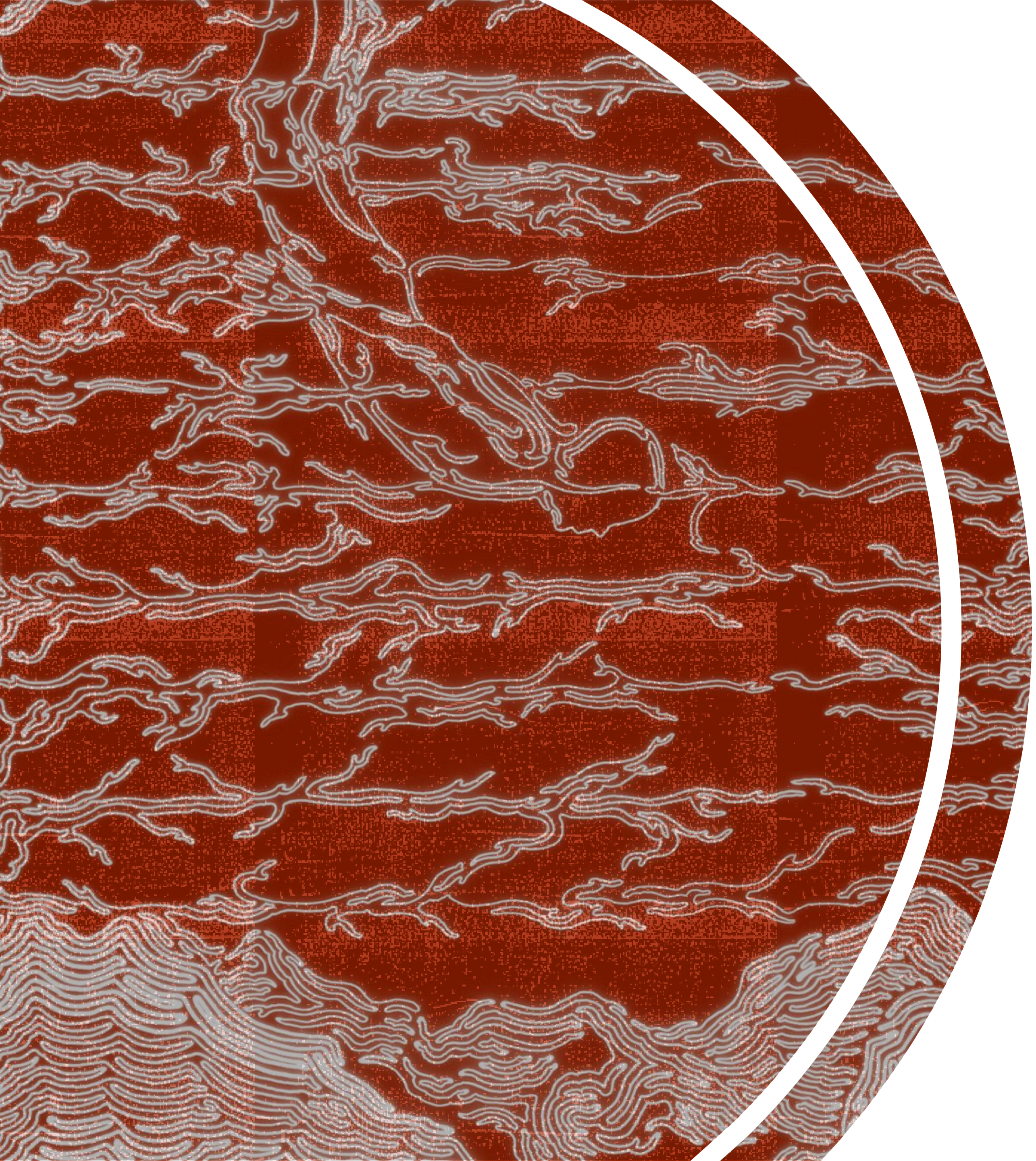


# Introduction to Computational Topology


Hsien-Chih Chang


Lecture 11, October 19, 2021

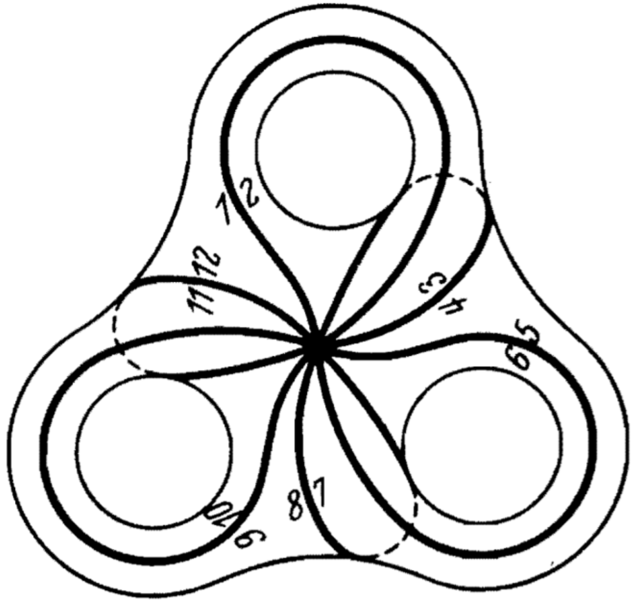
# Administrivia

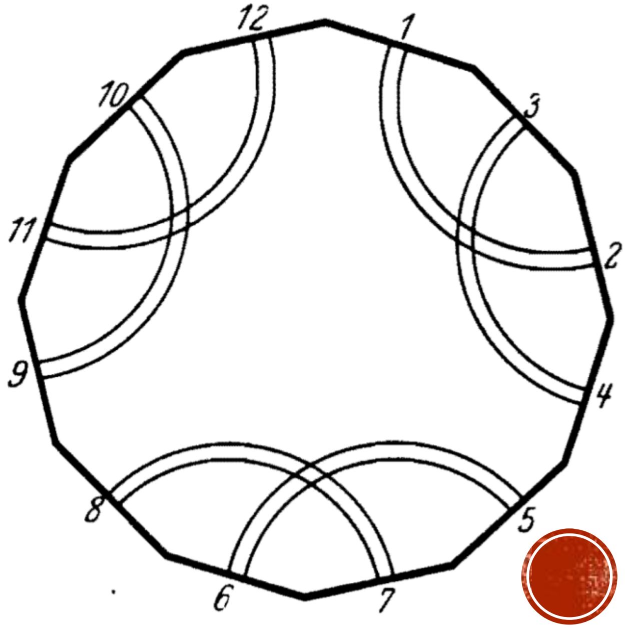
- Homework β will be out soon
- Remember to submit your project proposal!

# Homology

# Fundamental Groups of Surfaces

- $\pi_1(\Sigma(g,0)) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$
- $\pi_1(\Sigma(0,r)) = \langle a_1, \ldots, a_r \mid a_1 a_1 \ldots a_r a_r \rangle$



$$\left\langle \; a, b, c, d, e, p, q, r, t, k \;\middle|\; \begin{array}{lll} p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\ p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\ p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\ p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\ p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\ aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\ bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\ cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\ dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\ eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 & \end{array} \right\rangle$$

[Collins 1986]

# Undecidability of $\pi_1$ [Novikov 1955] [Boone 1958]

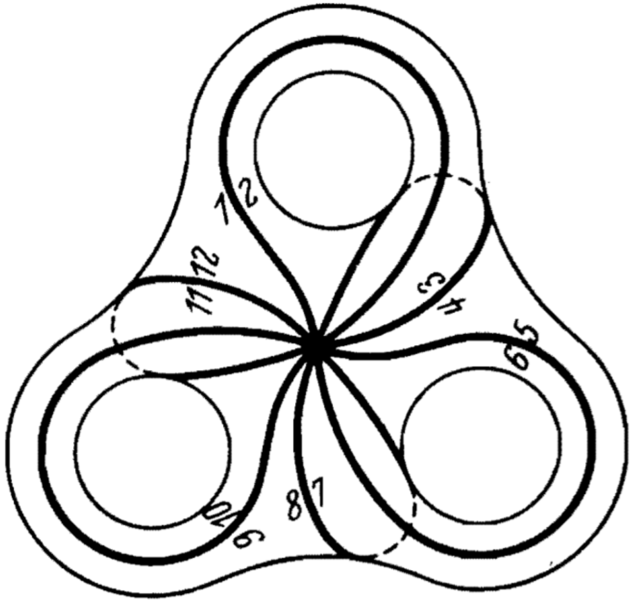
Checking if a 2-complex has trivial $\pi_1$ is undecidable

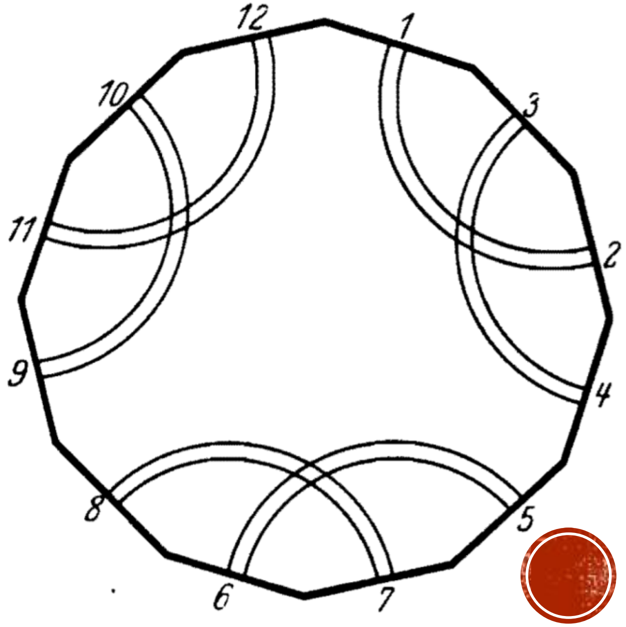
# What if we use Vector Spaces?

