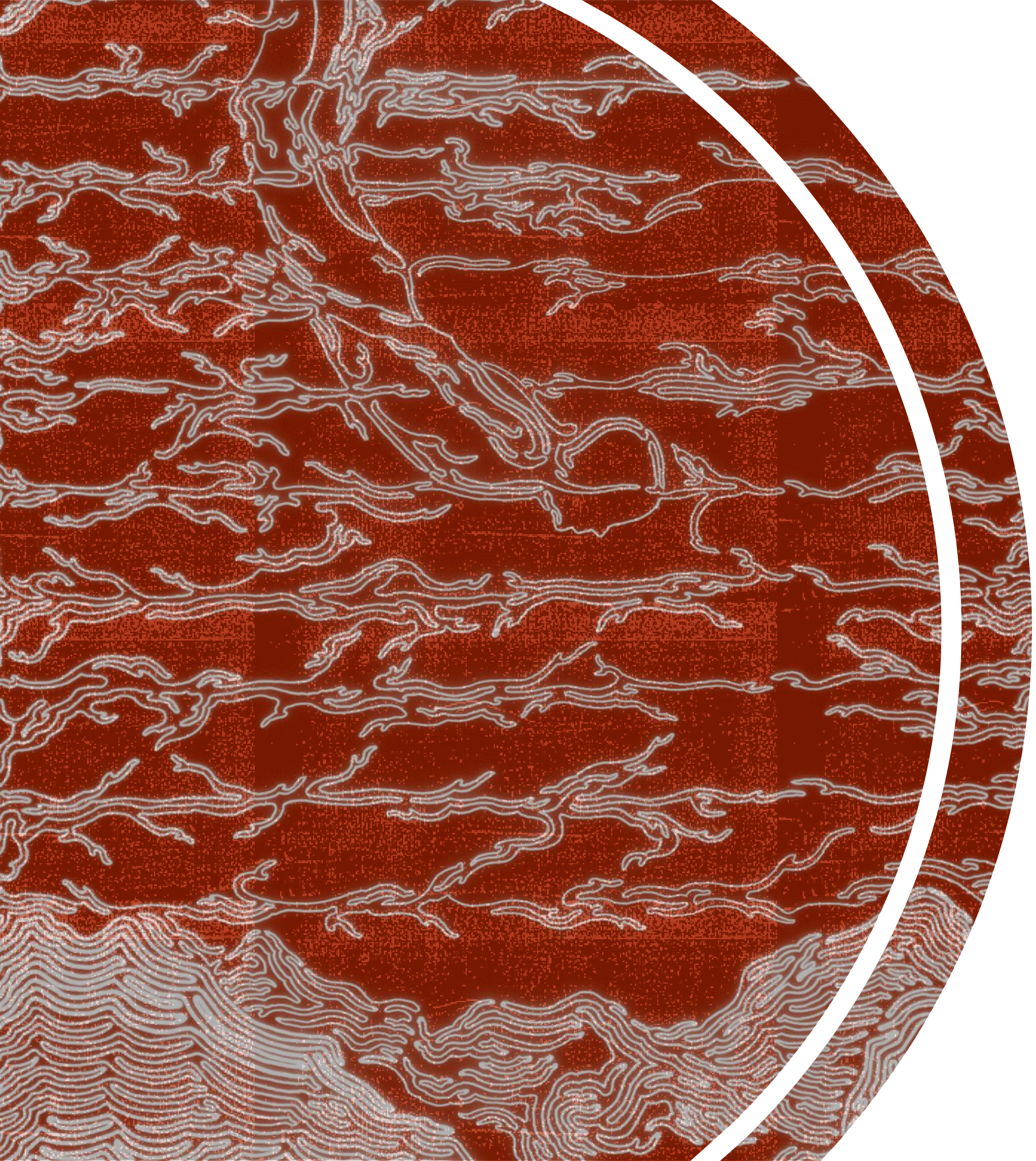

INTRODUCTION TO COMPUTATIONAL TOPOLOGY

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LECTURE 12, OCTOBER 21, 2021

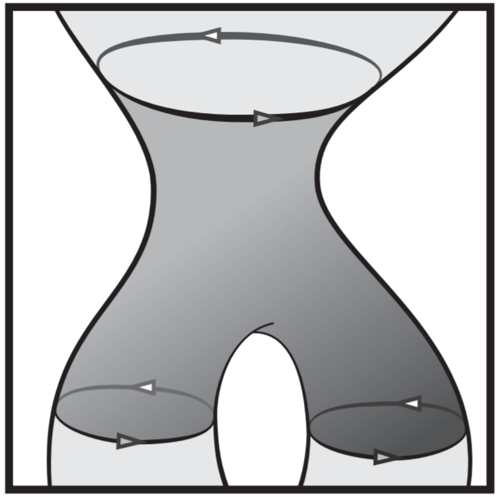
Administrivia

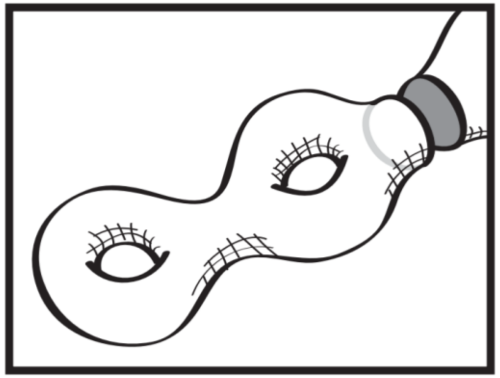
- Homework β is out, due 11/15

Homology Groups

- Simplicial homology group

$$H_n(X) = Z_n(X) \,/\, B_n(X)$$
$$= \ker \partial_n \,/\, \mathrm{im}\, \partial_{n+1}$$
$$= \text{Cycles/Boundaries}$$





