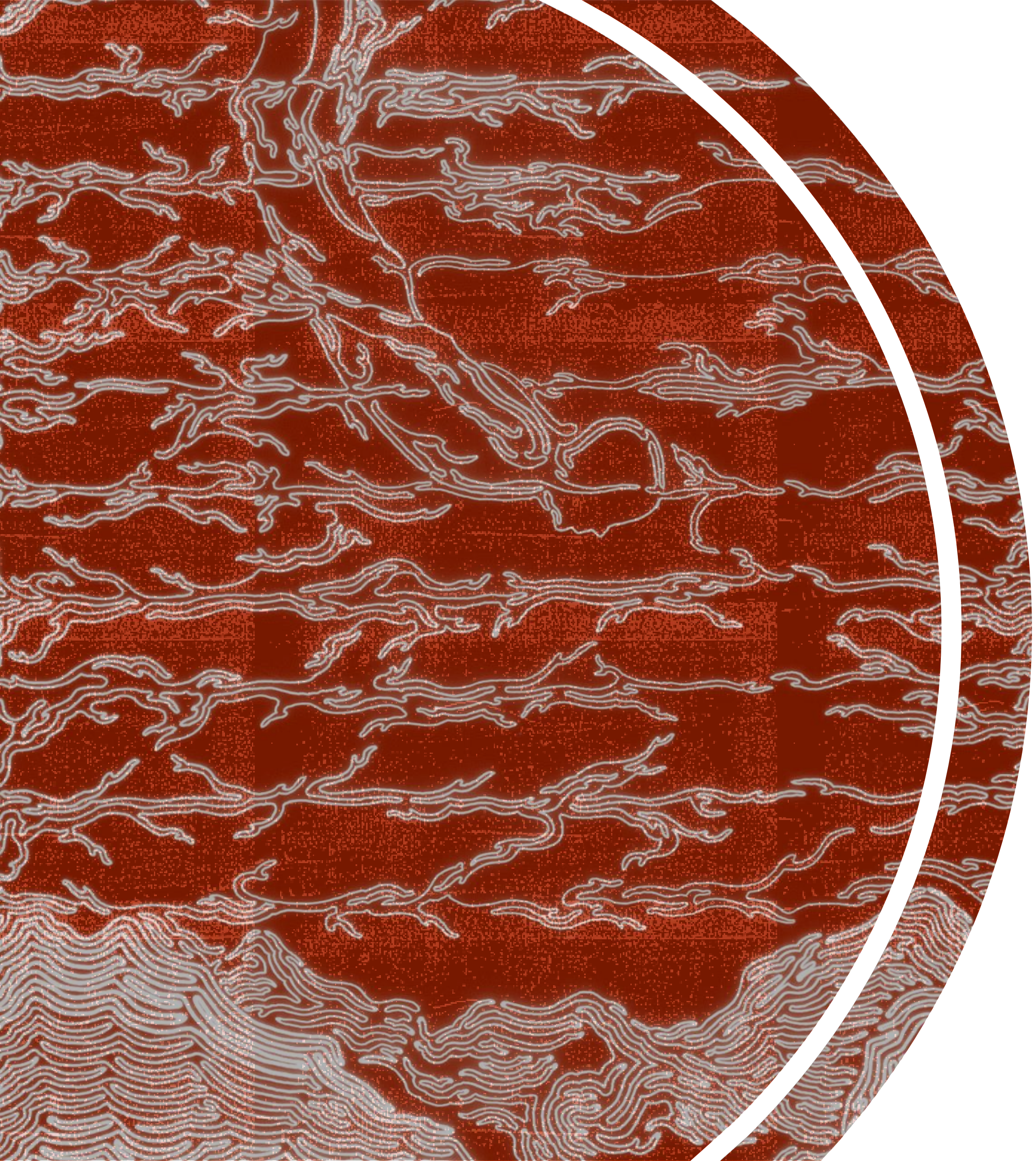


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 8, October 7, 2021

Administrivia

- Homework a is out, due 11/15 (end of term)

Homotopy Equivalence

Last Time on Algebraic Topology

- [γ] is the class of closed paths homotopic to γ in space X

$$\pi_1(X, x_0) = \{[\gamma] : \text{closed path } \gamma \text{ in } X \text{ starting and ending at } x_0\}$$

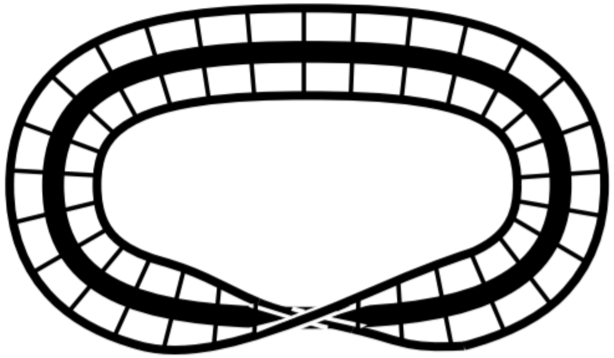
Does $\pi_1(X)$ Classify Spaces?

Equivalence

- Homeomorphism
  - f: X ⟶ Y continuous bijection
  - g: Y ⟶ X continuous bijection
  - $f \bullet g = id_X$
  - $g \bullet f = id_Y$

- Homotopy equivalence
  - f: X ⟶ Y continuous
  - g: Y ⟶ X continuous
  - $f \bullet g$ homotopic to id_X
  - $g \bullet f$ homotopic to id_Y