# Spark of Idea

- $\langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$

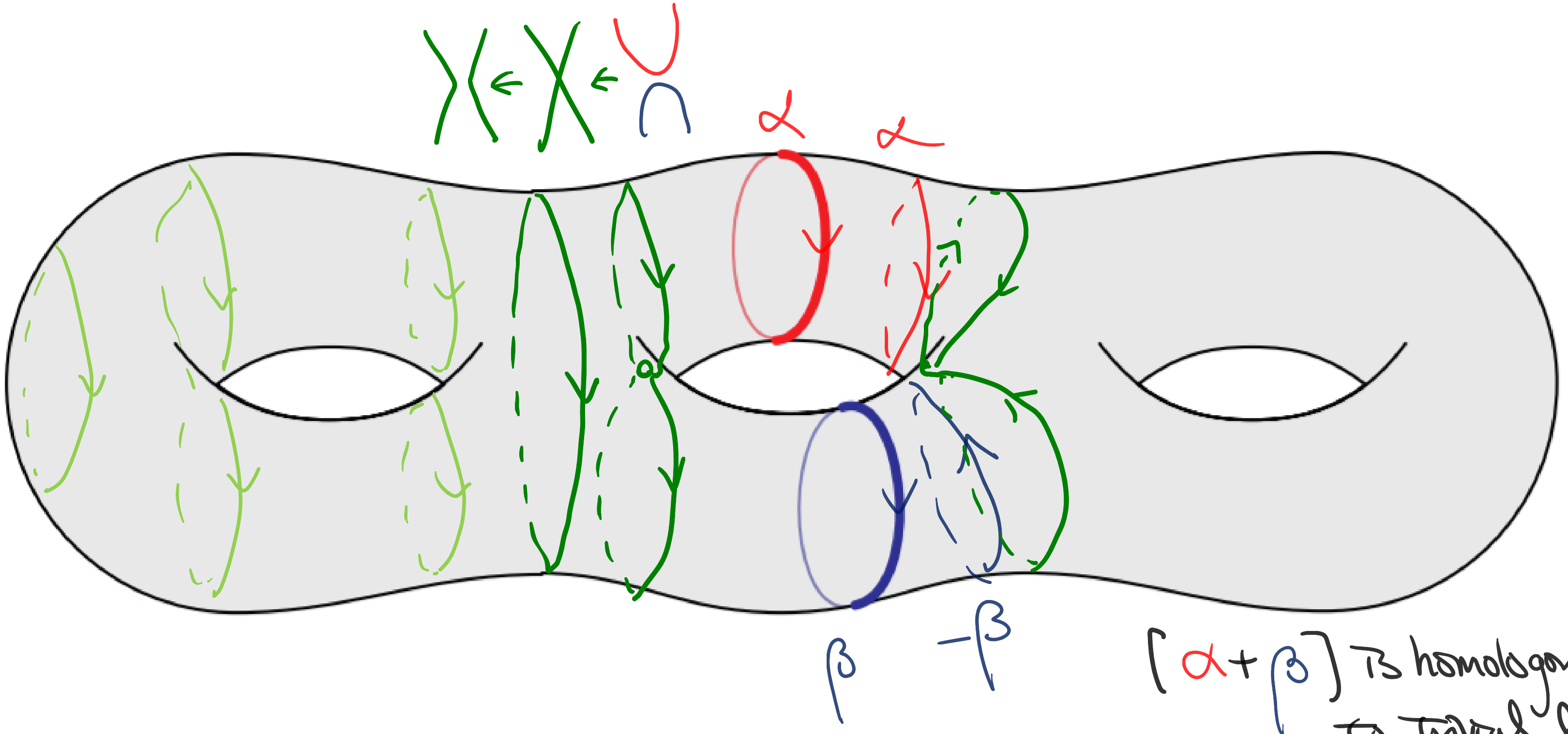
- $\mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g \rangle \,/\, \langle a_1 + b_1 - a_1 - b_1 \ldots \rangle$ 0



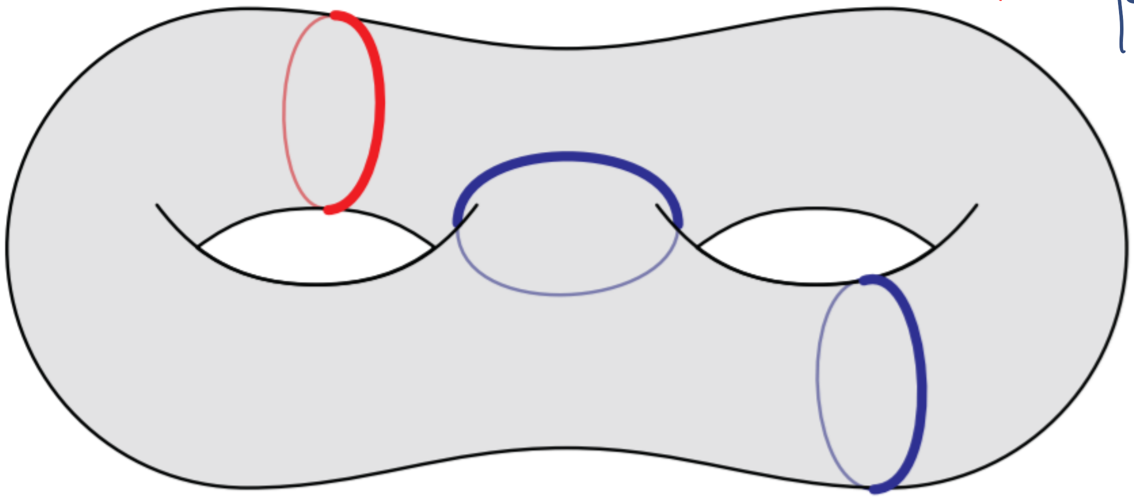
# Homology Equivalence

- Two "cycles" are **homologous** if together they are the boundary of some region

