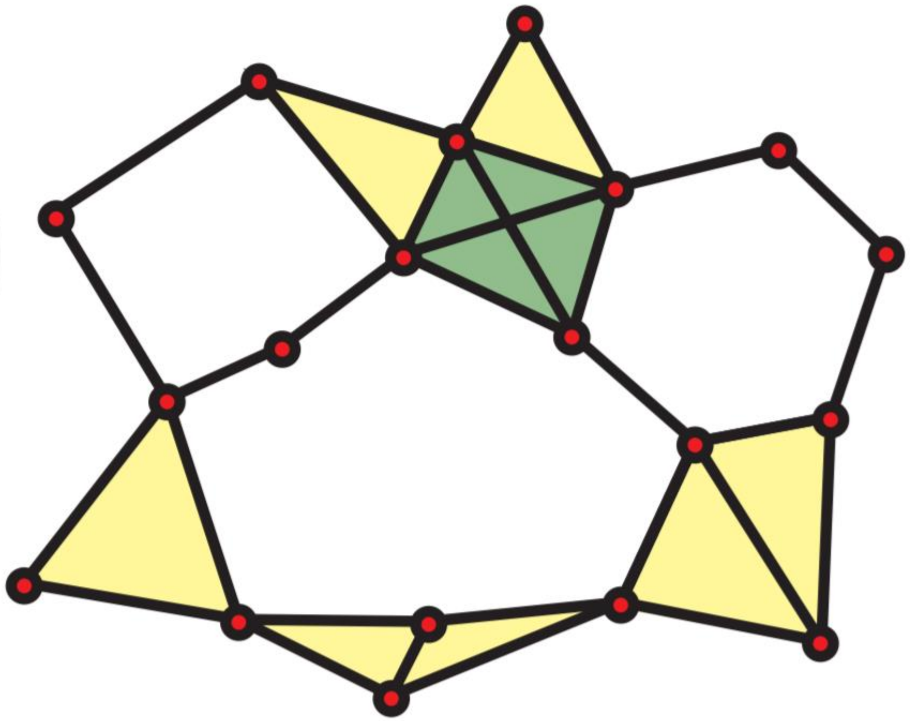
Lecture 13, October 26, 2021

Shape of Point Clouds

Čech complex

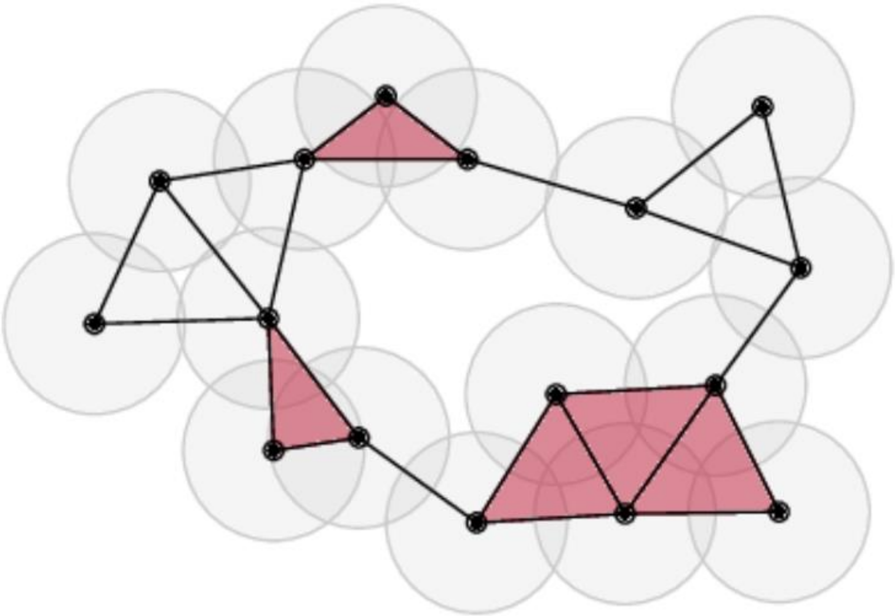
[Čech 1930s] [Alexandroff 1928]

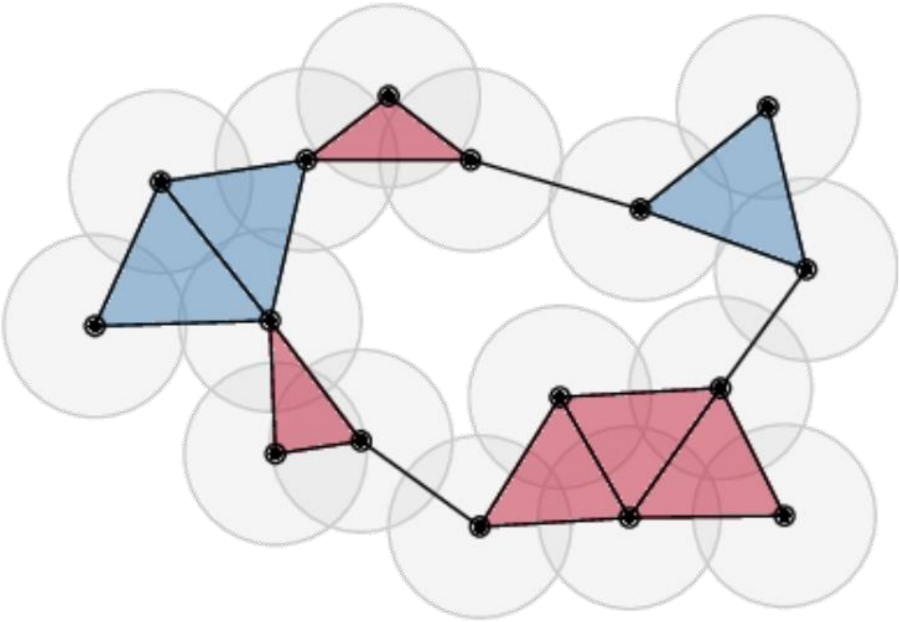




[Ghrist 2008]

