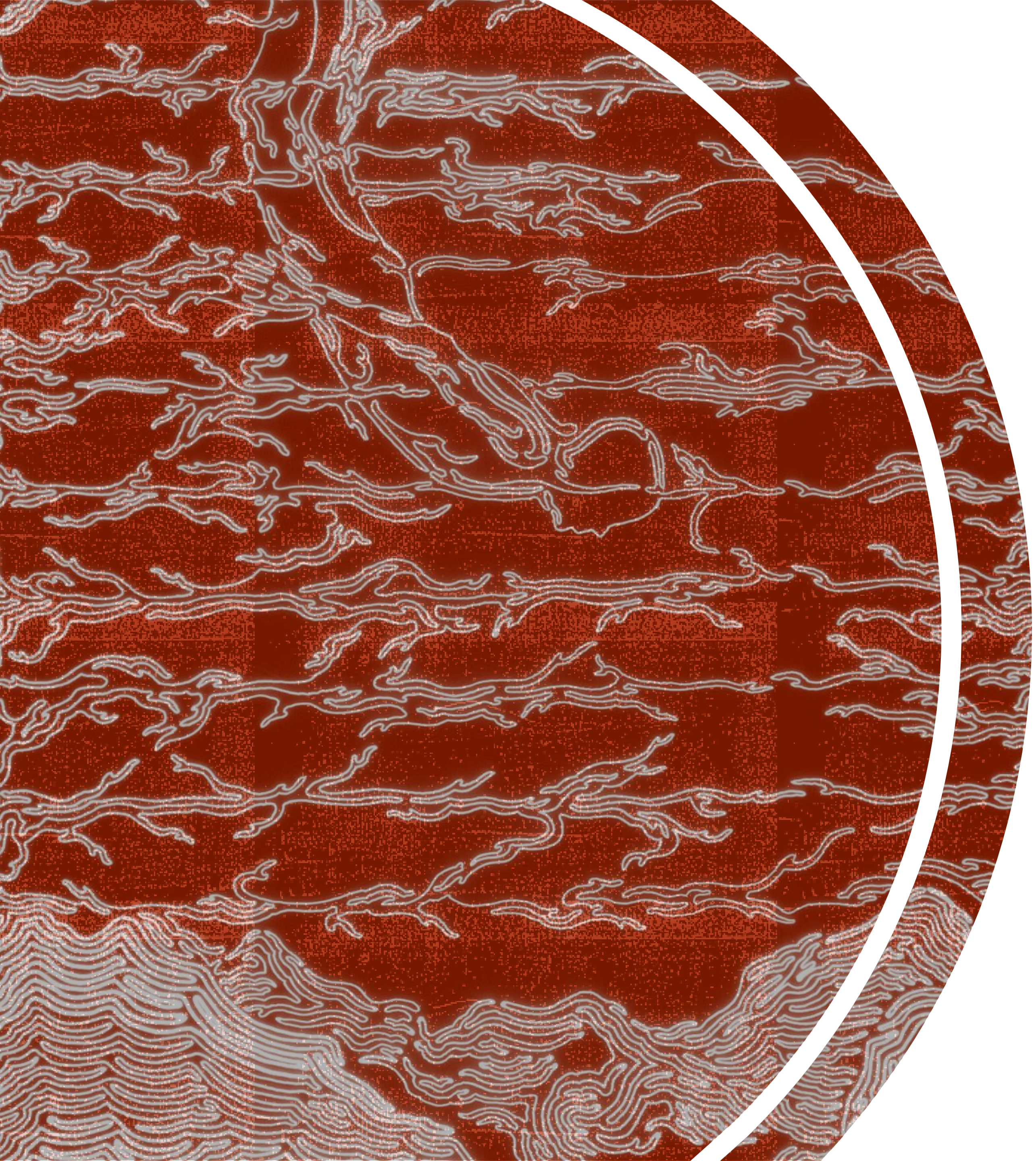


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 7, October 5, 2021

Administrivia

- Homework a will be out later today.