Euler Characteristic Redux

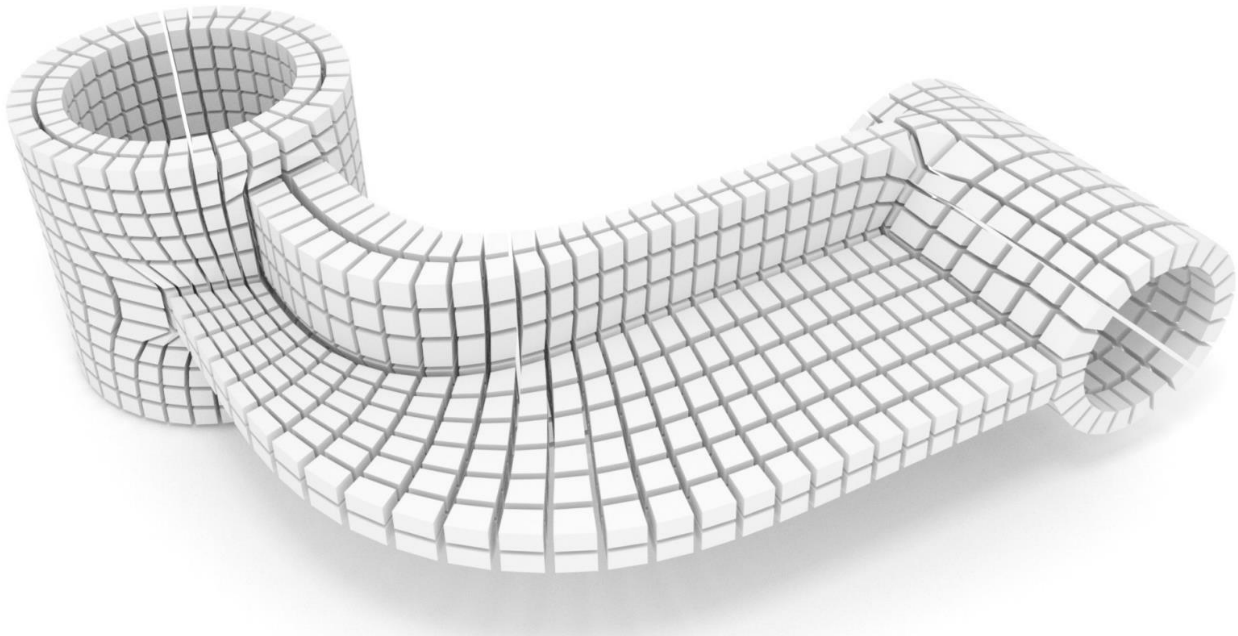
Dimension

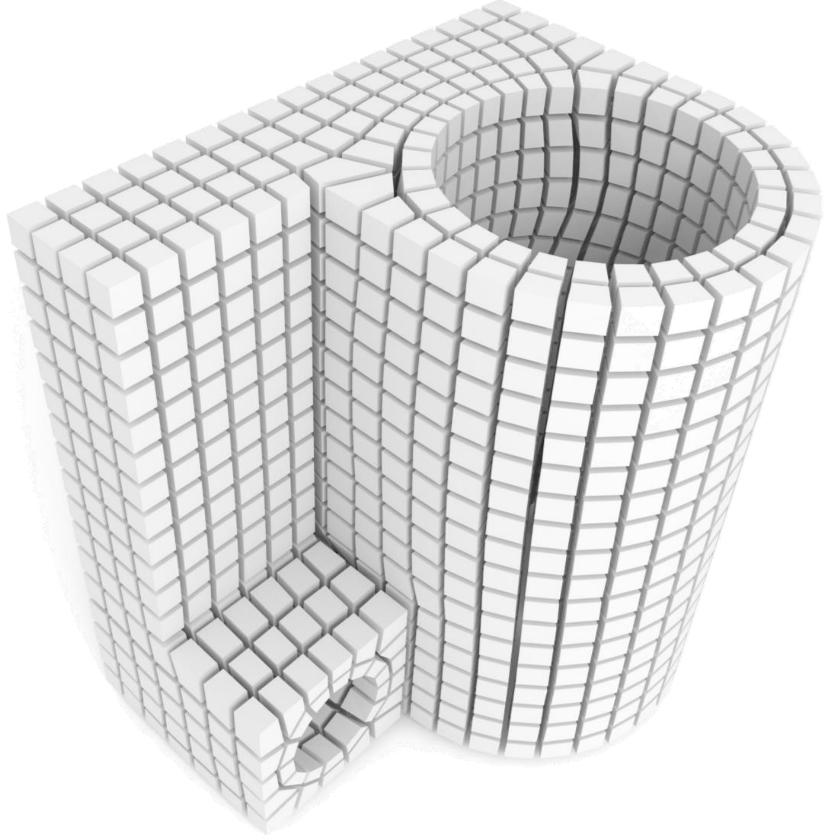
- Betti numbers β_n: dim $H_n(X)$

$H_0(T) = \mathbb{Z}$ $H_1(T) = \mathbb{Z}^2$ $H_2(T) = \mathbb{Z}$

$H_0(K) = \mathbb{Z}$ $H_1(T) = \mathbb{Z}_2 \times \mathbb{Z}$ $H_2(K) = \mathbb{Z}$

- **Theorem.** Betti numbers for 3d complex can be computed in linear* time.
  [Delfinado-Edelsbrunner 1995]