Vietoris-Rips complex [Vietoris 1927]

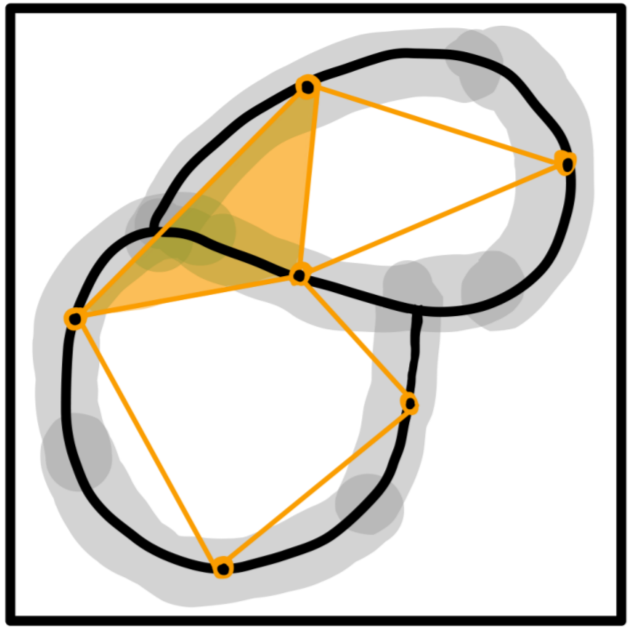


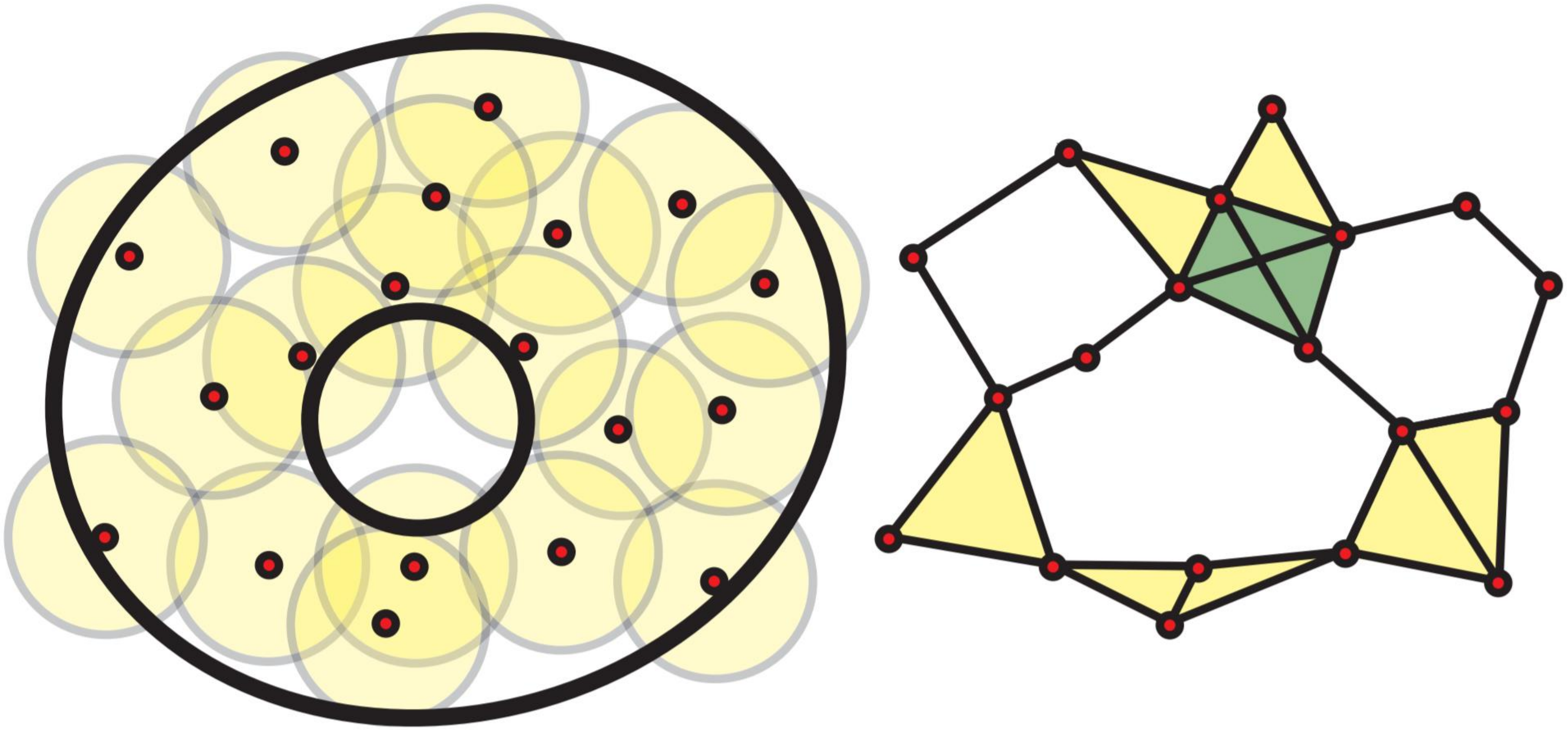


VR Lemma. $\check{C}_\varepsilon(X) \subseteq VR_\varepsilon(X) \subseteq \check{C}_{2\varepsilon}(X)$

Nerve

- Nrv(U) = {common intersections of subsets in U}