HOMOTOPY

- Homotopy of closed curves
  - H: $S^1 \times [0,1] \longrightarrow R^2$

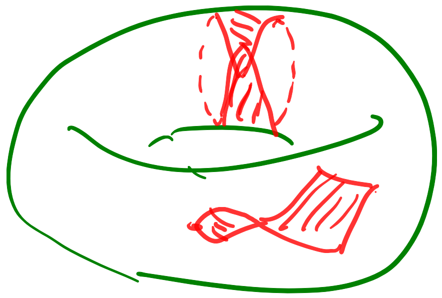
$H(\cdot, 0) = \gamma_1$ $\quad$ $H(\cdot, 1) = \gamma_2$

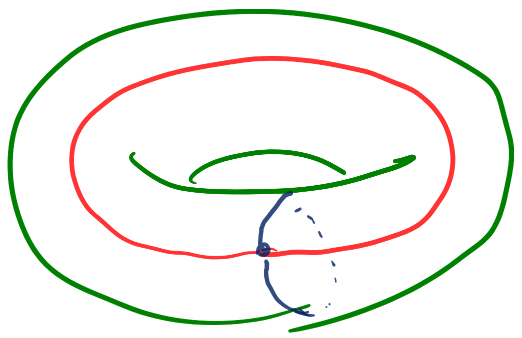
- Homotopy of two functions f and g from X to Y
  - H: $X \times [0,1] \longrightarrow Y$

Torus

$\gamma_1 . \gamma_2 : X \longrightarrow Y$

[0,1]×[0,1]





Are Two Curves Homotopic?

Homotopy Testing

- Cut surface into polygonal schema
- Keep track of how the curve crosses the cuts