[Liu-Zhang-Chien-Soloman-Bommes 2018]

Euler-Poincaré Formula

$\sum_n (-1)^n \cdot \dim C_n(X) =$ $\chi(X) = \Sigma_n (-1)^n \cdot \dim H_n(X)$

Proof of Euler-Poincaré Theorem.

$$\dim C_n = \dim Z_n + \dim B_{n-1} = \dim H_n + \dim B_n + \dim B_{n-1}$$

$$\dim H_n = \dim Z_n - \dim B_n$$

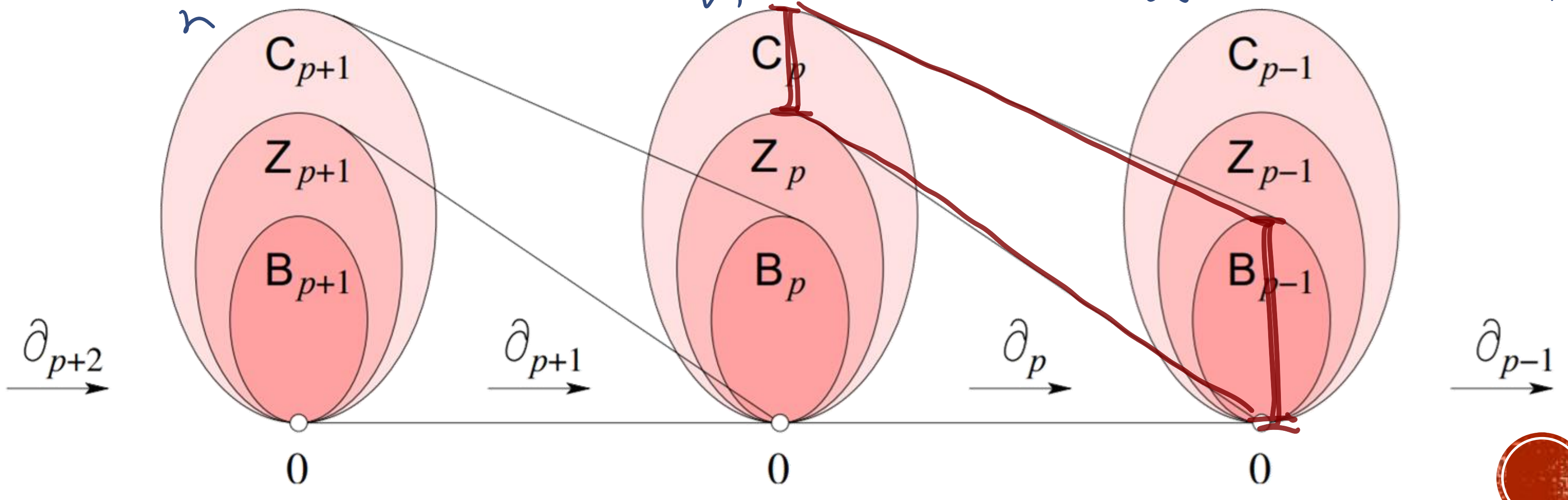
$$\chi(\Sigma) = \sum_n (-1)^n \dim C_n = \sum_n (-1)^n \left(\dim H_n + \dim B_n + \dim B_{n-1} \right)$$

$$= \sum_n (-1)^n \dim H_n + \sum_n (-1)^n \dim B_n - \sum_n (-1)^{n-1} \dim B_{n-1}$$

$$= \sum_n (-1)^n \dim H_n$$

$$+ \dim B_{-1} \quad (\emptyset)$$

$$+ \dim B_N \quad (\emptyset)$$



Computing Homology using Long Exact Sequence

Brouwer Fixed-Point Theorem [Bohl 1904] [Brouwer 1909]

Every map $f: D^n \rightarrow D^n$ has a fixed point

Proof of Brouwer Fixed-Point Theorem.

$h : D^n \to D^n$ has no fix-pt.

$r(x) :=$ intersection of ray $\overrightarrow{h(x)\,x}$ & $\partial D^n \cong S^{n-1}$

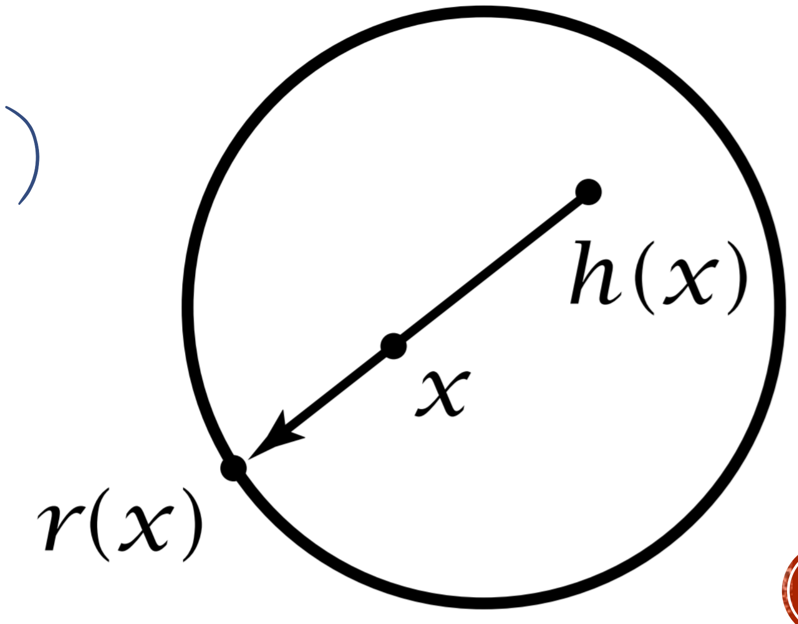
r is a retraction. $\partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n$ $\quad i \circ r = \underline{id}$

$$H_{n-1}(\partial D^n) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$$

$$\mathbb{Z} \qquad\qquad 0 \qquad\qquad \mathbb{Z}$$

How do we know?