Homotopy Equivalence



$X :=$ Möbius band M

$Y :=$ circle S^1

$f : X \to Y$

$g : Y \to X$

$f \circ g : Y \to Y \to X$

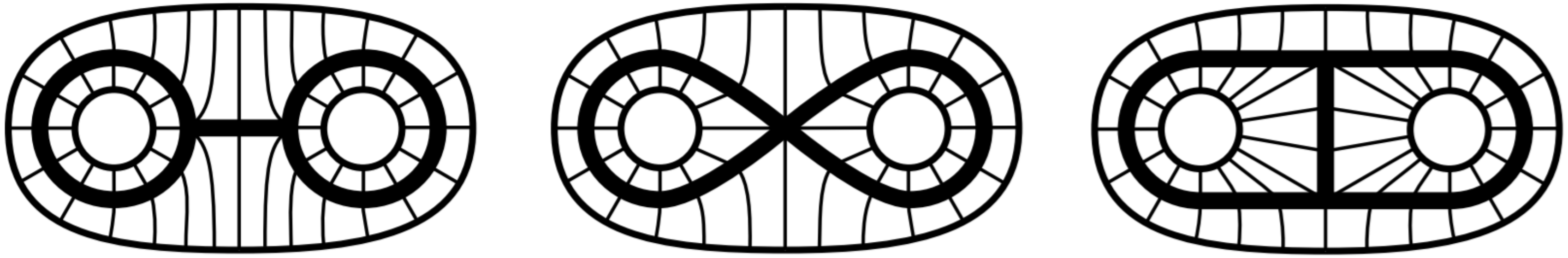
$= g(f(\cdot))$

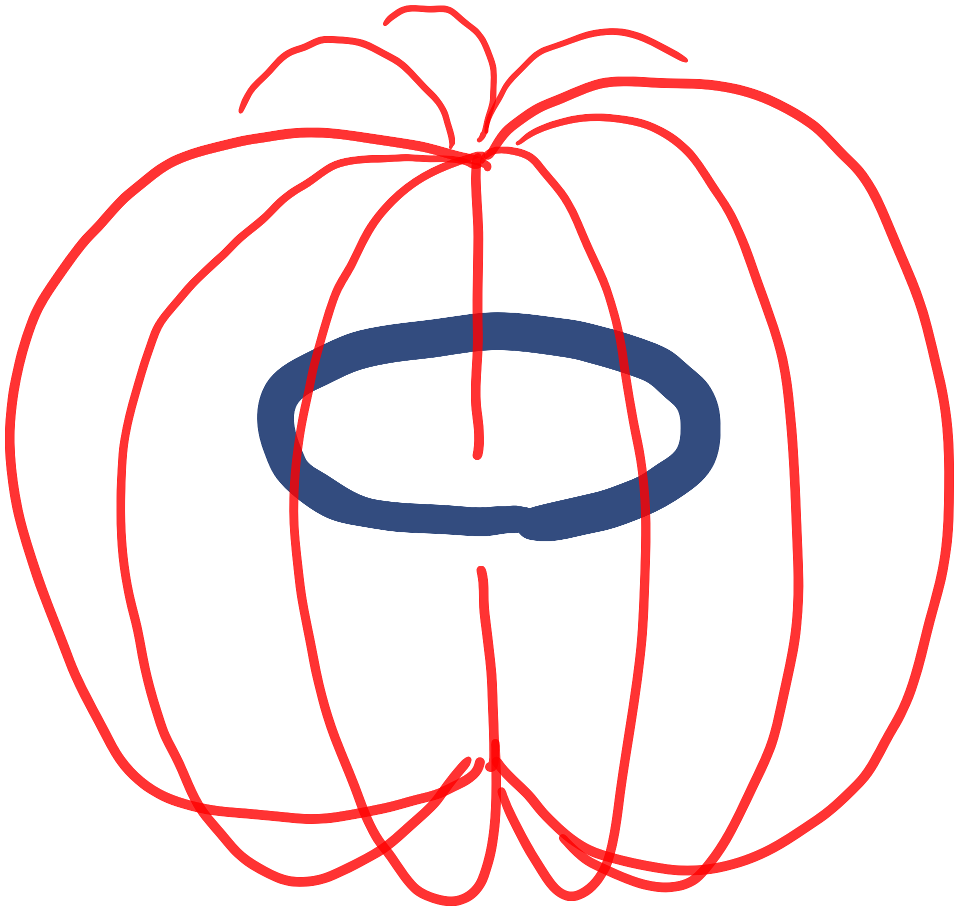
retraction from $M \to S^1$

$g \circ f : Y \to X \to Y$

identity on S^1

Homotopy Equivalence



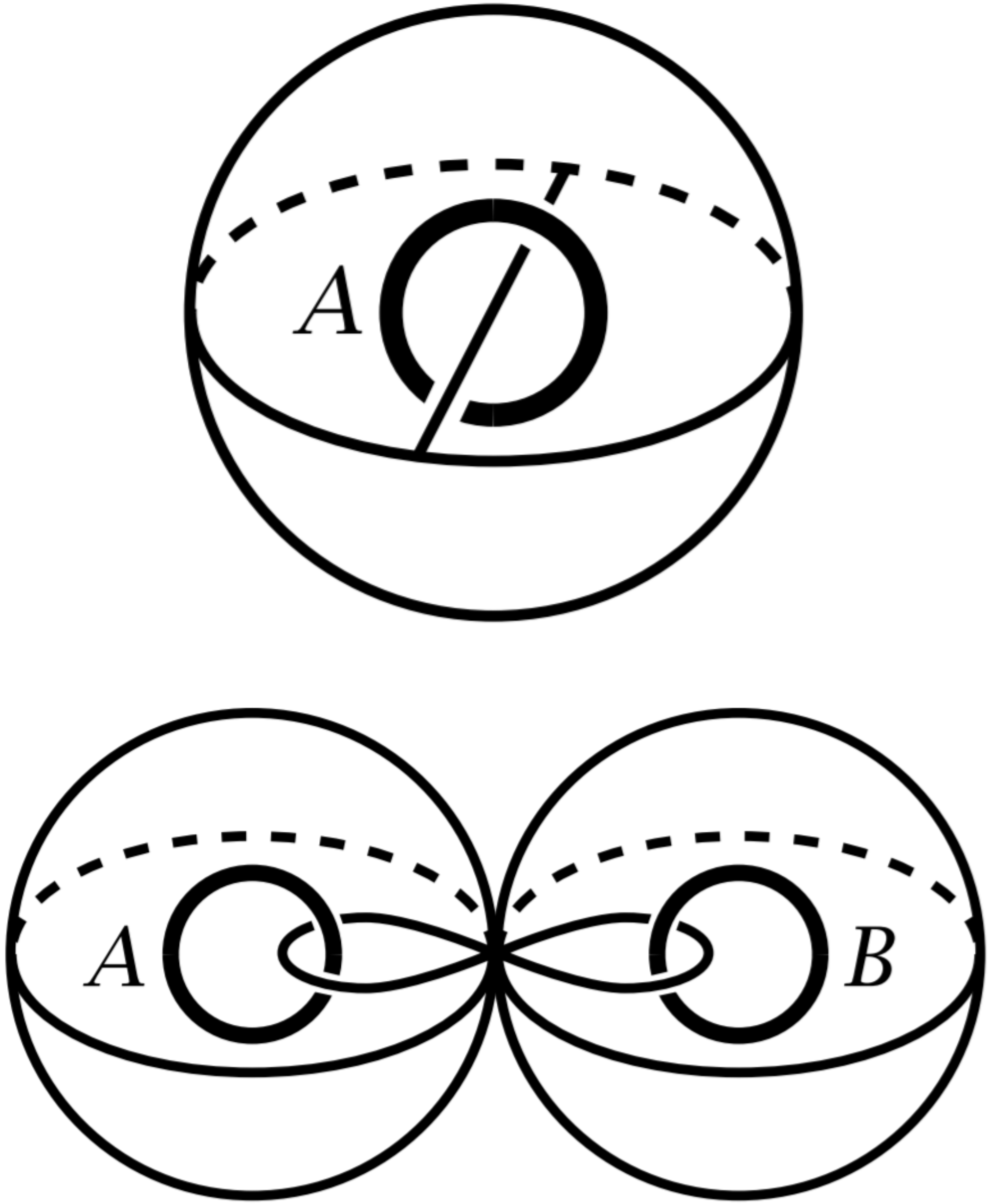


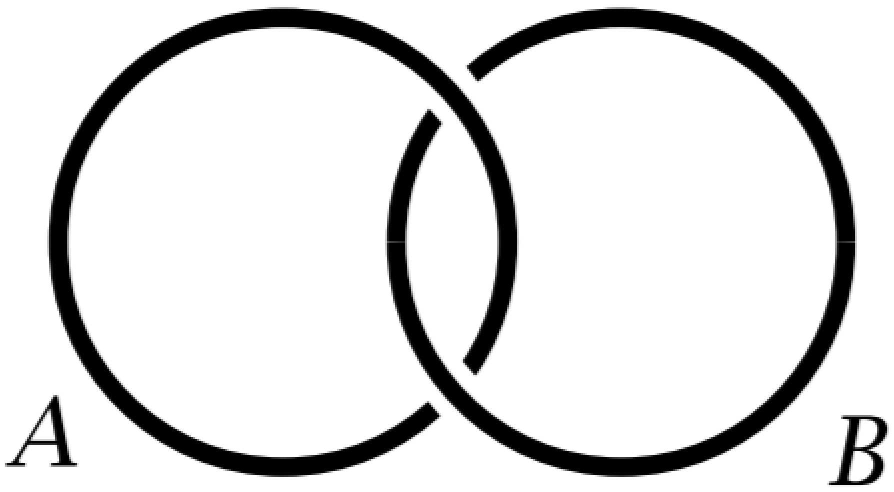