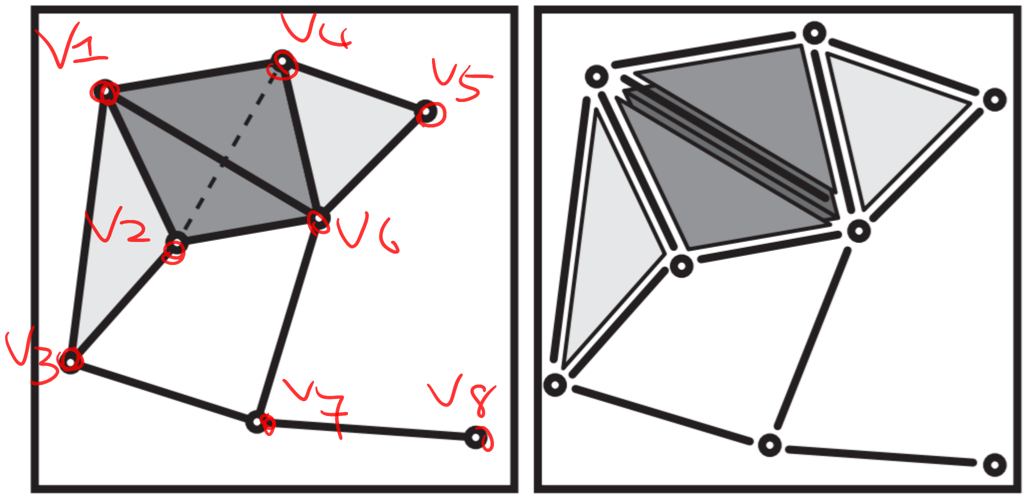



[α+β] is homologous to trivial loop.

⟺ α and −β are homologous.



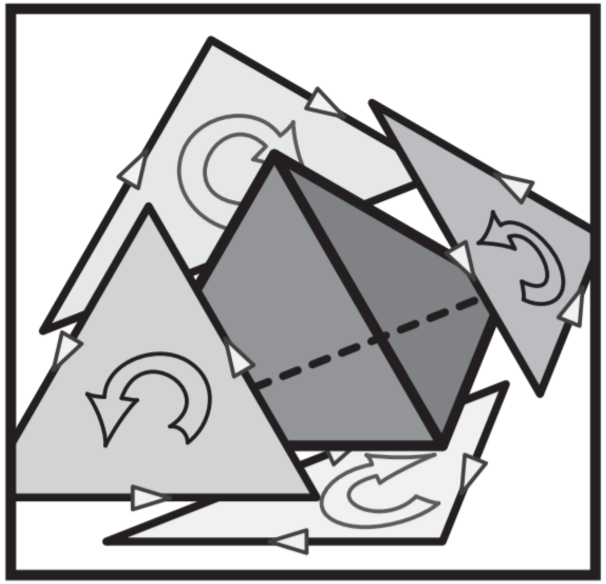
# Chain Complex




- Vector spaces over elements

- 0-complex $\mathbb{Z}\langle v_1 \dots v_n \rangle \ni v_1 - 2v_3 + 4v_8$
- 1-complex $\mathbb{Z}\langle e_1 \dots e_m \rangle$
- 2-complex $\mathbb{Z}\langle f_1 \dots f_k \rangle$
- …

# Boundary Map



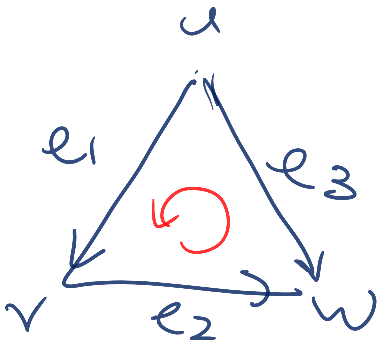
$\sigma = [v_0, \ldots, v_n]$

- $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$

$\partial_n(\sigma) = \sum_i (-1)^i \cdot \sigma[v_0, \ldots \hat{v}_i \ldots v_n]$