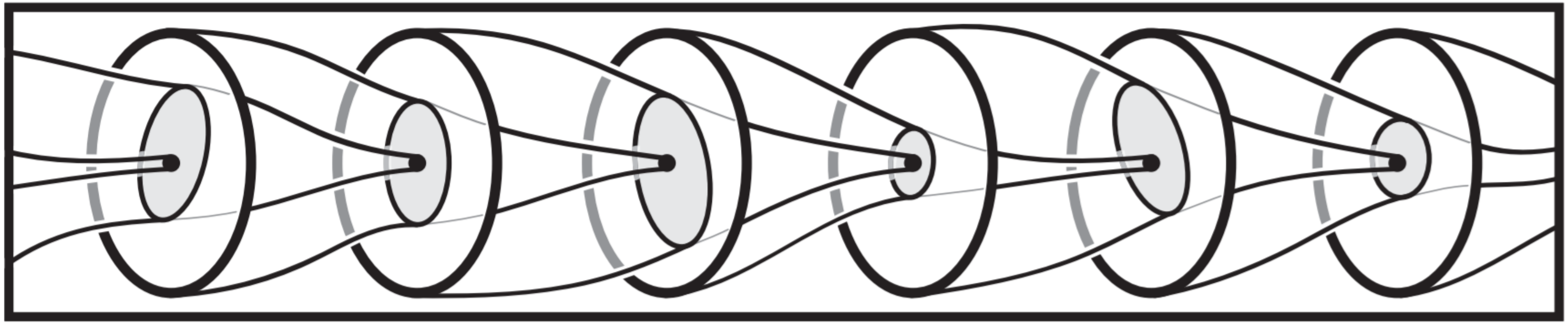
$\xrightarrow{\quad} \!\!\!\!\!\!\times$.



Why is $H^{n-1}(S^{n-1}) = \mathbb{Z}$?

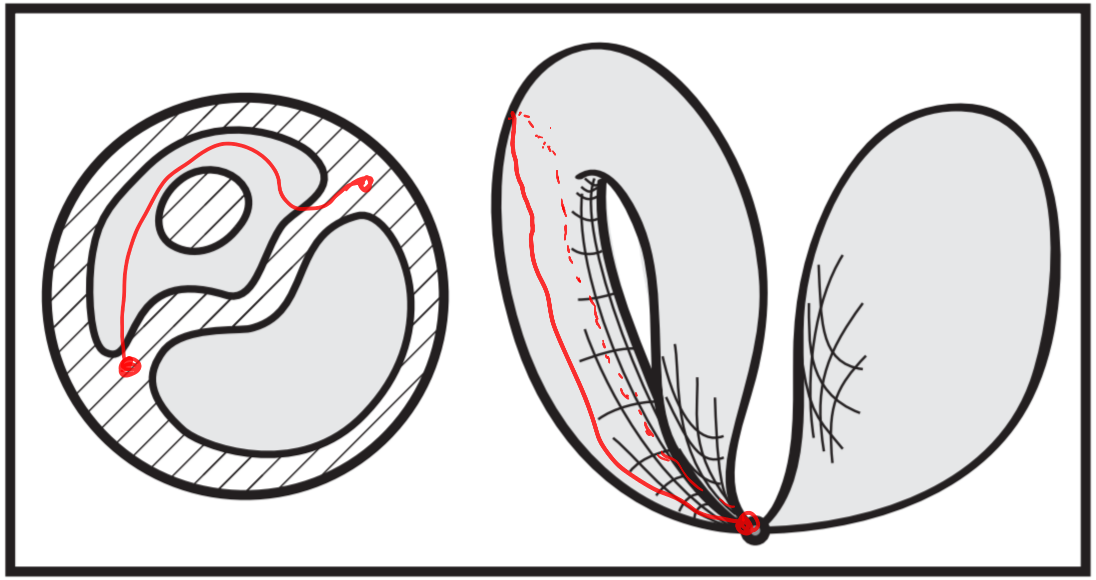
Exact Sequence

- $\ldots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots$ is exact if
  - $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$ for all n



Relative Homology

- $C_n(X, A) = C_n(X) \,/\, C_n(A)$
- $\partial_n \colon C_n(X, A) \to C_{n-1}(X, A)$



- **Theorem.** If A is a closed subcomplex of X, then

$$H_n(X, A) = H_n(X/A) \quad \text{if } n>0.$$

Connecting Map

- $\partial_*: H_n(X, A) \longrightarrow H_{n-1}(A)$

$[\gamma] \in H_n(X, A)$

$\exists \beta \in C_n(X) : j(\beta) = \gamma$

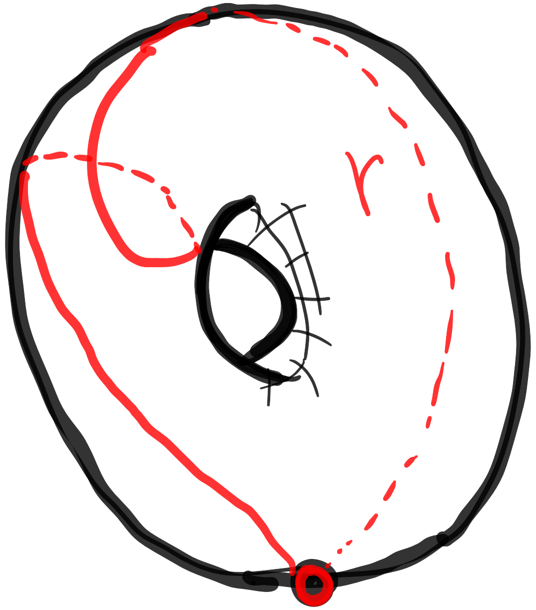
$\partial\gamma = \partial j(\beta) = j(\partial\beta) = 0$