Visualization Exercise

- Complement of circles

Visualization Exercise

- Complement of circles



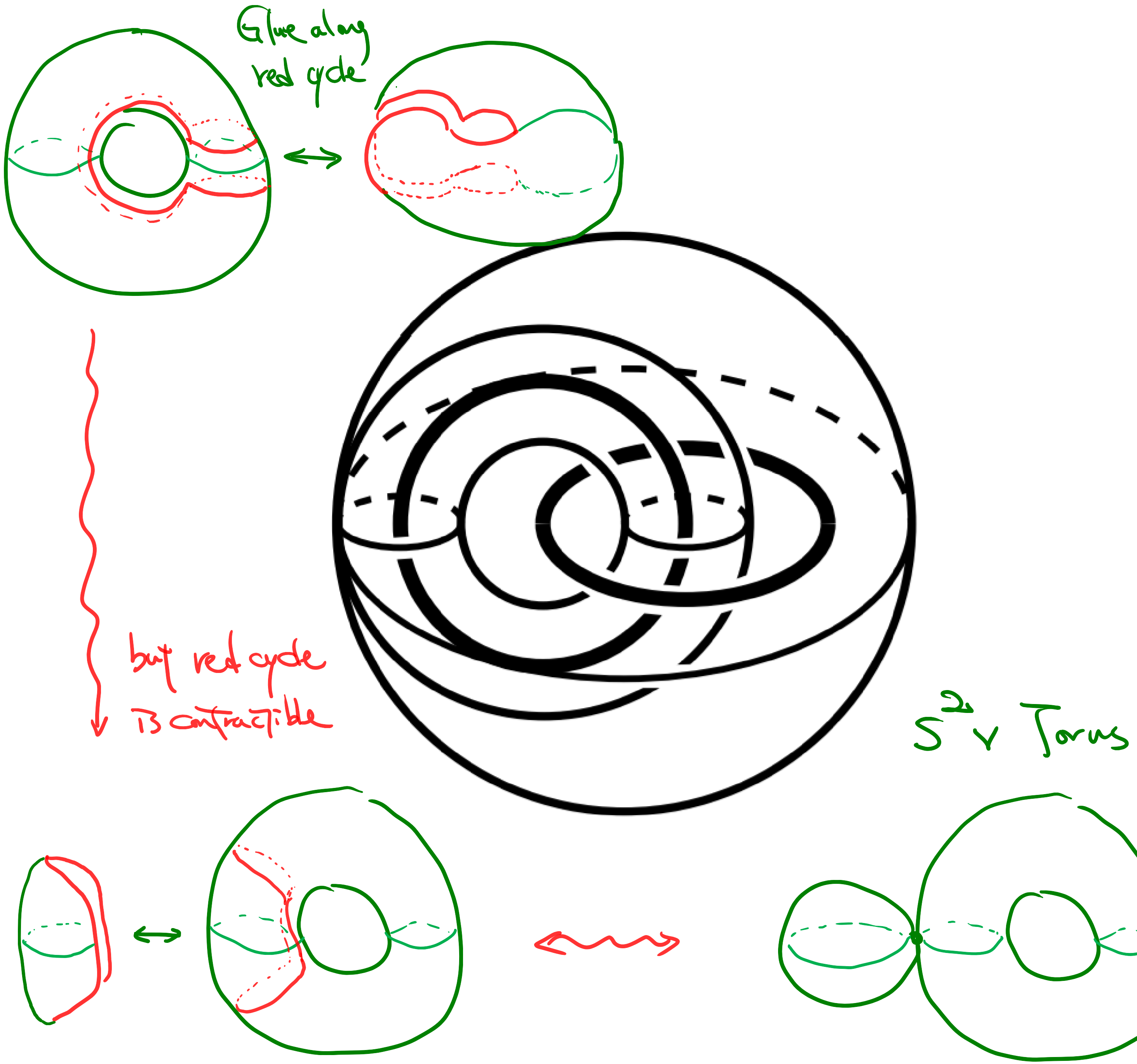


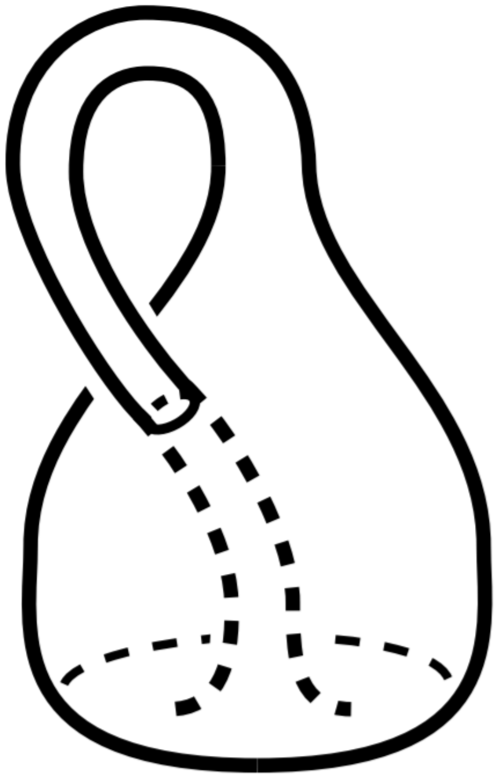
Visualization Exercise

- Complement of linked circles

Visualization Exercise

- Complement of linked circles





Visualization Exercise

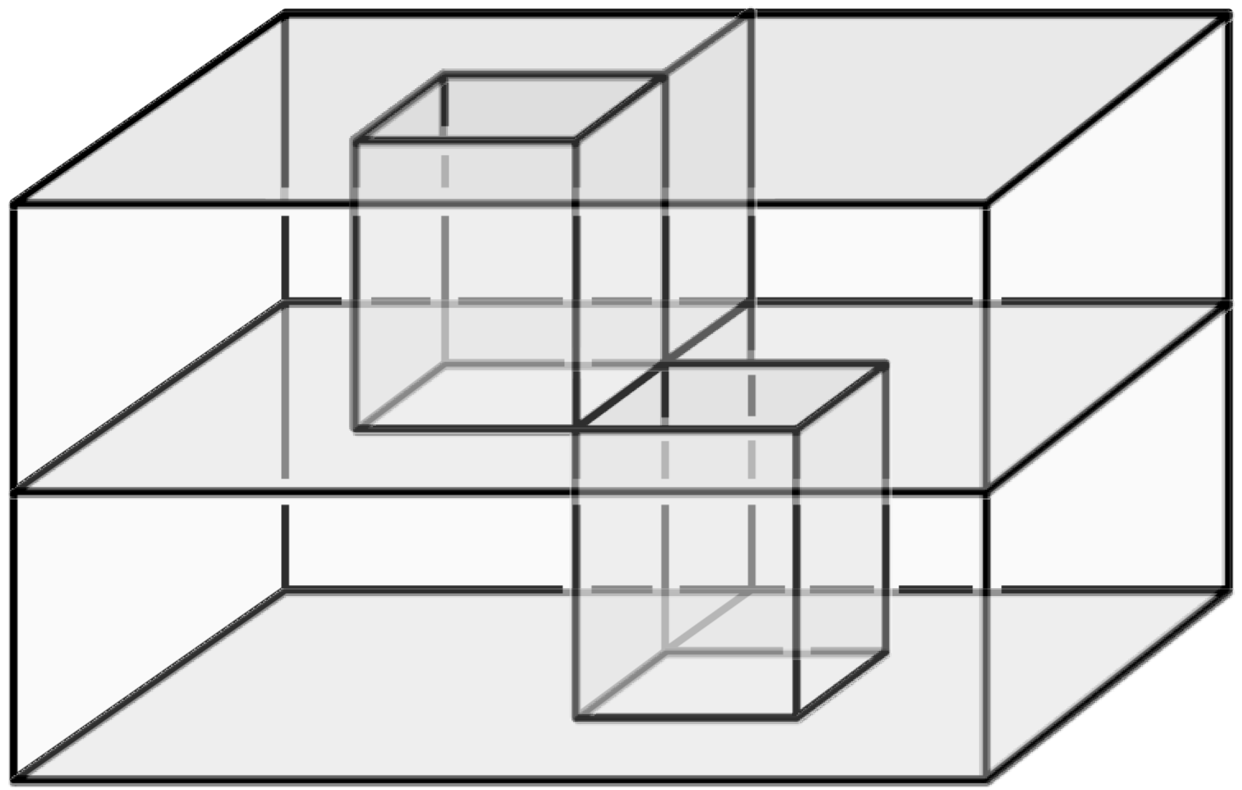
- Complement of Klein bottle

Visualization Exercise

- Complement of Klein bottle

Visualization Exercise

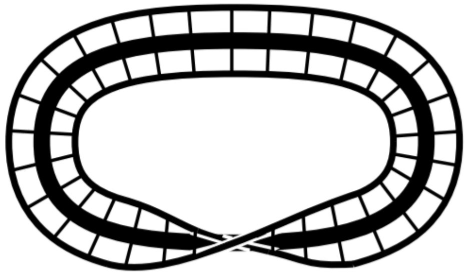
- House with two rooms



Homotopy ≠ Homotopy Equivalence

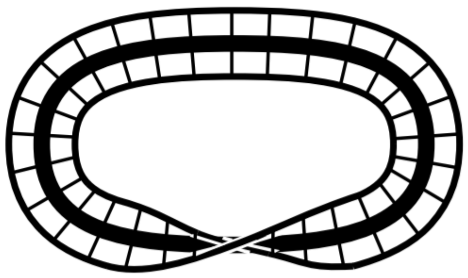
- Homotopy:
  Morph within the same space

- Homotopy Equivalence:
  Morph between identity and maps between two spaces



Where is the Homotopy?