  = {simplices}, naturally forming a complex



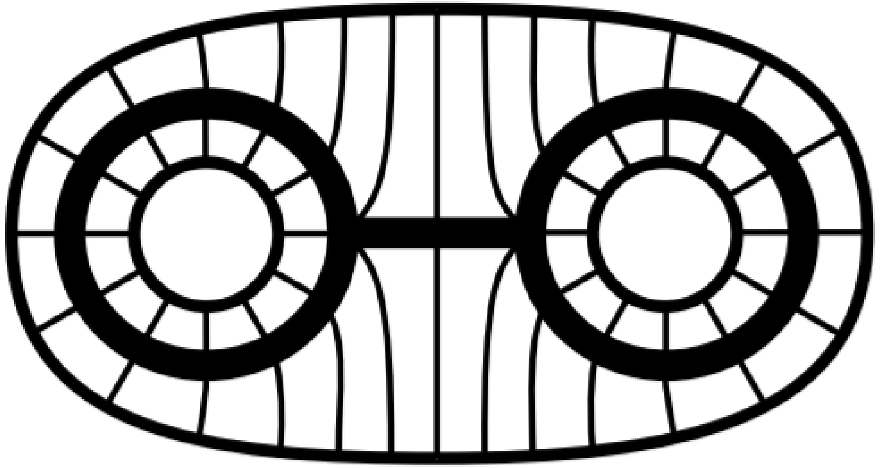
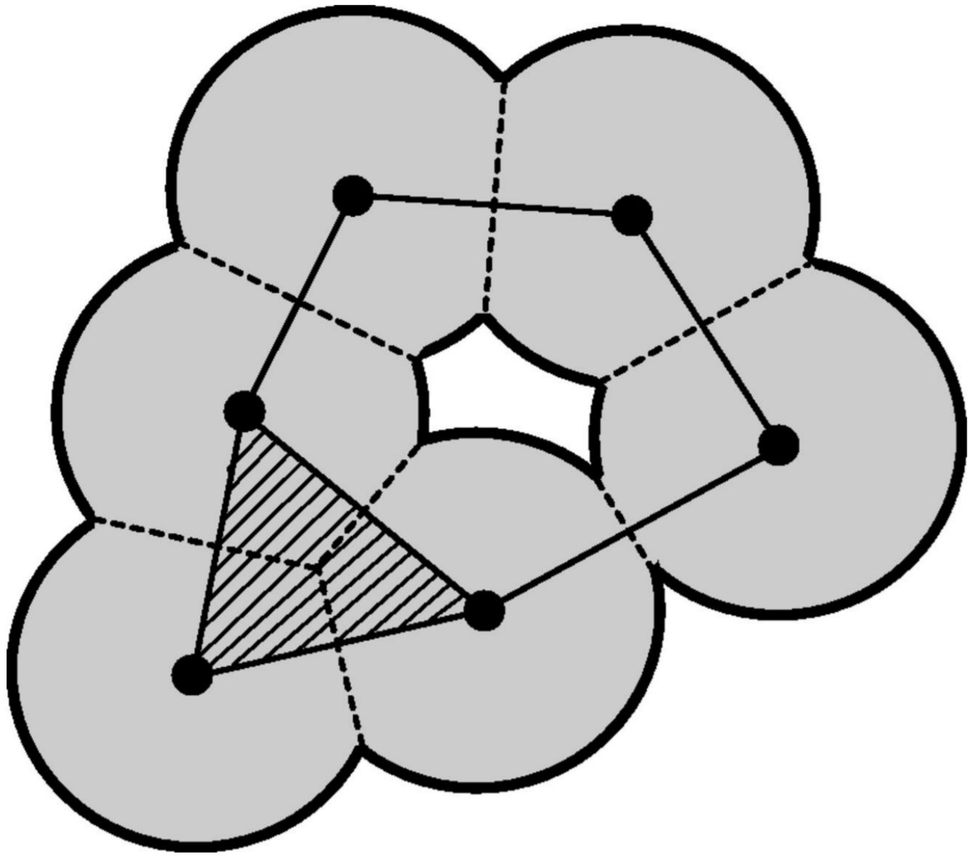


Nerve Theorem

[Borsuk 1948] [Alexandroff 1928]

Nrv(U) and ∪U are homotopic equivalent if all common intersections of subsets in U are contractible

Proof of Nerve Theorem.