Thus $\exists \alpha \in C_{n-1}(A) : i(\alpha) = \partial\beta$

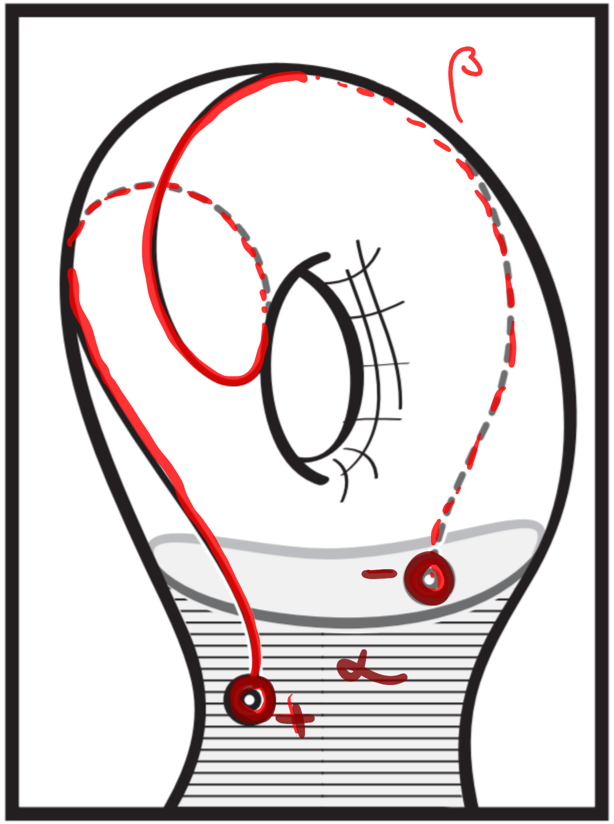
Define $\partial_*[\gamma] := [\alpha]$

"Snake Lemma":

$$\begin{array}{ccccccccc} & & & & \beta & & \gamma & & \\ 0 & \to & C_n(A) & \to & C_n(X) & \xrightarrow{j} & C_n(X,A) & \to & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \to & C_{n-1}(A) & \xrightarrow{i} & C_n(X) & \to & C_n(X,A) & \to & 0 \\ & & \alpha & & \partial\beta & & & & \end{array}$$



X/A



$A \xrightarrow{i} X \xrightarrow{j} (X, A)$

329. Witold Hurewicz: *On duality theorems.*

Let A be a locally compact space, B a closed subset of A, and $H^n(A)$, $H^n(B)$, $H^n(A-B)$ the n-dimensional cohomology groups of the sets A, B and $A-B$ (with integers as coefficients). Consider "natural homomorphisms" $H^n(A)\to H^n(B)\to H^{n+1}(A-B)\to H^{n+1}(A)\to H^{n+1}(A-B)$. It can be shown that the kernel of each of these homomorphisms is the image of the preceding homomorphism. This statement contains Kolmogoroff's generalization of Alexander's duality theorem and has many applications. Using the preceding theorem one can prove: If A and B are compact spaces of dimensions n and m respectively, the necessary and sufficient condition that the topological product $A\times B$ be of dimension $n+m$ is the existence of an open set $U\subset A$ and an open set $V\subset B$ such that $H^n(U)$ and $H^m(V)$ contain elements α and β satisfying the following conditions: If the integer d is a factor of the order of α, then $\beta\not\equiv 0$ modulo d (that is, there is no element γ of $H^m(V)$ satisfying $\beta=d\gamma$); if the integer e is a factor of the order of β, then $\alpha\not\equiv 0$ modulo e. (Received May 3, 1941.)

Long Exact Sequence [Hurewicz 1941]

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \ldots$$

is an exact sequence

Exact Seq. of Ball & Sphere

$$\cdots \to H_3(\partial D^3) \xrightarrow{\bar{i}_*} H_3(D^3) \xrightarrow{\bar{j}_*} H_3(D^3/\partial D^3) \xrightarrow{\partial_*} H_2(\partial D^3) \xrightarrow{\bar{i}_*} H_2(D^3) \xrightarrow{\bar{j}_*} H_2(D^3/\partial D^3) \to \cdots$$

($H_3(D^3) = 0$, $H_2(D^3) = 0$)

$$0 \to H_3(D^3/\partial D^3) \to H_2(\partial D^3) \to 0$$

($D^3/\partial D^3 = S^3$, $\partial D^3 = S^2$)

$$\cdots \cong H_3(S^3) \cong H_2(S^2) \cong H_1(S^1) \cong \mathbb{Z}$$

(induction; exact seq.; already proved.)