- Retraction
  - $r: X \to A$
  - $r|_A = id_A$
- Inclusion
  - $i: A \to X$
  - $i|_A = id_A$

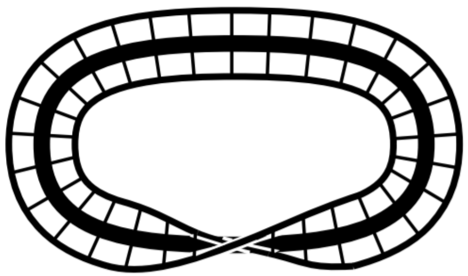


- Deformation retract
  - $f_t: X \to X$
  - $f_1(X) = A$
  - $f_t|_A = id_A$
  - $f_0 = id_X$

= Homotopy from id_X to $r \cdot i$

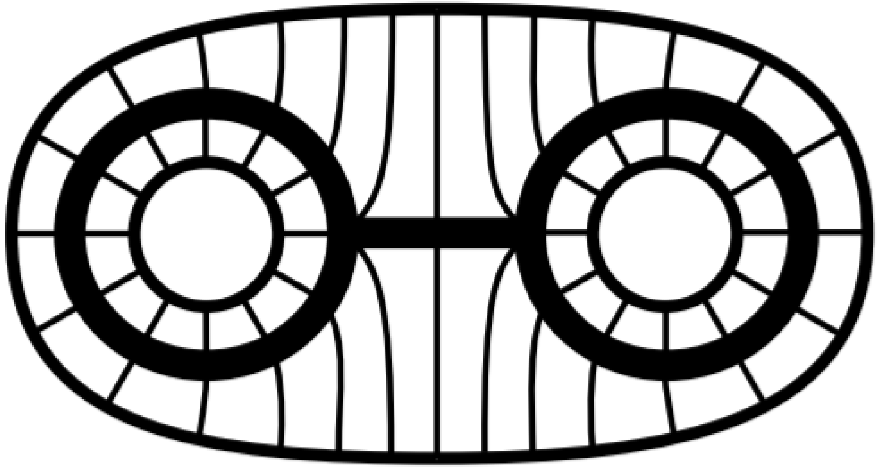
Where is the Homotopy Equivalence?

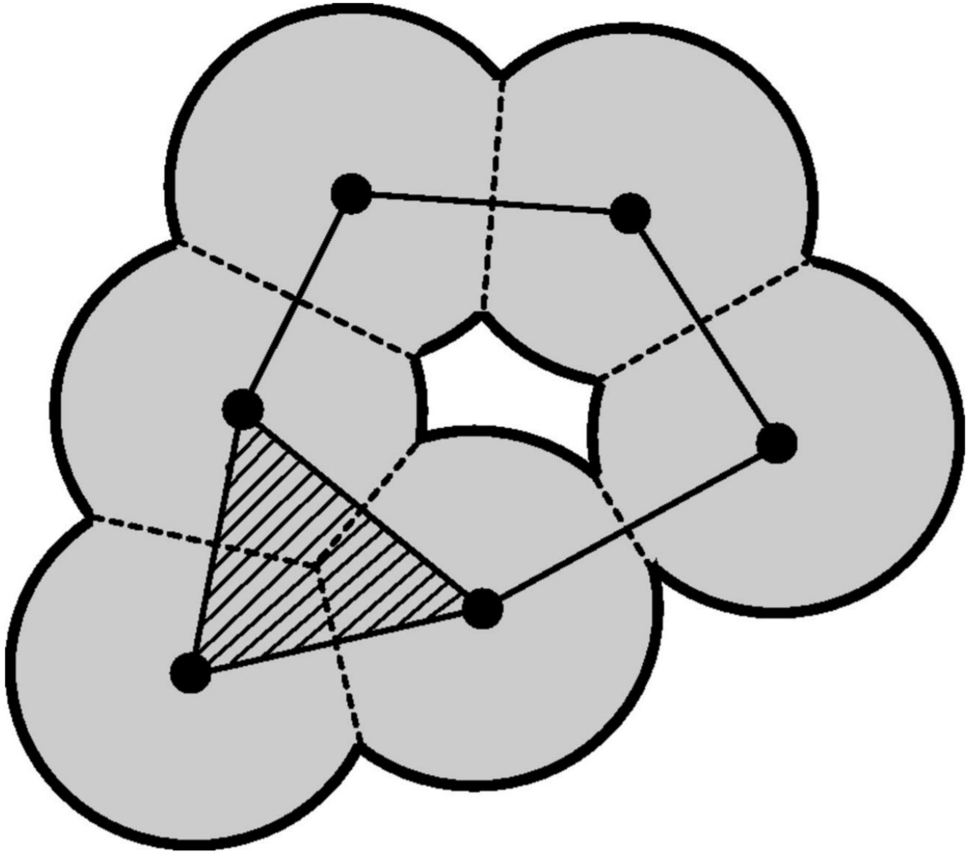
- Retraction
  - $r: X \to A$
  - $r|_A = id_A$
- Inclusion
  - $i: A \to X$
  - $i|_A = id_A$



- $i \bullet r = id_A$
- $r \bullet i$ homotopic to id_X
  - Through deformation retract from X to A
    = homotopy from id_X to $r \bullet i$

Proposition. Deformation retract provides homotopy equivalence between space X and subspace A.





Vietoris-Smale Mapping Theorem

[Vietoris 1927] [Smale 1957]

If f: X ⟶ Y is surjective and proper, and
all preimage $f^{-1}(y)$ is contractible,
then X and Y are homotopically equivalent.

Intermission

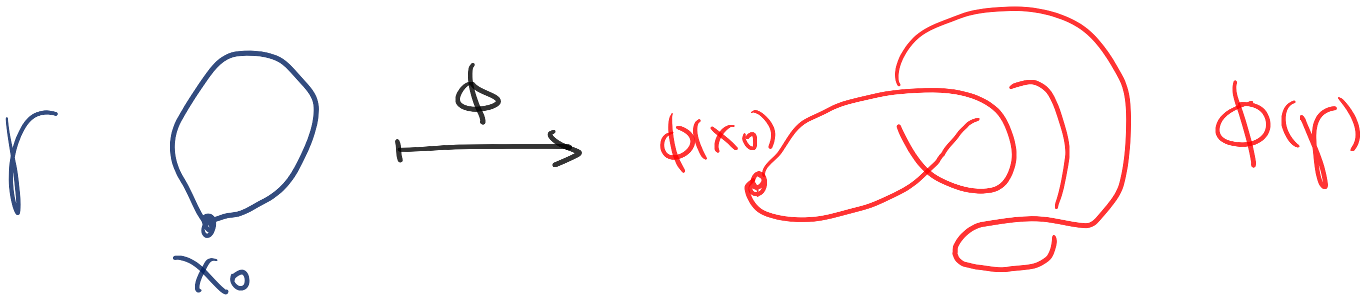
Food for Thought.

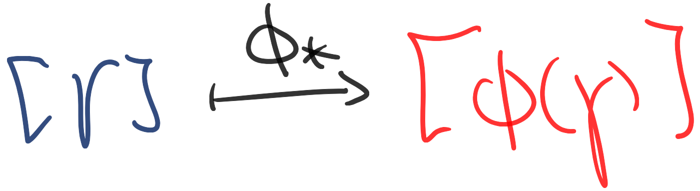
Does trivial π_1 imply contractibility?