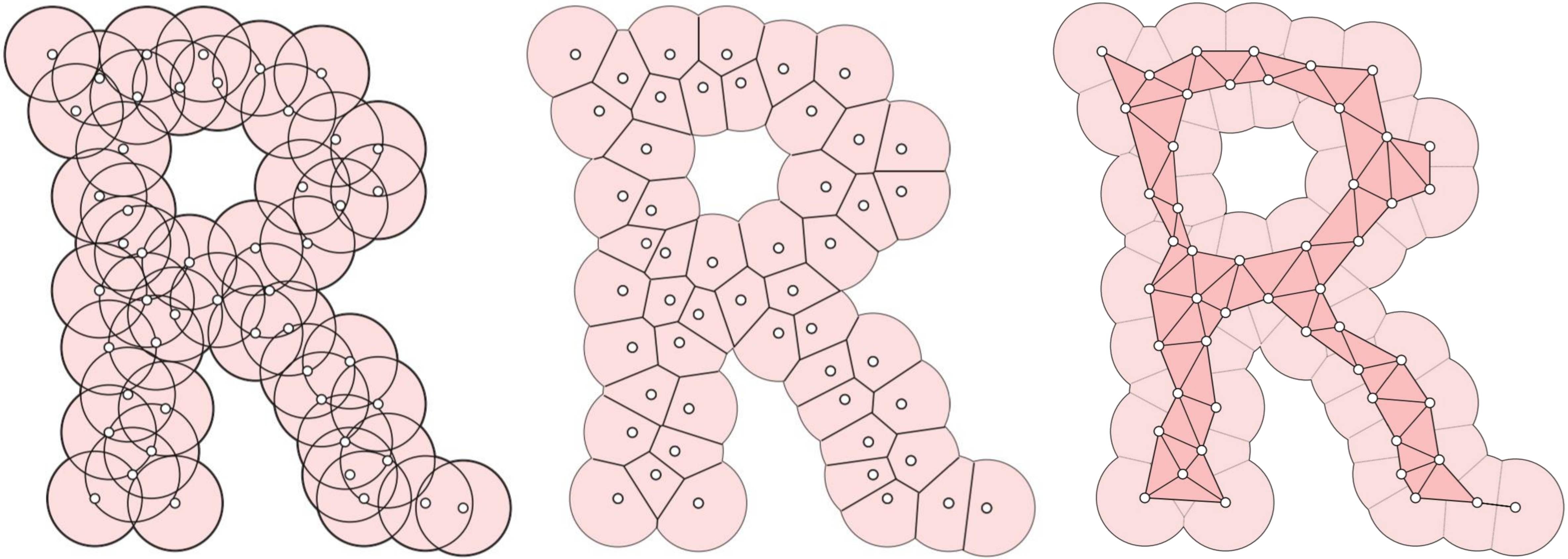
Vietoris-Smale Mapping Theorem

[Vietoris 1927] [Smale 1957]

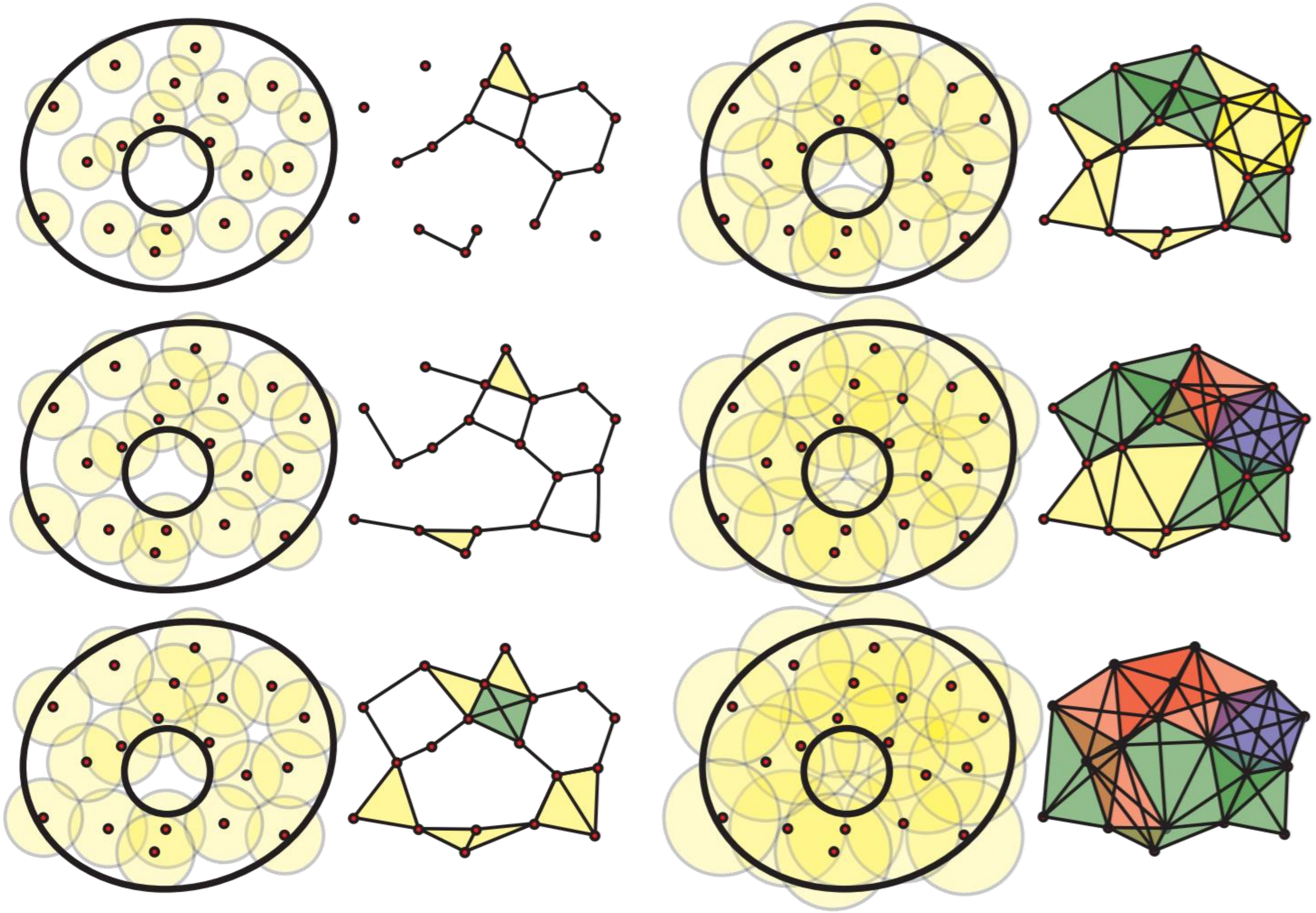
If f: X ⟶ Y is surjective and proper, and all preimage $f^{-1}(y)$ is contractible, then X and Y are homotopically equivalent.