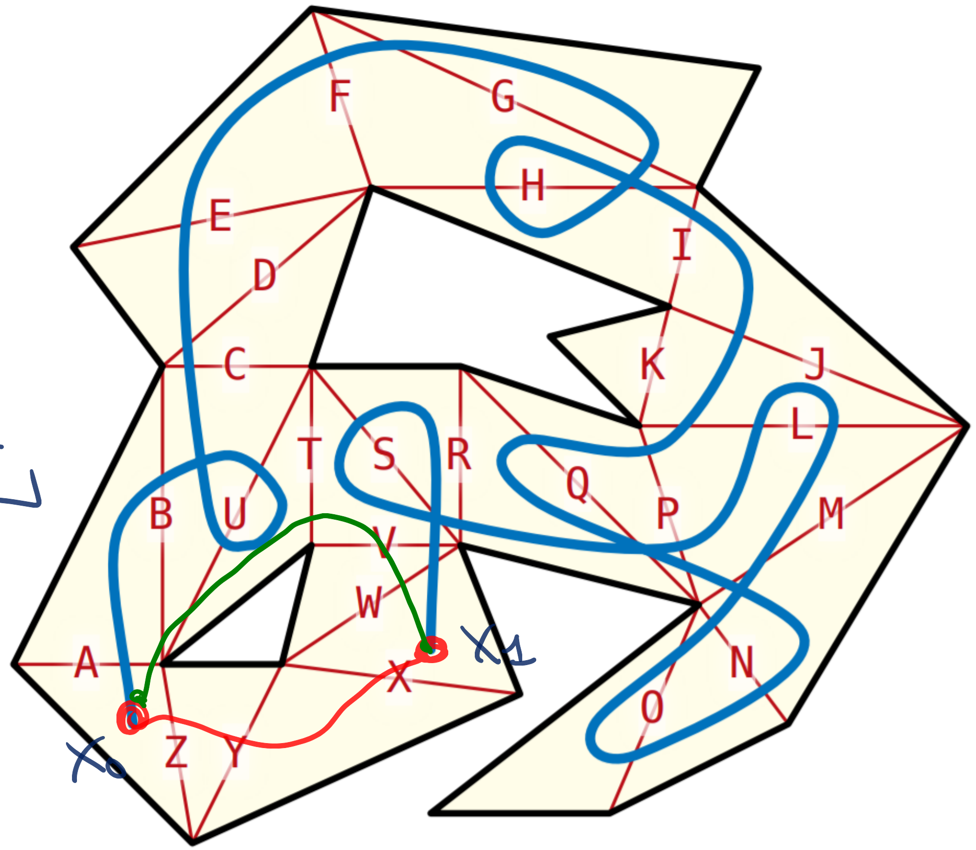
Path

Homotopy between paths $p_1 . p_2 : [0.1] \to \Sigma$

$H : [0.1] \times [0.1] \to \Sigma$

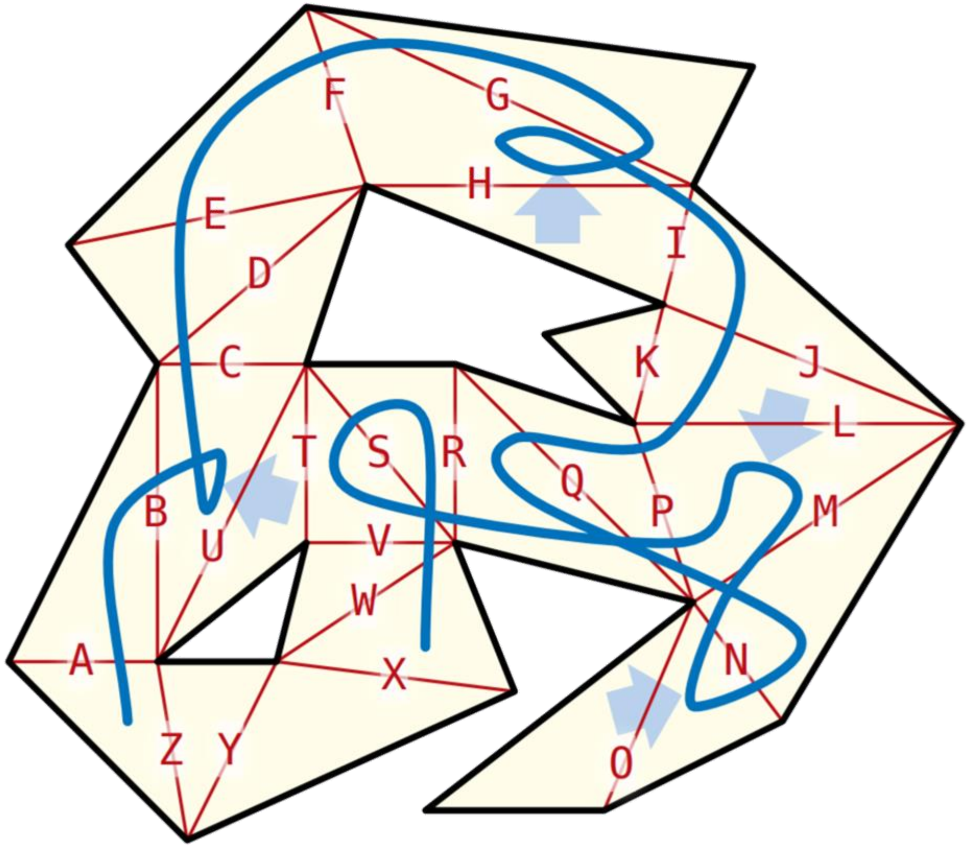
$H(\cdot, 0) = p_1 \quad H(\cdot, 1) = p_2$

$H(0, t) = X_0 \, , \; H(1, t) = X_1$



Homotopy Testing

- Cut surface into polygonal schema
- Keep track of how the curve crosses the cuts
- Reduce the crossing sequence