Induced Homomorphism

Induced Homomorphism

- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$





Proposition. ϕ_* is a group homomorphism.

Lemma. Retraction from X to A induces an injective inclusion map $i_*: \pi_1(A) \longrightarrow \pi_1(X)$.

$r: M \to S^1$

$i: S^1 \to M$

$r_*: \pi_1(M) \to \pi_1(S^1)$

$i_*: \pi_1(S^1) \to \pi_1(M)$

$i \circ r: S^1 \to S^1 = id_{S^1}$ $\quad (i \circ r)_* = i_* \circ r_*$

$\pi_1(S^1) \xrightarrow{i_*} \pi_1(M) \xrightarrow{r_*} \pi_1(S^1) = (id_{S^1})_*$

$\mathbb{Z} \longleftrightarrow \mathbb{Z} \qquad \mathbb{Z}$

Lemma. Deformation retract from X to A induces an isomorphism $i_*: \pi_1(A) \longrightarrow \pi_1(X)$.

$r: M \to S^1$ $\qquad$ $i: S^1 \to M$

$r_*: \pi_1(M) \to \pi_1(S^1)$ $\qquad$ $i_*: \pi_1(S^1) \longrightarrow \pi_1(M)$

$r \circ i : M \to M \overset{\text{homotopic to}}{\simeq} id_M$ $\qquad$ $(r \circ i)_* = r_* \circ i_*$

$$\pi_1(M) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(M) = (id_M)_*$$

$$\mathbb{Z} \longleftrightarrow \mathbb{Z} \qquad \mathbb{Z}$$

Theorem. Homotopy equivalence induces group isomorphism on π_1.

2D Brouwer Fixed-point Theorem

[Bohl 1904] [Brouwer 1909]

Every map from a disk to itself has a fixed point

Proof of 2D Brouwer Fixed-point Theorem.

Assume for contradiction. $h: D \to D$

$h(x) \neq x \quad \forall x \in D$.

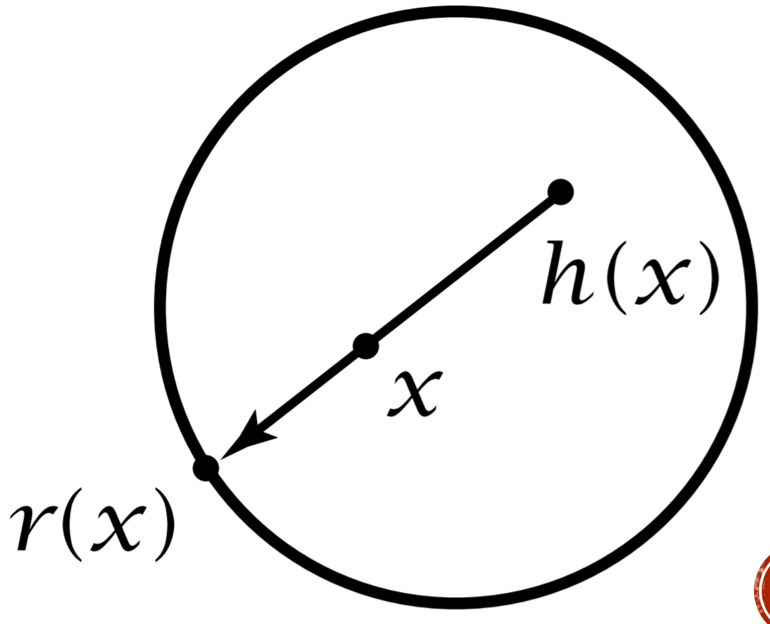
Define $r: D \to S^1$ $\quad i: S^1 \to D$

By lemma before, i_* is injective

(because r is a retraction)

$i_*: \pi_1(S^1) \to \pi_1(D)$

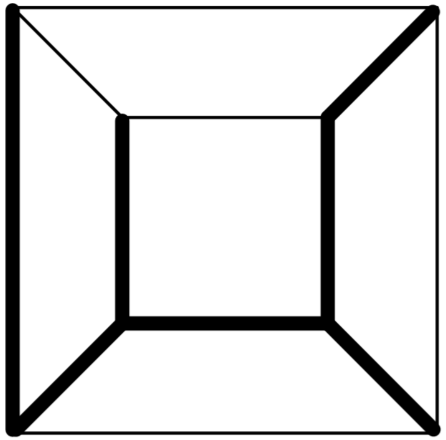
$\mathbb{Z} \hookrightarrow \emptyset$?? ✗

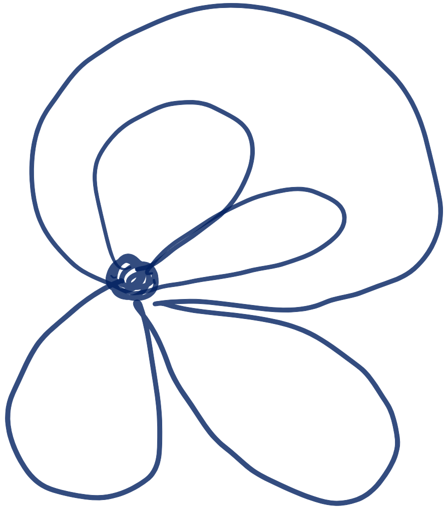


Computing Fundamental Groups

Can we compute $\pi_1(\Sigma(g,r))$?

π_1(Graph)





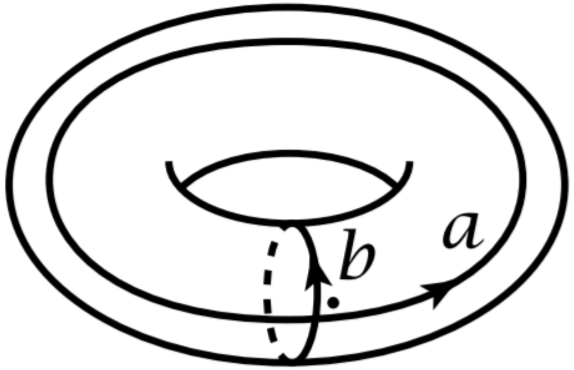
$$S^1 \vee S^1 \vee \dots \vee S^1$$

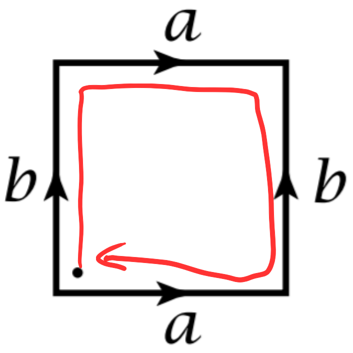
$$= \bigvee S^1$$

$$\pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{5 \text{ copies}}) = \langle a_1, a_2, \dots, a_5 \rangle$$

$$=: \mathbb{Z} * \dots * \mathbb{Z}$$

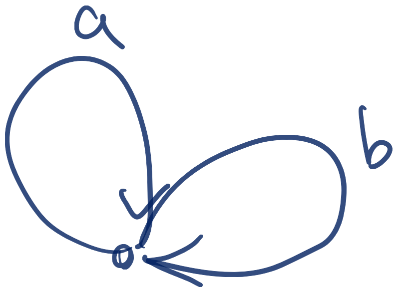
π_1(Torus)