$\partial_1 : C_1(X) \to C_0(X)$

$e \longmapsto v - u$

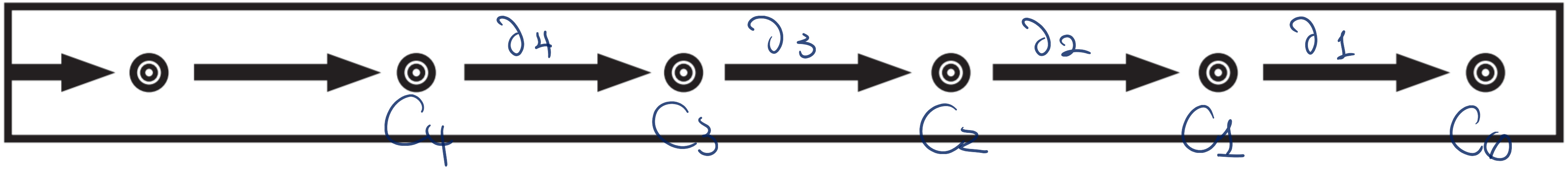


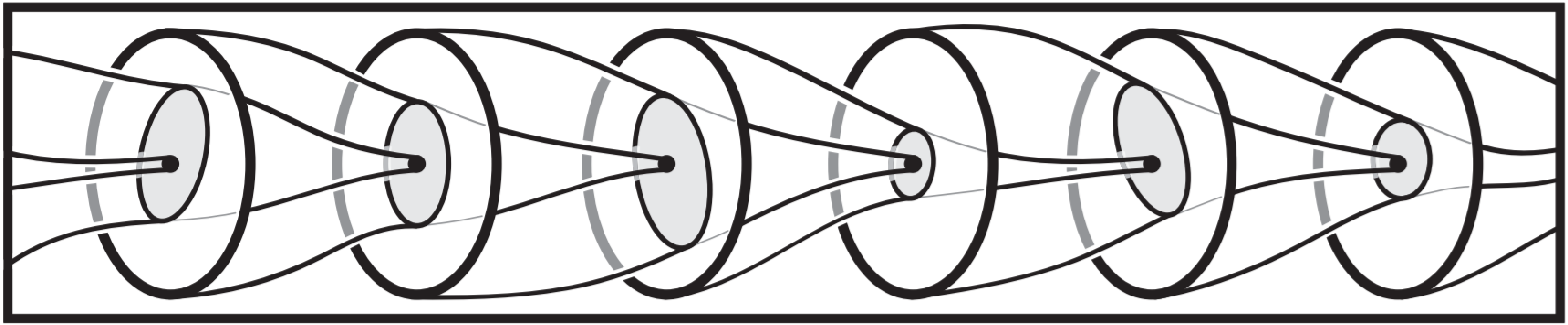

$\partial_2 : C_2(X) \to C_1(X)$

$\triangle \qquad e_1 + e_2 - e_3$

$\partial[u.v.w] = [u.v] + [v.w] - [u\,w] = [u.v] + [v.w] + [w.v]$

$(C_0, \partial_0)$





# Fundamental Lemma of Homology

$$\partial \bullet \partial = 0$$

*"Boundary of a region must be boundaryless"*

# Proof of $\partial \bullet \partial = 0$.

$\sigma$ e $n$-dim simplex $\sigma = [v_0, \dots, v_n]$

$$\partial_n \sigma = \sum_i (-1)^i \cdot \sigma[v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\partial_{n-1}(\partial_n \sigma) = \sum_i (-1)^i \circ \underline{\partial_{n-1} \sigma[v_0 \dots, \hat{v}_i, \dots v_n]}$$

$$= \sum_i (-1)^i \circ \sum_j (-1)^j \cdot \sigma[v_0 \dots \hat{v}_i \dots \hat{v}_j \dots v_n]$$

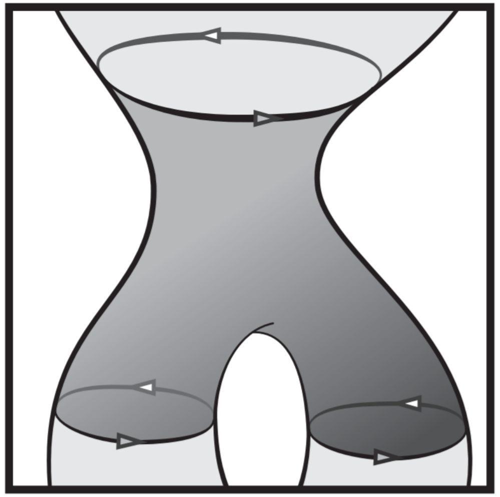
$$= \emptyset$$

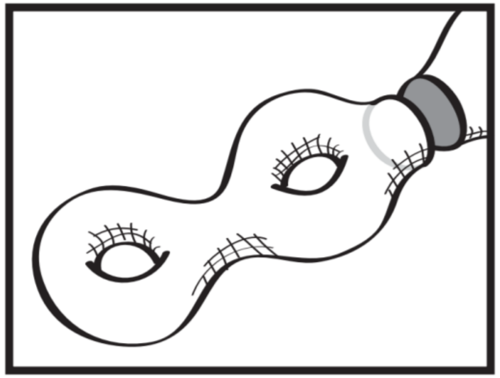
# Cycles and Boundaries

- Cycle space $Z_n(X) = \ker \partial_n$

- Boundary space $B_n(X) = \operatorname{im} \partial_{n+1}$

Fundamental Lemma : $B_n(X) \subseteq Z_n(X)$ .



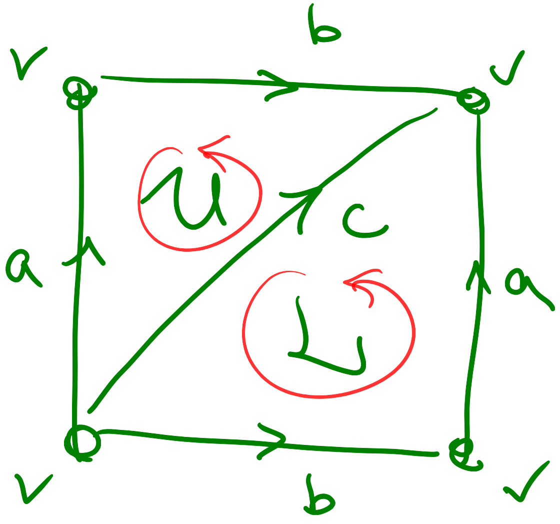


# Homology Groups

- Simplicial homology group

$$H_n(X) = Z_n(X) \,/\, B_n(X)$$
$$= \text{Cycles}/\text{Boundaries}$$

# Computing Homology




$\partial_2 : C_2 \to C_1$

$\partial_2(U) = -a-b+c$

$\partial_2(L) = a+b-c = -\partial_2(U)$

$\partial_2(C_2) = \langle -a-b+c \rangle \leq C_1$

$C_2 = \langle U, L \rangle$

$C_1 = \langle a, b, c \rangle$

$C_0 = \langle v \rangle$

$\partial_1 : C_1 \to C_0$

$\sigma \mapsto 0$

$H_1(\text{Torus}) = \ker\partial_1 / \operatorname{im}\partial_2$

$= \langle a,b,c \rangle / \langle -a-b+c \rangle$

$\cong \mathbb{Z}^2$

$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$H_2(\text{Torus}) = \ker\partial_2 / \operatorname{im}\partial_3 = \langle U+L \rangle \cong \mathbb{Z}$

$H_0(\text{Torus}) = \ker\partial_0 / \operatorname{im}\partial_1 = \langle v \rangle / 0 \cong \mathbb{Z}$

# Homology of Torus

- $H_n(X) = Z_n(X) / B_n(X)$
  $= \ker \partial_n / \operatorname{im} \partial_{n+1}$