Lemma. Every crossing sequence reduces uniquely.

Pf. Claim. $\exists \hat{w}$ s.t.

$$\begin{tikzcd} & x \arrow[dr] & \\ w \arrow[ur] \arrow[dr] & & \hat{w} \\ & y \arrow[ur] & \end{tikzcd}$$

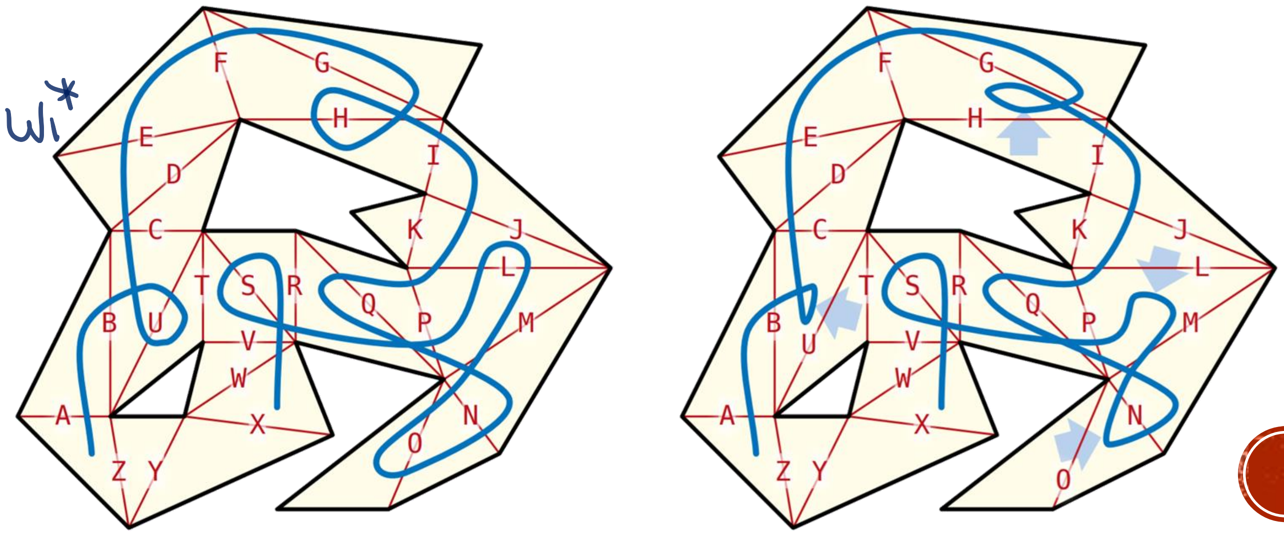
- $w = w_1 \circ x^+ x^- \circ w_2 \circ y^+ y^- \circ w_3$ $\qquad \hat{w} = w_1 \circ w_2 \circ w_3$
- $w = w_1 \circ x^+ x^- x^+ \circ w_3$ $\qquad \hat{w} = w_1 \circ x^+ \circ w_3$