$bab^{-1}a^{-1} = 1$

$ba = ab$

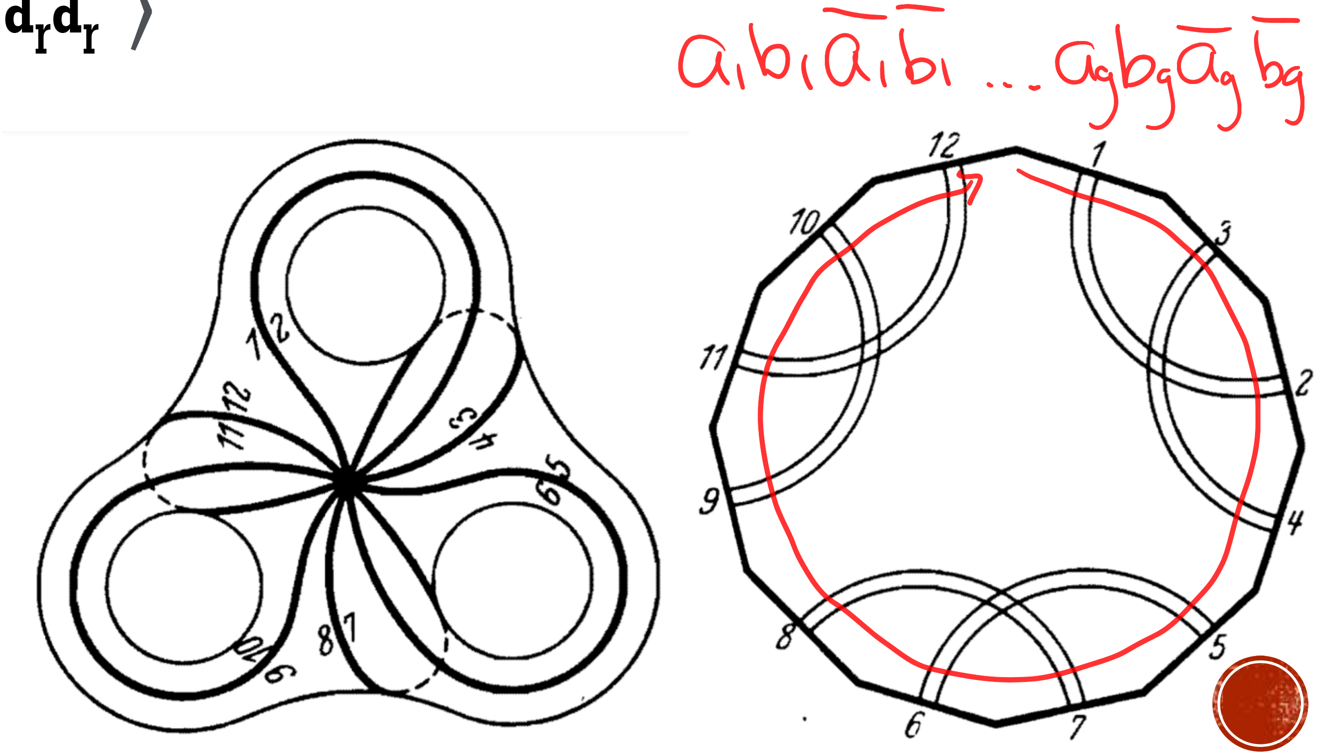


$$\pi_1(T) := \langle a, b \mid bab^{-1}a^{-1} \rangle$$

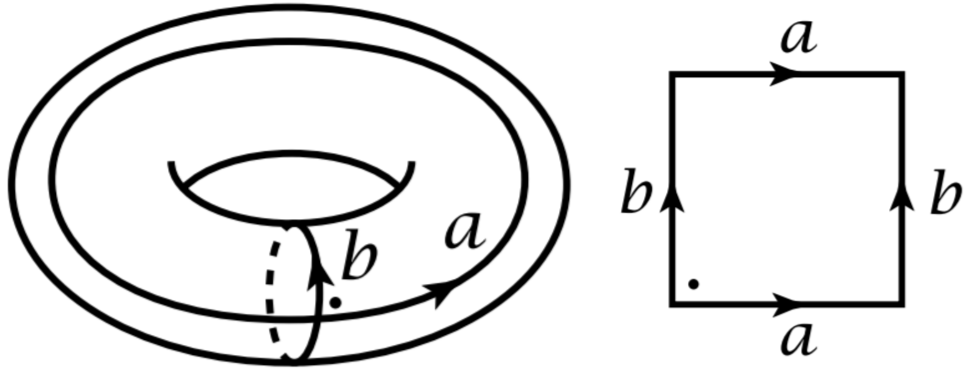
$$= \mathbb{Z} \times \mathbb{Z}$$

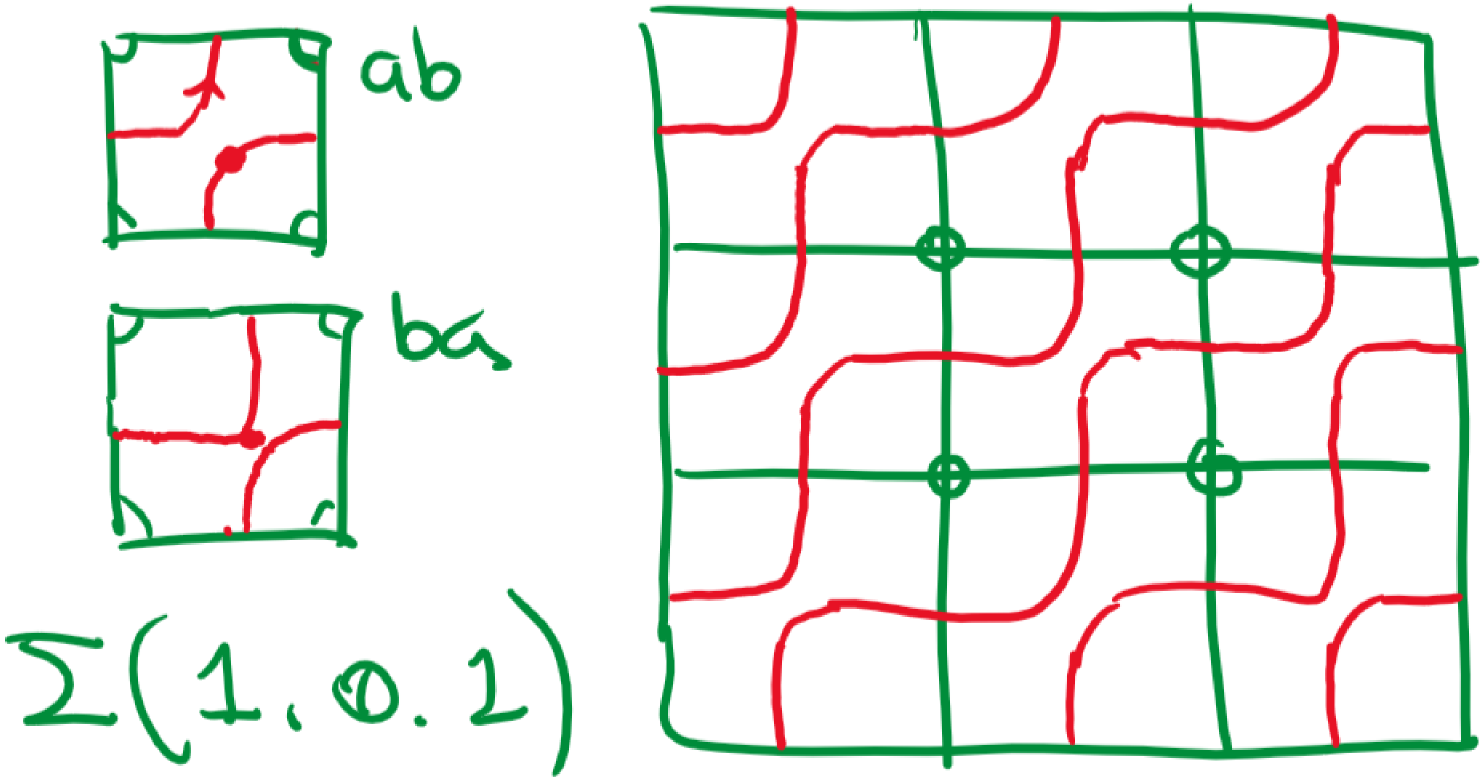
Fundamental Groups of Surfaces

- $\pi_1(\Sigma(g,0)) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$
- $\pi_1(\Sigma(0,r)) = \langle a_1, \ldots, a_r \mid a_1 a_1 \ldots a_r a_r \rangle$



What about Punctures?

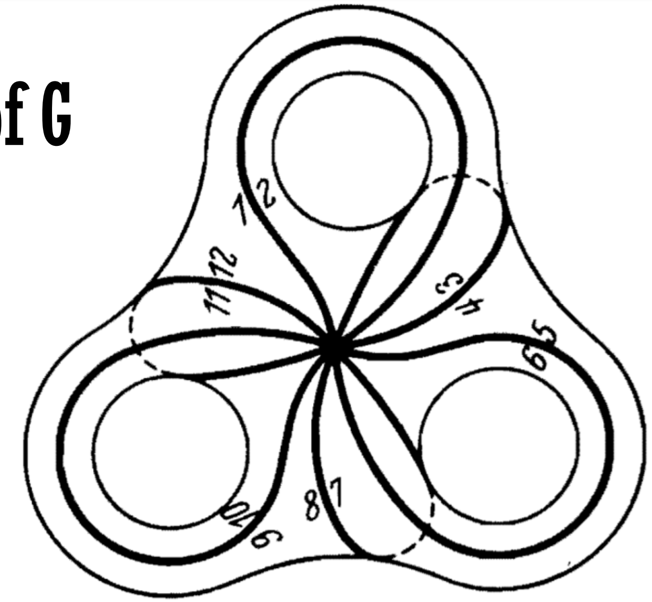


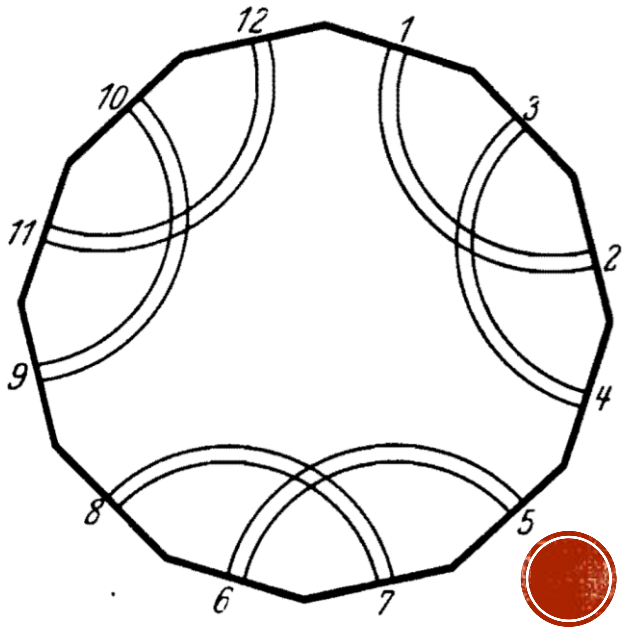


Fundamental Groups of 2-Complex

- $\pi_1(\Sigma) = \langle C \mid F \rangle$
  - C: cotree edges
  - F: faces

- $\pi_1(\Sigma)$ is independent to the choice of G





$$
\left\langle \begin{array}{l} a,b,c,d,e,p,q,r,t,k \quad \Big| \\
\begin{array}{lll}
p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 &
\end{array} \end{array} \right\rangle
$$

[Collins 1986]

Undecidability of π_1 [Novikov 1955] [Boone 1958]

Checking if a 2-complex has trivial π_1 is undecidable

$\pi_1(X)$ is Homotopic Invariant but Useless for Computation

Choose your own adventure:

more (A)lgorithms on curve homotopy, or
something (B)etter than fundamental groups