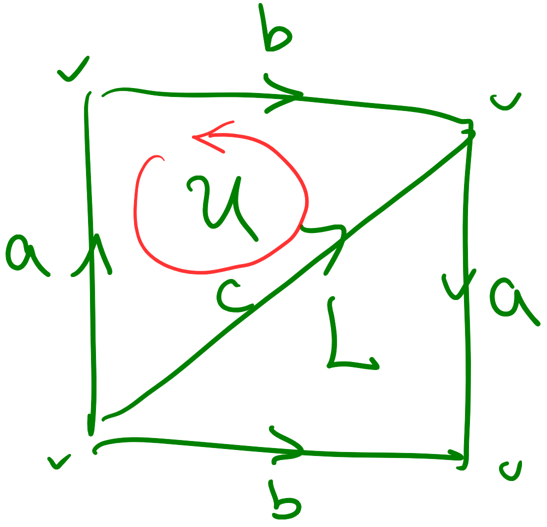
# Intermission

## Food for Thought.

What do vector spaces give us?

# Homology of Klein Bottle

- $H_n(X) = Z_n(X) \,/\, B_n(X)$
  $= \ker \partial_n / \operatorname{im} \partial_{n+1}$




$C_0 = \langle v \rangle$

$C_1 = \langle a.b.c \rangle$

$C_2 = \langle U.L \rangle$

$\partial_1 : \langle a.b.c \rangle \mapsto 0$ .

$\partial_2(U) = -a-b+c$

$\partial_2(L) = -a+b-c$

$$H_1(K) = \ker\partial_1 / \operatorname{im}\partial_2$$

$$= \langle a, b, \cancel{c}^{\,-a-b+c} \rangle \Big/ \langle -a-b+c,\ \overset{-2a}{-a\cancel{+b}-c} \rangle$$

$$= \left( \langle a, b \rangle / \langle -2a \rangle \right) \times \frac{\cancel{\langle -a-b+c \rangle}^{\,0}}{\cancel{\langle -a-b+c \rangle}}$$

$$= \langle a \rangle / 2\langle a \rangle \times \langle b \rangle / 0$$

$$= \mathbb{Z}_2 \times \mathbb{Z}$$

# What is Good about Homology?

# Christmas Wishlist

- Fast computation
- Functoriality:
  - $f: X \to Y$ implies $f_*: H_n(X) \to H_n(Y)$ for all $n$
  - $(f \circ g)_* = f_* \circ g_*$, $id_* = id$, $\partial \circ f_* = f_* \circ \partial$, …
- Invariance:
  - $f, g: X \to Y$ homotopic implies $f_* = g_*$
- …

VOL. 31, 1945 *MATHEMATICS: EILENBERG AND STEENROD* 117

## *AXIOMATIC APPROACH TO HOMOLOGY THEORY*

BY SAMUEL EILENBERG AND NORMAN E. STEENROD

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN

Communicated February 21, 1945

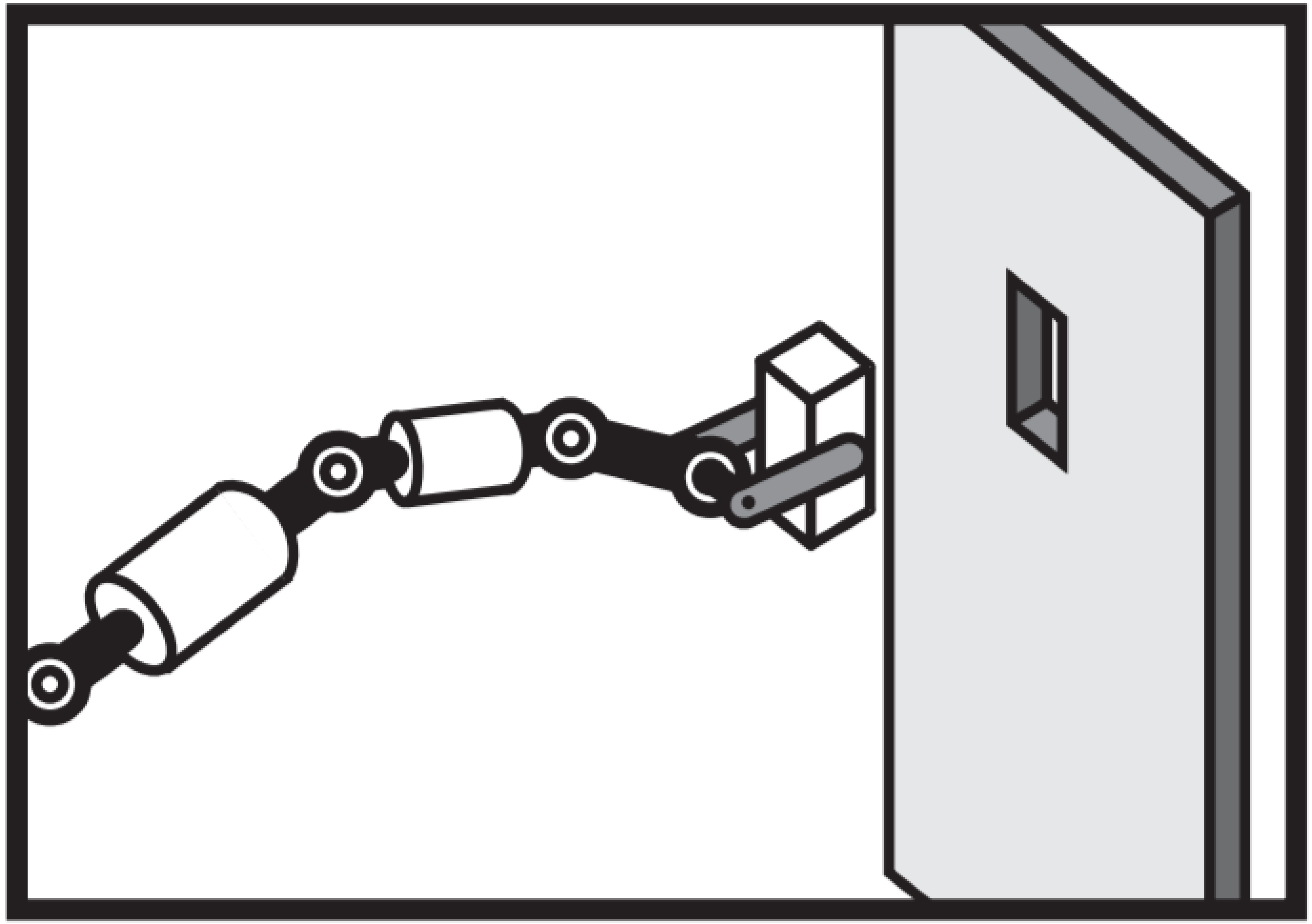
*1. Introduction.*—The present paper provides a brief outline of an axiomatic approach to the concept: homology group. It is intended that a full development should appear in book form.