$w_2^* \leftsquigarrow w \rightsquigarrow w_1^*$

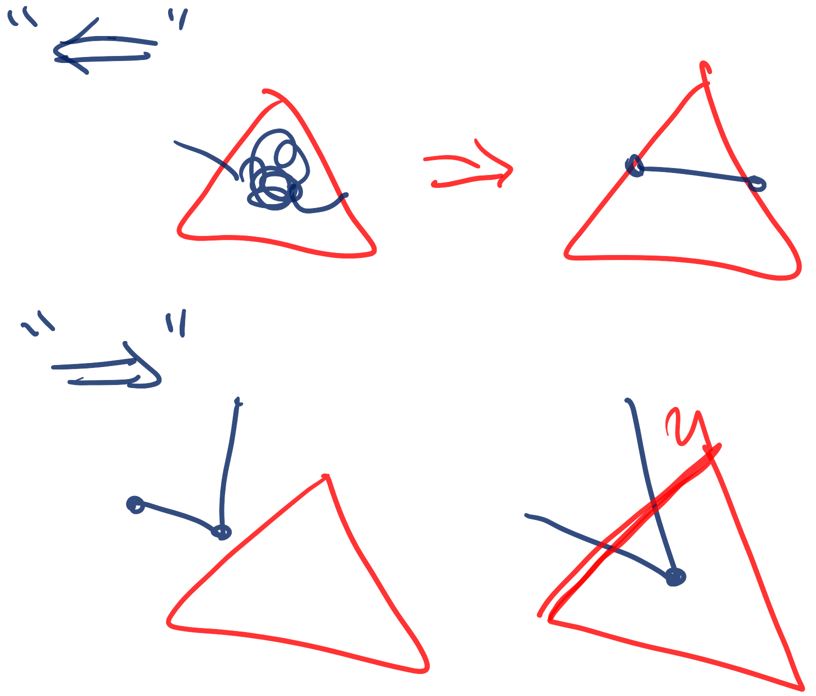
$w_2^* \leftarrow z_1 \leftarrow z_2 \cdots \leftarrow \overset{z_i}{\overset{\parallel}{w}} \rightarrow \cdots \rightarrow z_n \rightarrow w_1^*$