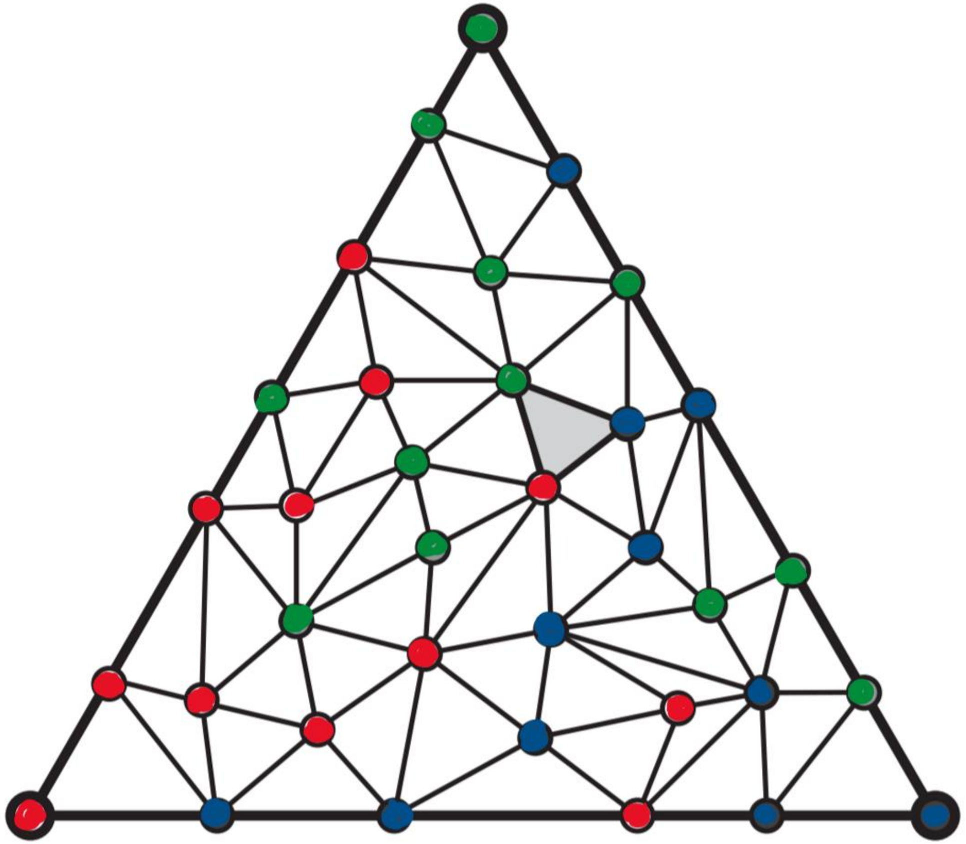
Claim. Any exact $0 \to A \to B \to 0$ implies $A \cong B$.

Intermission

Food for Thought.

Hmm. Is this actually a math course in disguise? (Yes.)