The usual approach to homology theory is by way of the somewhat complicated idea of a complex. In order to arrive at a purely topological concept, the student of the subject is required to wade patiently through a large amount of analytic geometry. Many of the ideas used in the constructions, such as orientation, chain and algebraic boundary, seem artificial. The motivation for their use appears only in retrospect.

Since, in the case of homology groups, the definition by construction is so unwieldy, it is to be expected that an axiomatic approach or definition by properties should result in greater logical simplicity and in a broadened

# SINGULAR HOMOLOGY THEORY [Eilenberg-Steenrod 1945]

"Homology group has everything you want from Christmas."

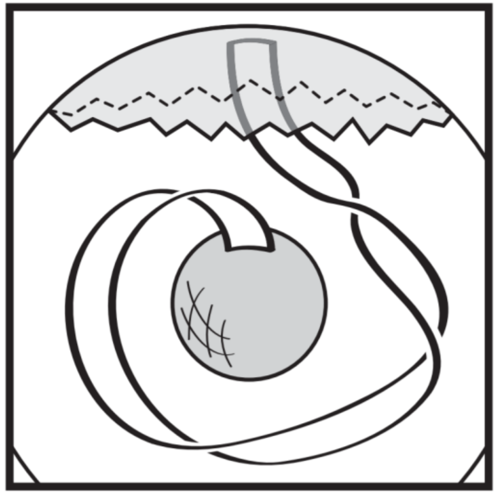


# Robot Arm Puzzle

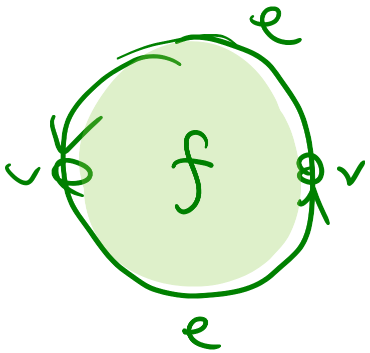
$K : (S^1)^N \to SO_3$

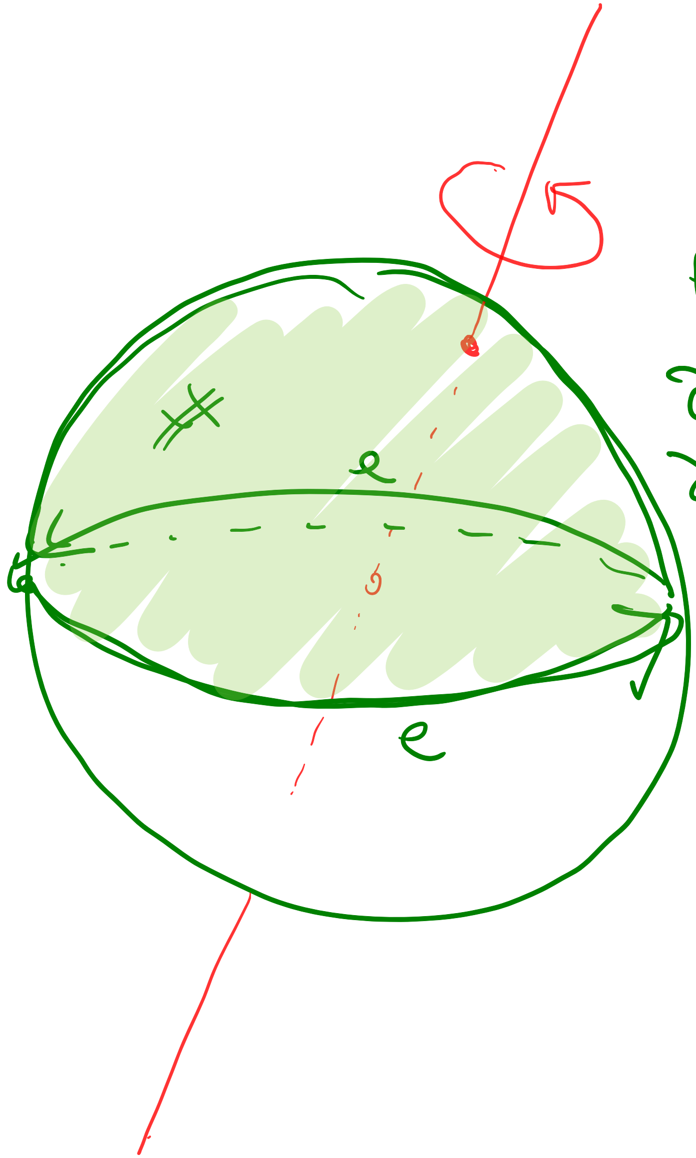
# Robot Arm Puzzle

- What is $H_1(SO_3)$?