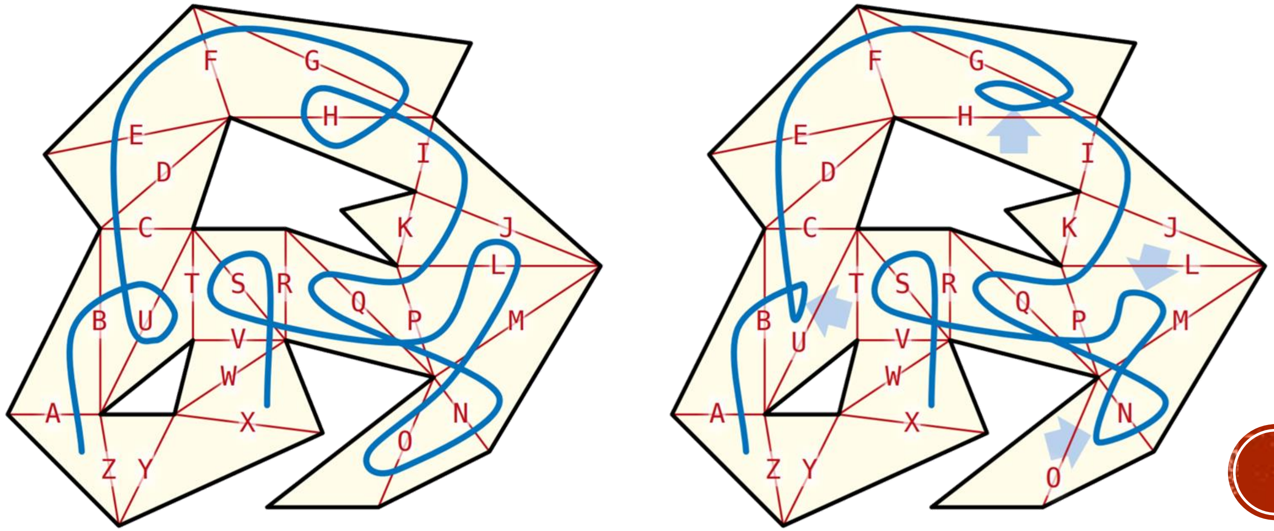
$\left.\begin{array}{l} z_{i-1} \leftarrow z_i \rightarrow z_{i+1} \\ z_{i-1} \rightarrow \hat{z}_i \leftarrow z_{i+1} \end{array}\right\}$ by Claim.

$w_2^* \rightarrow \hat{z}_1 \rightarrow \cdots \rightarrow \hat{z}_i \leftarrow \cdots \leftarrow w_1^*$