Alpha complex

[Edelsbrunner-Kirkpatrick-Seidel 1983] [Edelsbrunner 1995]



But how about the Underlying Space?



[Ghrist 2008]

Cool Idea

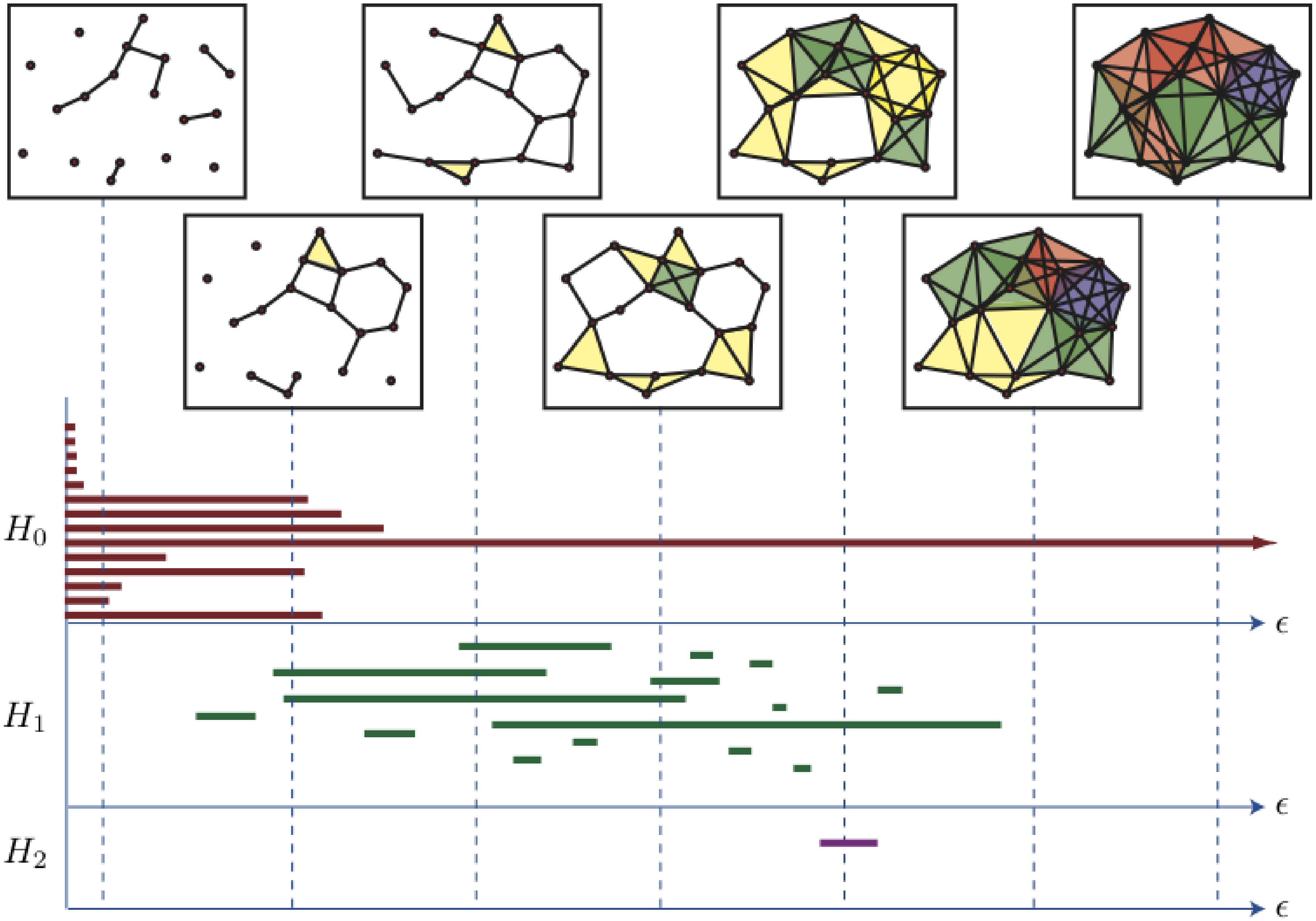
- Don’t guess ε; take all!