$SO_3 \cong$




$C_0 = \langle v \rangle$

$C_1 = \langle e \rangle$

$C_2 = \langle f \rangle$

$C_3 = \langle B \rangle$

$H_1 = \ker\partial_1 / \operatorname{im}\partial_2$

$\partial_1 : e \mapsto 0$

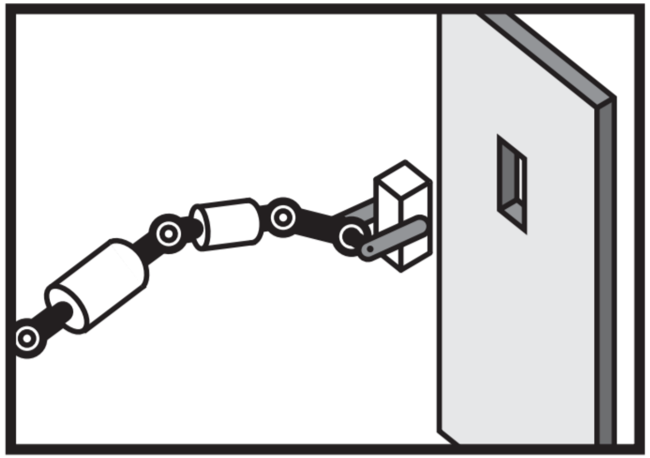
$\partial_2 : f \mapsto 2e$

$\ker\partial_1 = \langle e \rangle$

$\operatorname{im}\partial_2 = \langle 2e \rangle$

$H_1(SO_3) = \mathbb{Z}/2\mathbb{Z}$

$= \mathbb{Z}_2$



$$SO_3 \xrightarrow{s} S_1^N \xrightarrow{\kappa} SO_3$$

$$H_1(SO_3) \xrightarrow{s_*} H_1(S_1^N) \xrightarrow{\kappa_*} H_1(SO_3)$$

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] \arrow[rr, bend right, "Id"'] & \mathbb{Z}^N \arrow[r] & \mathbb{Z}_2 \end{tikzcd}$$

$x:$

$2x = 0$

# Robot Arm Puzzle

- What is $H_1(SO_3)$?
- Given $\kappa: S_1^N \to SO_3$, is there $s: SO_3 \to S_1^N$, such that $s \circ \kappa = id$?

# Euler Characteristic Redux

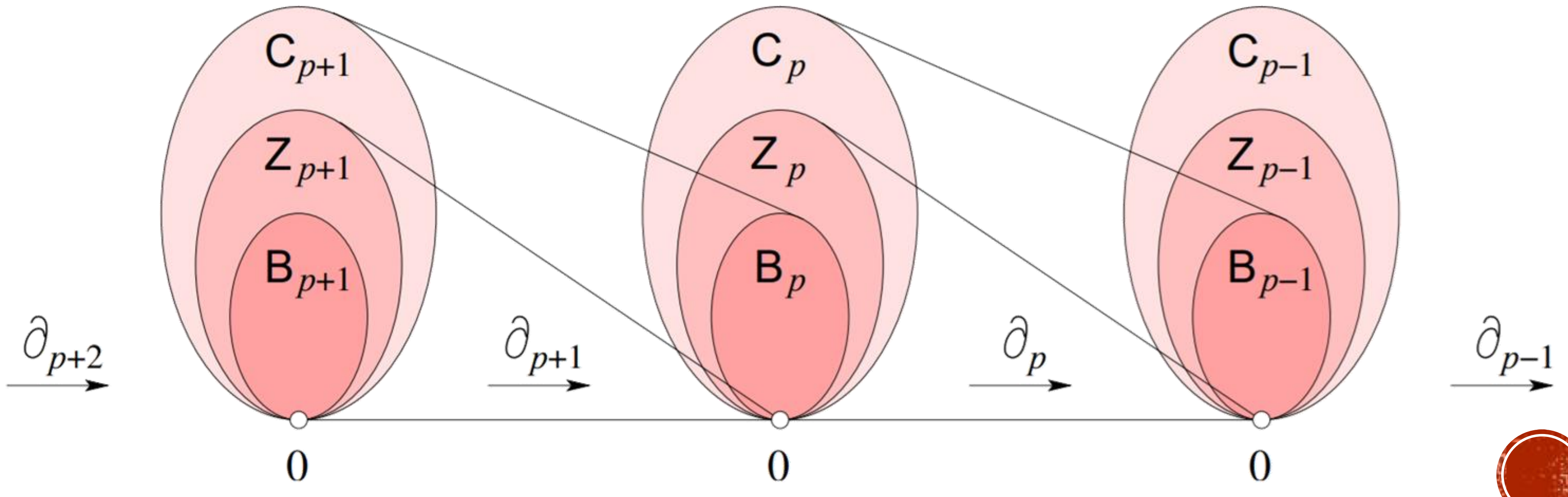
# Dimension

- Betti numbers $\beta_n$: dim $H_n(X)$

# Euler-Poincaré Formula

$$\chi(X) = \Sigma_n (-1)^n \cdot \dim H_n(X)$$

# Proof of Euler-Poincaré Theorem.



# Order Now for Christmas!

## On Thursday.

Homology in action.

(Optional: homology groups has internal structures.)