Fixed-Point Theorems

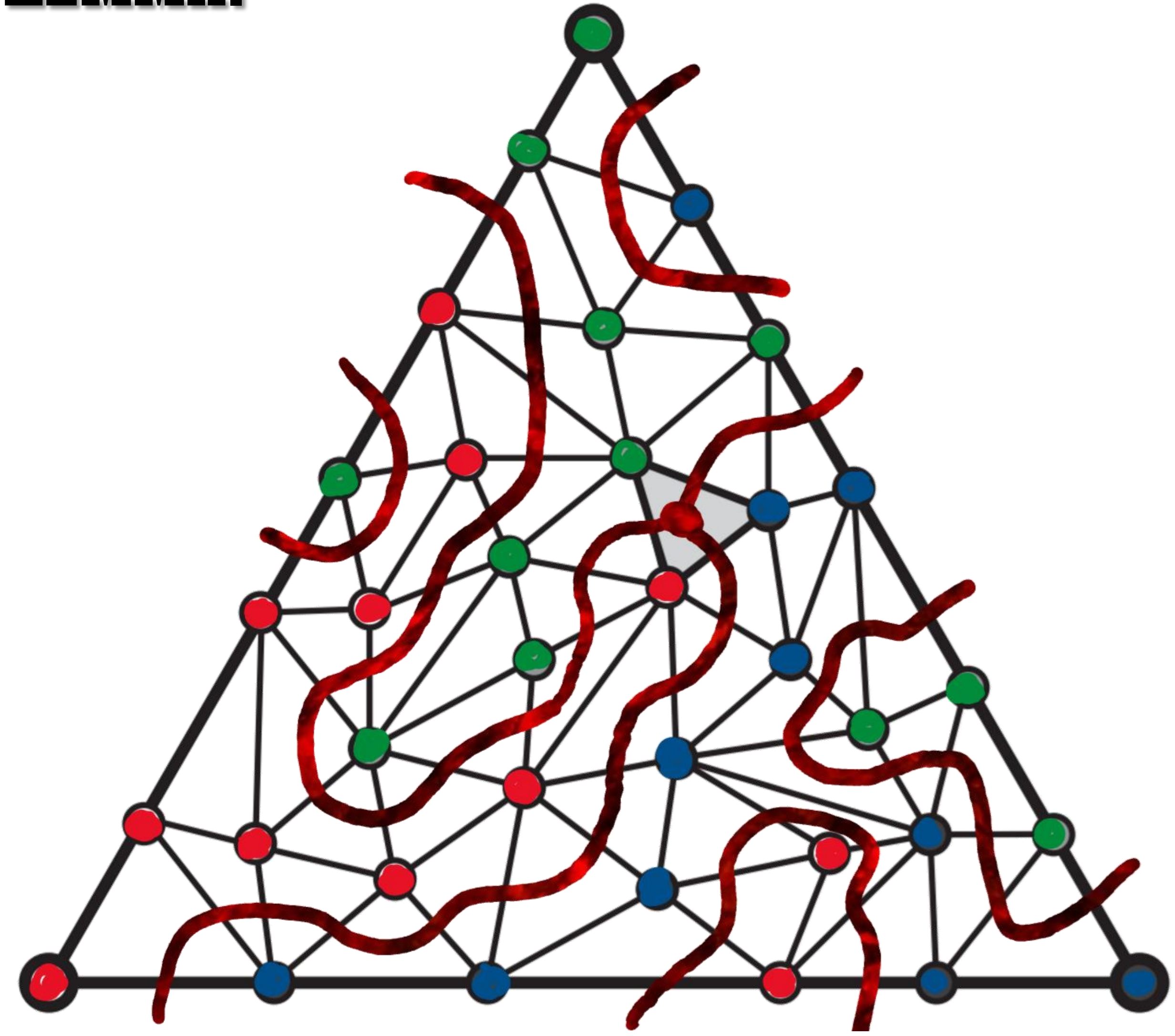


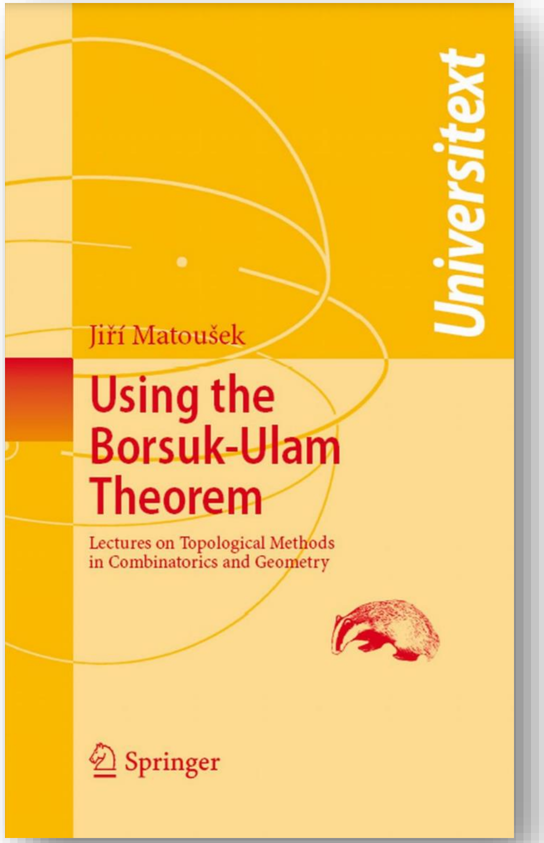
Sperner's Lemma

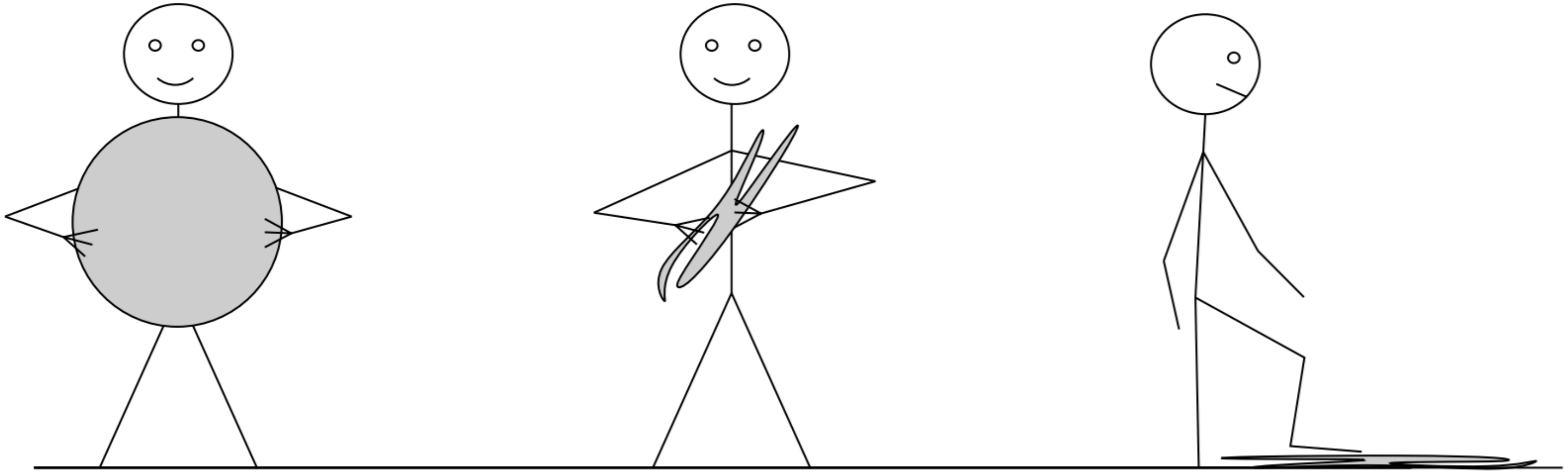
[Sperner 1928] [Knaster–Kuratowski–Mazurkiewicz 1929]

Every properly [d]-colored triangulation of Δ^d
contains a simplex with all the colors

Proof of 2d Sperner's Lemma.