Intermission

Food for Thought.

Why are common intersections of balls contractible?

Persistent Homology



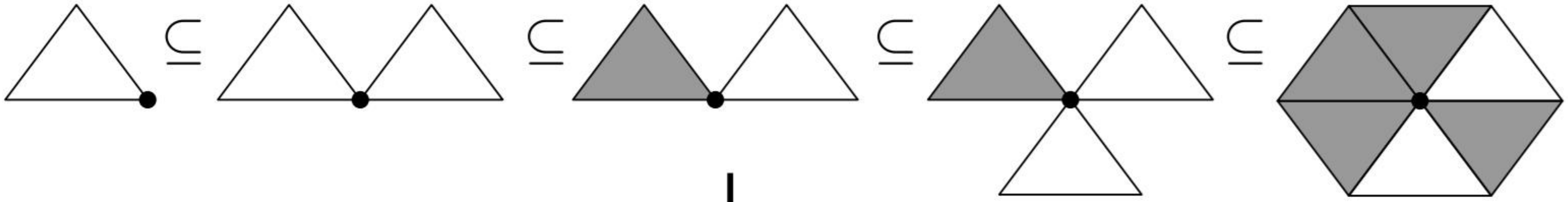
[Ghrist 2008]

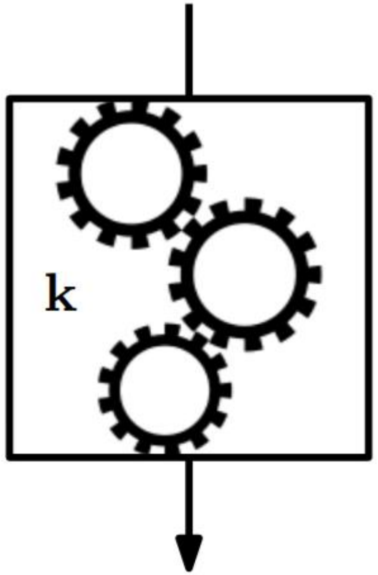
Barcodes

- Summary of homology data at all scales of ε
- Existence
- Computation
- Stability

Persistent Homology

[Edelsbrunner-Letscher-Zomorodian 2002]



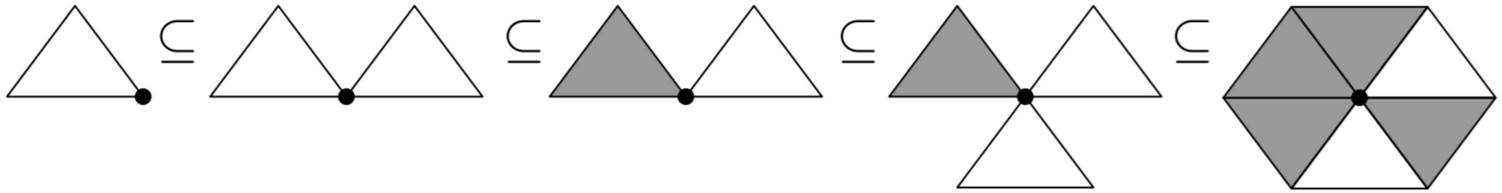


(degree-1 homology)

$$\mathbf{k} \xrightarrow{\binom{1}{0}} \mathbf{k}^2 \xrightarrow{(0\ 1)} \mathbf{k} \xrightarrow{\binom{0}{1}} \mathbf{k}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} \mathbf{k}^2 \ \cdots$$

[Oudot 2020]