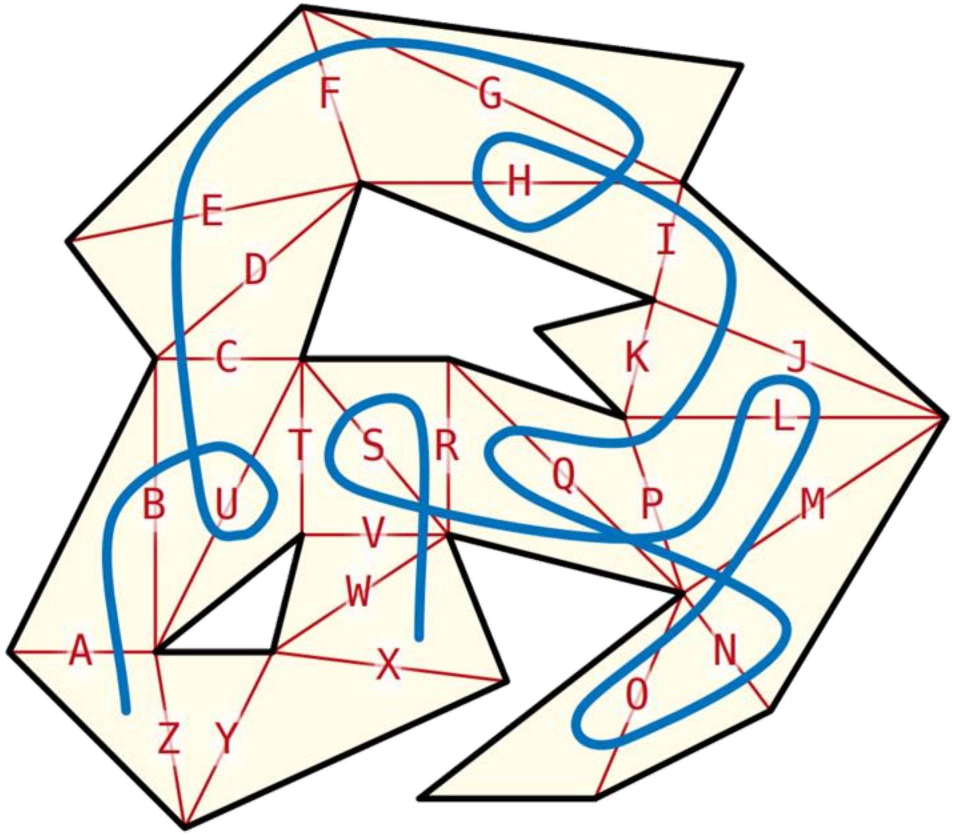


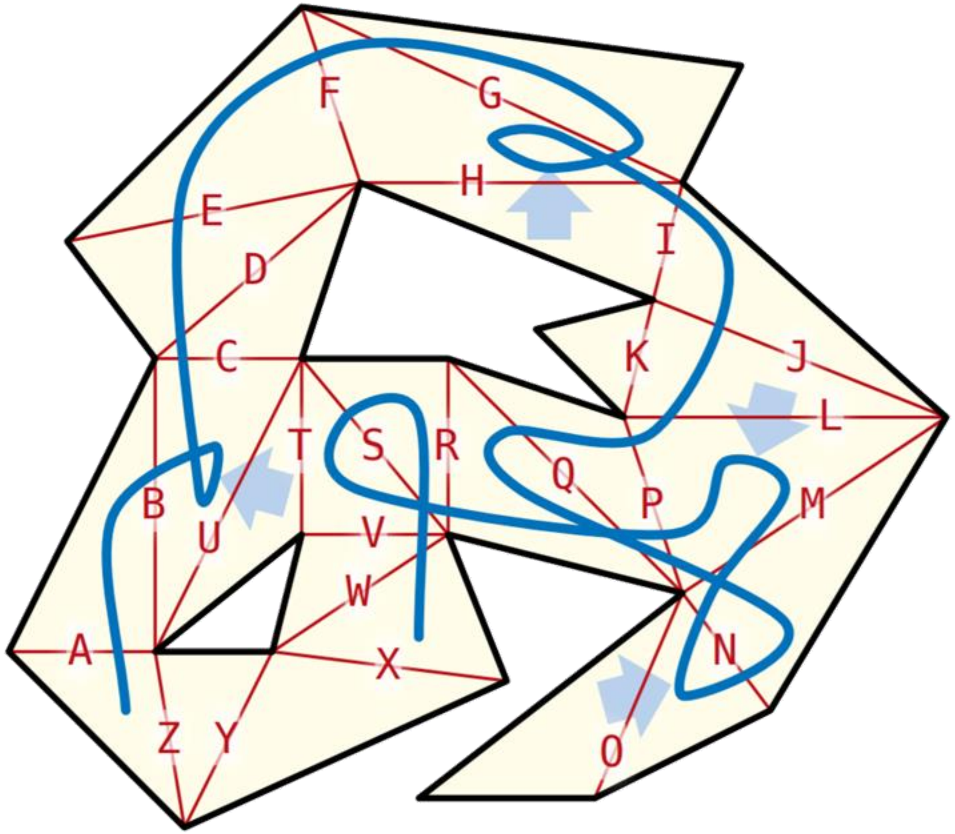
Proposition. Two curves are homotopic if and only if they share the same reduced crossing sequence.





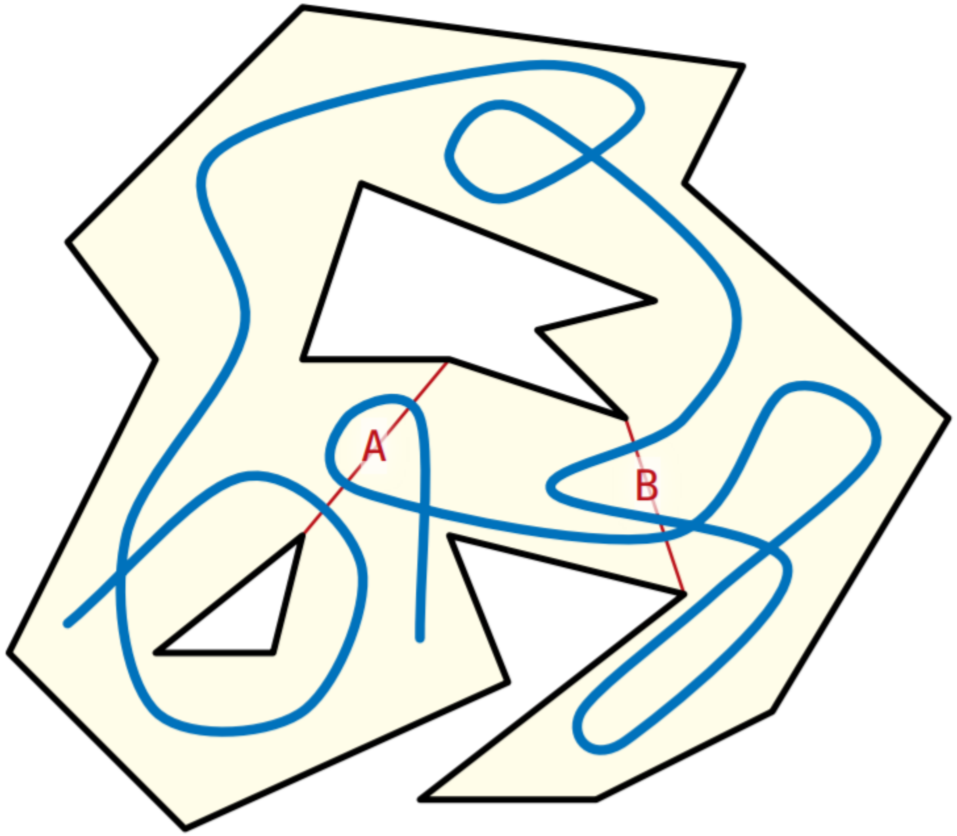
Theorem. Homotopy testing between two k-edge planar polygonal curves takes O(n log n + nk) time.