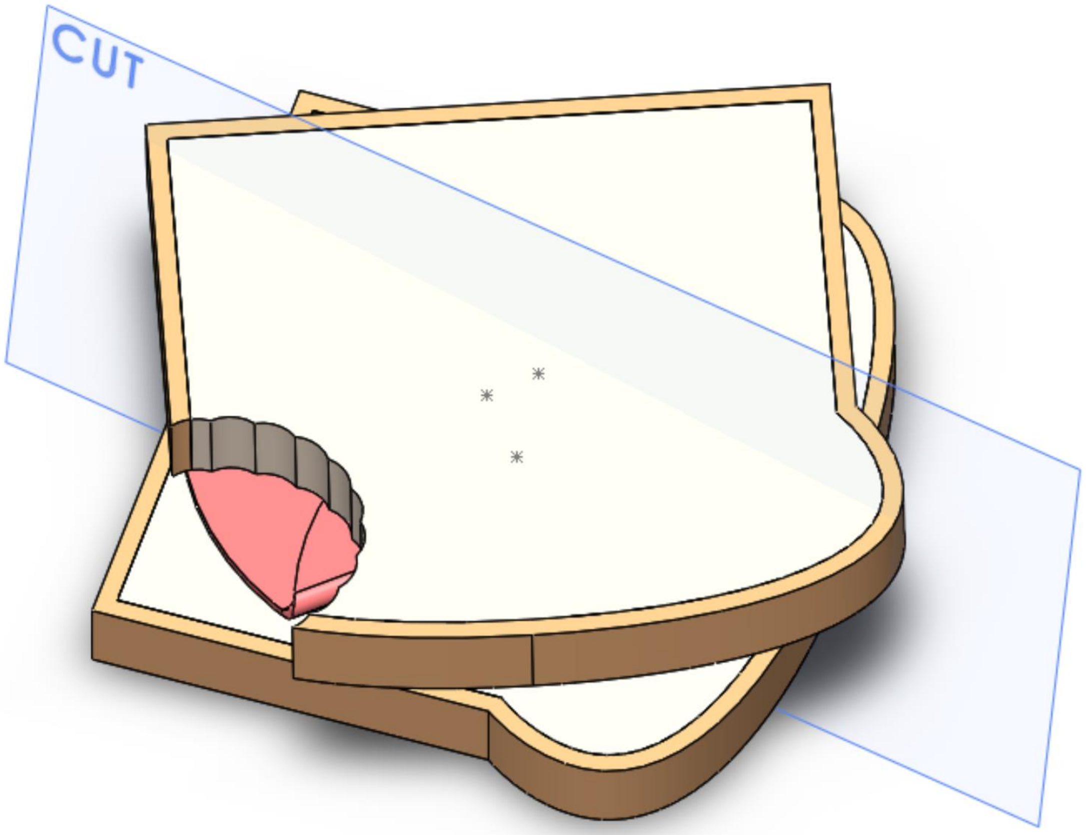
Borsuk-Ulam Theorem [Borsuk 1933]

Every map $f: S^n \to R^n$ has a point x where

$$f(x) = f(-x)$$

Equivalent Formulations

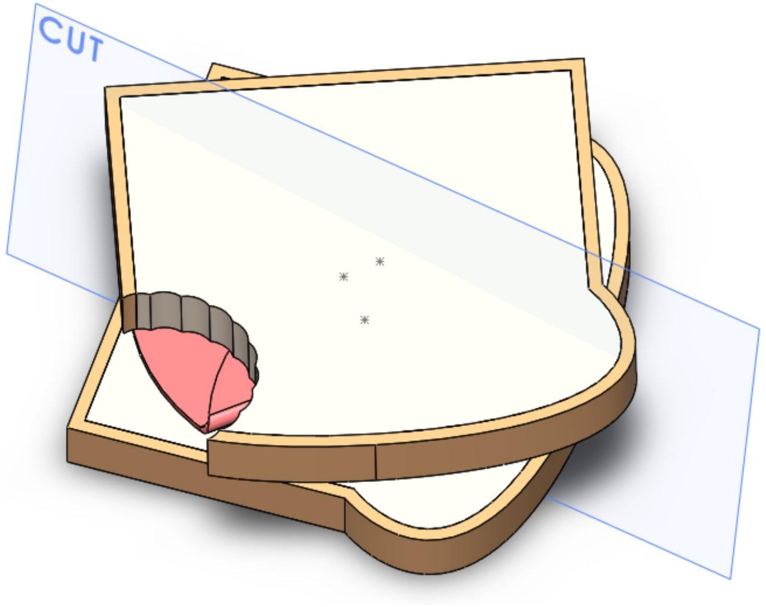
- Every map $f: S^n \rightarrow R^n$ that is antipodal has a point x where $f(x) = 0$
- There is no antipodal map $g: S^n \rightarrow S^{n-1}$
- There is no map $h: D^n \rightarrow S^{n-1}$ that is antipodal on ∂D^n



Ham Sandwich Theorem [Banach 1938] [Stone-Tukey 1942]

A ham sandwich has a straight cut that divides the ham and two breads evenly

Proof of Ham Sandwich Theorem.



$f : S^3 \to \mathbb{R}^3$

$\frac{\alpha_1}{c} x + \frac{\alpha_2}{c} y + \frac{\alpha_3}{c} z = \cancel{c}\ 1$

$f(\alpha_1, \alpha_2, \alpha_3) := (\text{bread}, \text{ham}, \text{cheese})$ on one side of the cut.

$f(\alpha_1, \alpha_2, \alpha_3) = f(-\alpha_1, -\alpha_2, -\alpha_3)$ by Borsuk-Ulam.

Necklace Splitting Theorem [Alon-West 1986]

Every (open) necklace with d colors of jewels can be divided between two thieves using at most d cuts

Proof of Necklace Splitting Theorem.

moment curve $(t, t^2, t^3), \ldots, t^d$

$\ldots + \alpha_d x^d$

$\alpha_1 x + \alpha_2 y + \alpha_3 z = 1$

$\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 = 1$

$\Rightarrow$ hyperplane $\cap$ moment curve at most ~~3~~ d times.

Spicy Chicken Theorem [Karasev-Hubard-Aronov 2014]

A spicy fat chicken can be cut into p pieces, each with equal amount of meat and spice

Fair Partitioning through Topology Tools

Next Time.

More homology in action, on data analysis