Persistent Homology

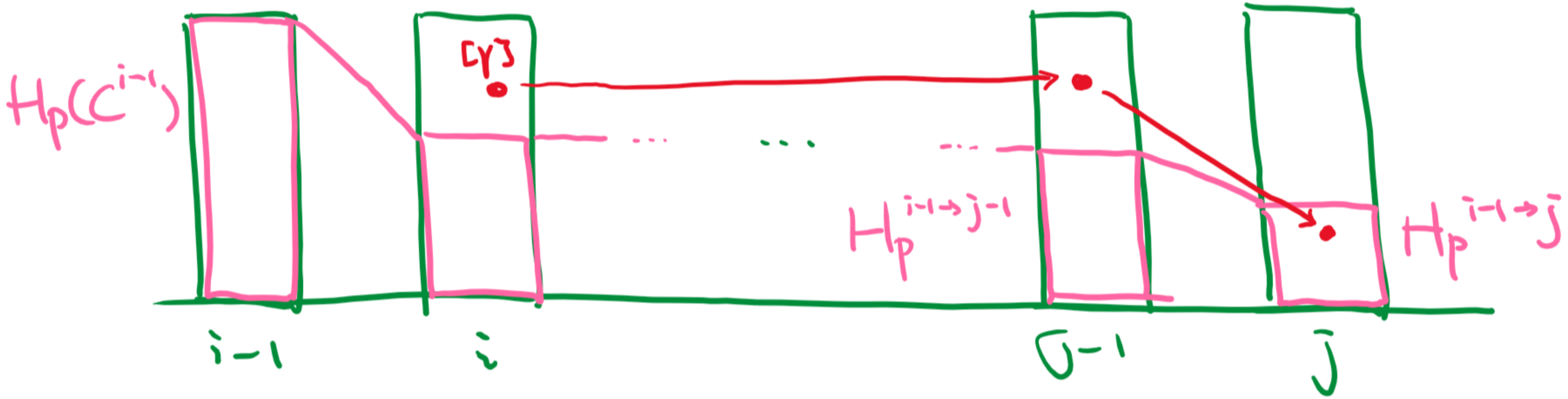


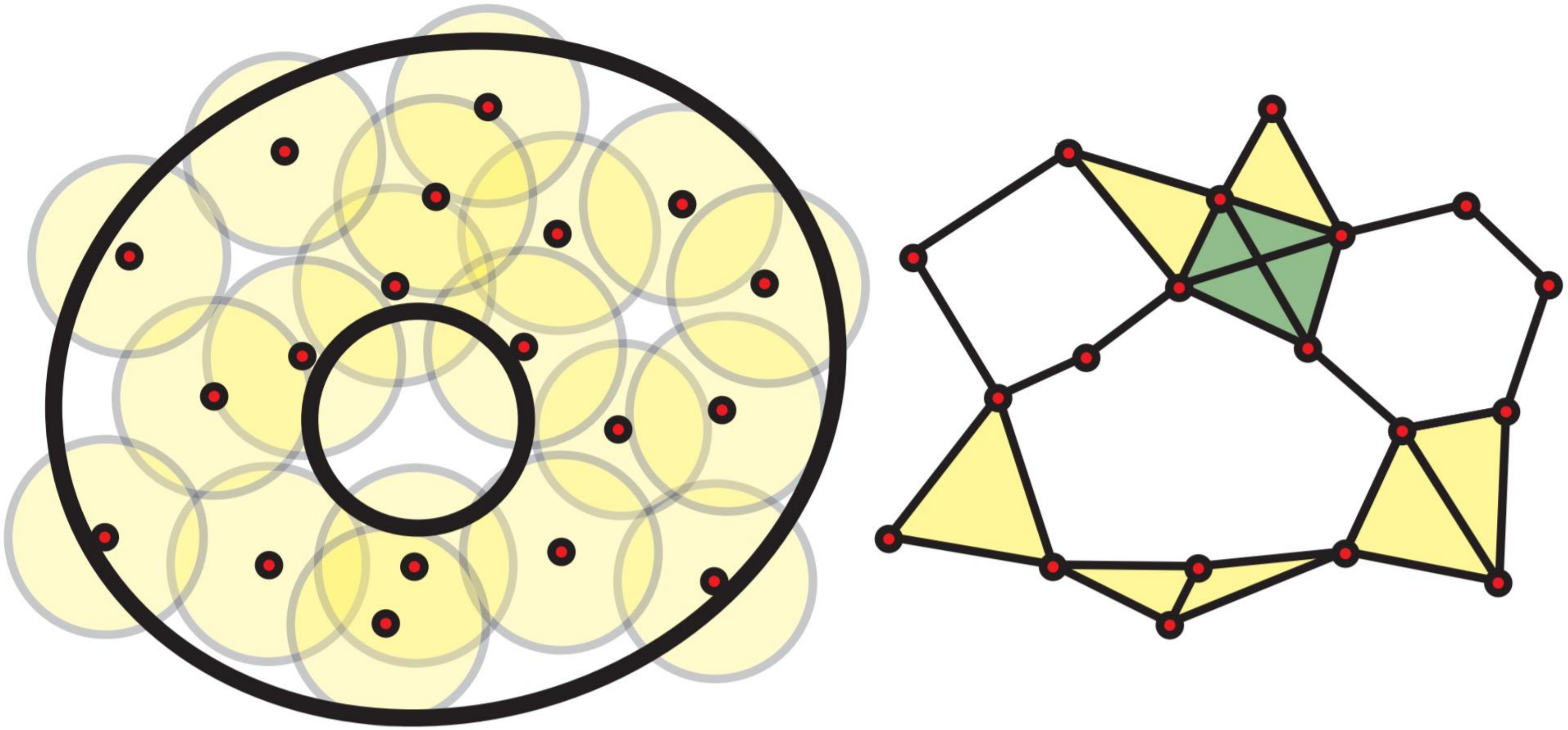
$$\mathbf{k} \xrightarrow{\binom{1}{0}} \mathbf{k}^2 \xrightarrow{(0\ 1)} \mathbf{k} \xrightarrow{\binom{0}{1}} \mathbf{k}^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \mathbf{k}^2 \ \cdots$$

$$H_p^{i \to j}(C) = \mathrm{im}(\ H_p(C^i) \to H_p(C^j)\ )$$

Birth and Death

- [γ] is born at i if [γ] is in $H_p(C^i)$ but not in $H_p(C^{i-1})$
- [γ] dies at j if [γ] merges with older cycles:
  - $i^*[\gamma]$ not in $H_p^{i-1\to j-1}$ but in $H_p^{i-1\to j}$

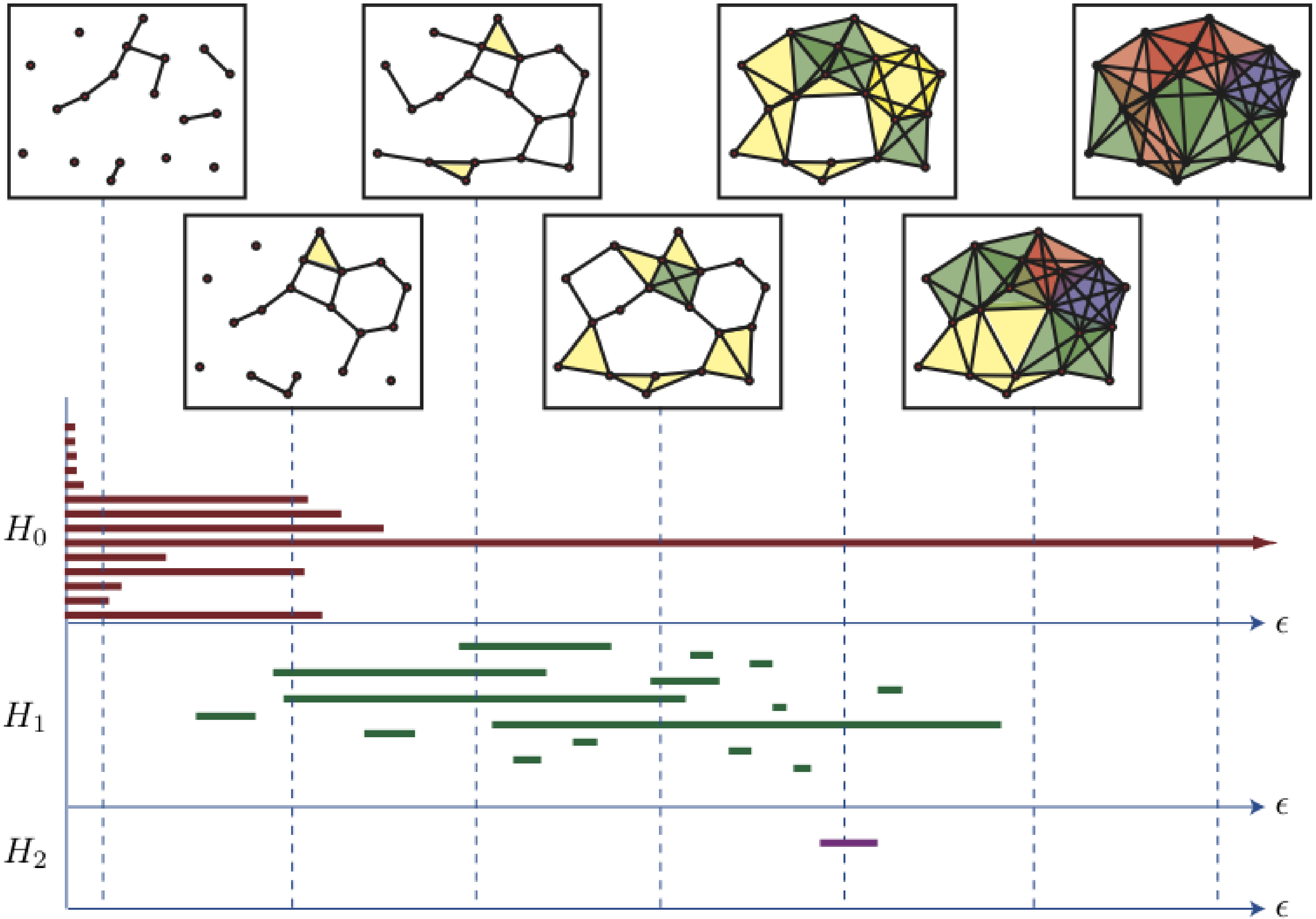




Barcode Theorem [Zomorodian-Carlsson 2005]

Barcode exists; more precisely,

H(C) decomposes into $\oplus_i$ F[b_i, d_i]



[Ghrist 2008]

Persistence

- Pers[γ] = $\varepsilon_j - \varepsilon_i$

Computation

PAIR(Δ):

$\gamma \leftarrow \partial\Delta$ // $(p-1)$-cycle

$\sigma \leftarrow$ youngest $(p-1)$-simplex in γ.

while (σ, Δ') paired & $\gamma \neq \emptyset$:

- $\gamma' \leftarrow$ cycle killed by Δ'
- $\gamma \leftarrow \gamma + \gamma'$
- $\sigma \leftarrow$ youngest $(p-1)$-simplex in γ

if $\gamma \neq \emptyset$:

- Δ is negative, pair (σ, Δ)

else:

- Δ positive

Topological Summary of Point Cloud in a Diagram

Next Time.

Stability of persistence; sketching topology

Čech Homology

- $\check{C}_n(U) = \langle$common intersections of n+1 subsets in U$\rangle = C_n(Nrv(U))$
- $\partial_n U_J = \Sigma_i (-1)^i U_{J-i}$

- **Theorem.** Čech homology = normal homology if every U_J are homologically trivial