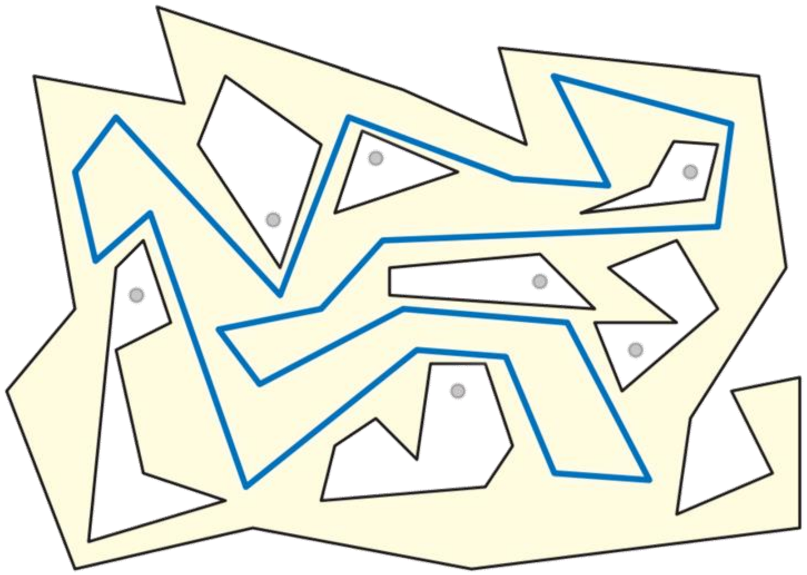
Observations

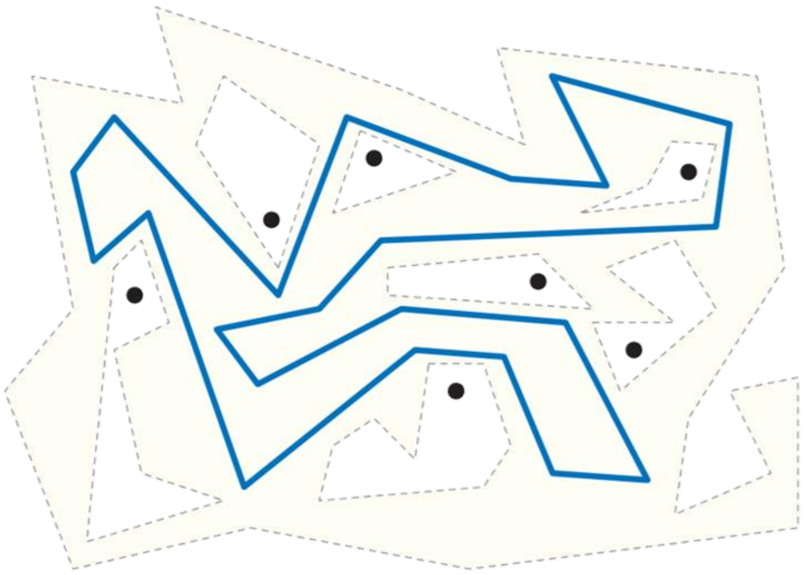
- System of loops are enough

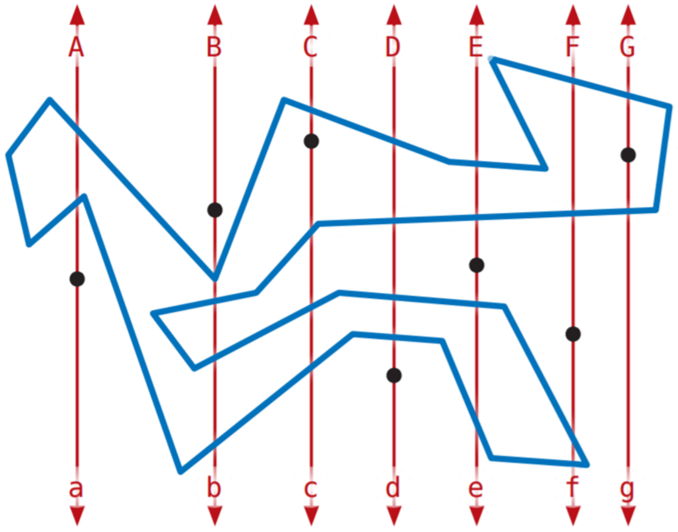


Observations

- Triangulation doesn't matter; replace it with punctures

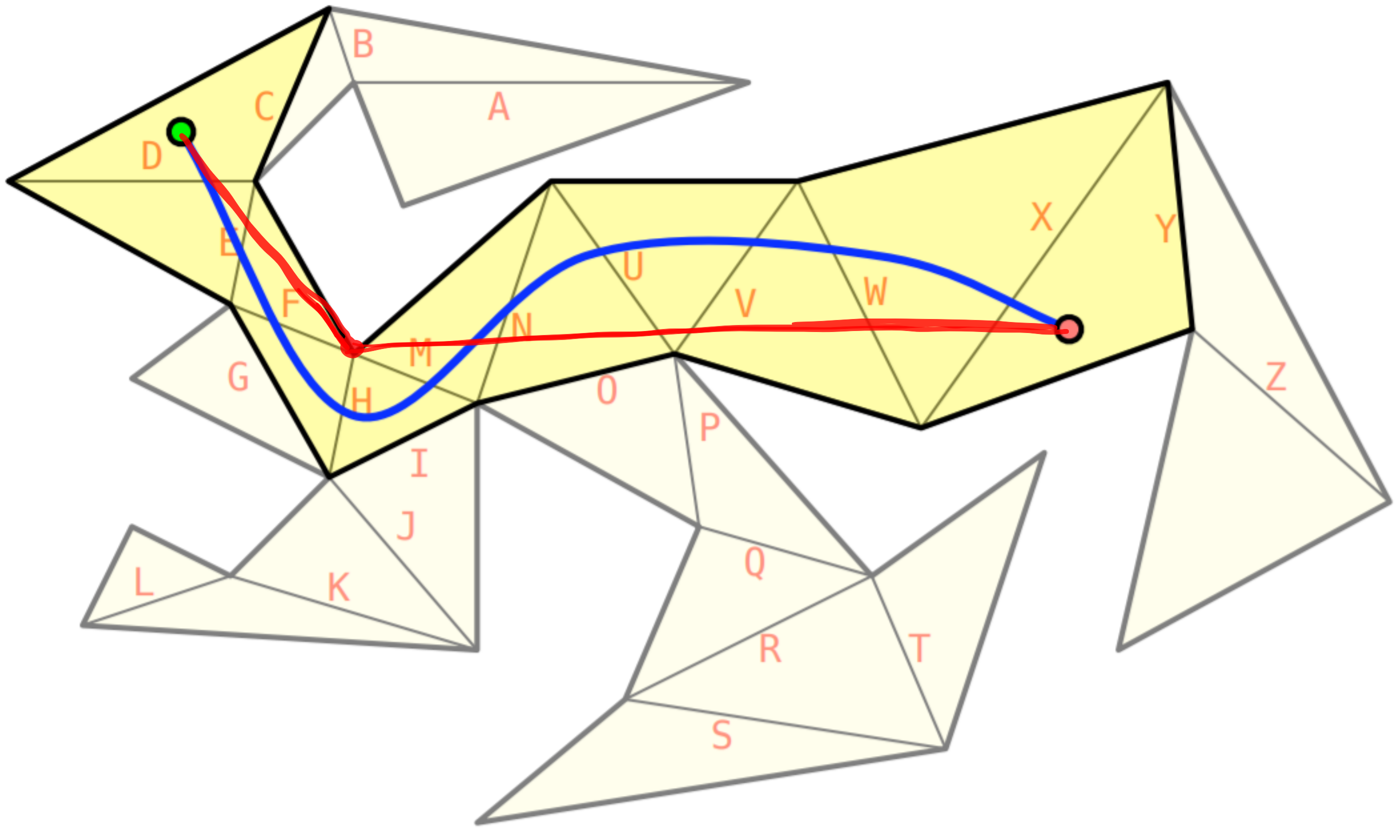






A partition of $\mathbb{R}^2 \setminus S$ into vertical slabs and a loop with crossing sequence AbcDeffeDcbbcDEFgGFEDCbA.

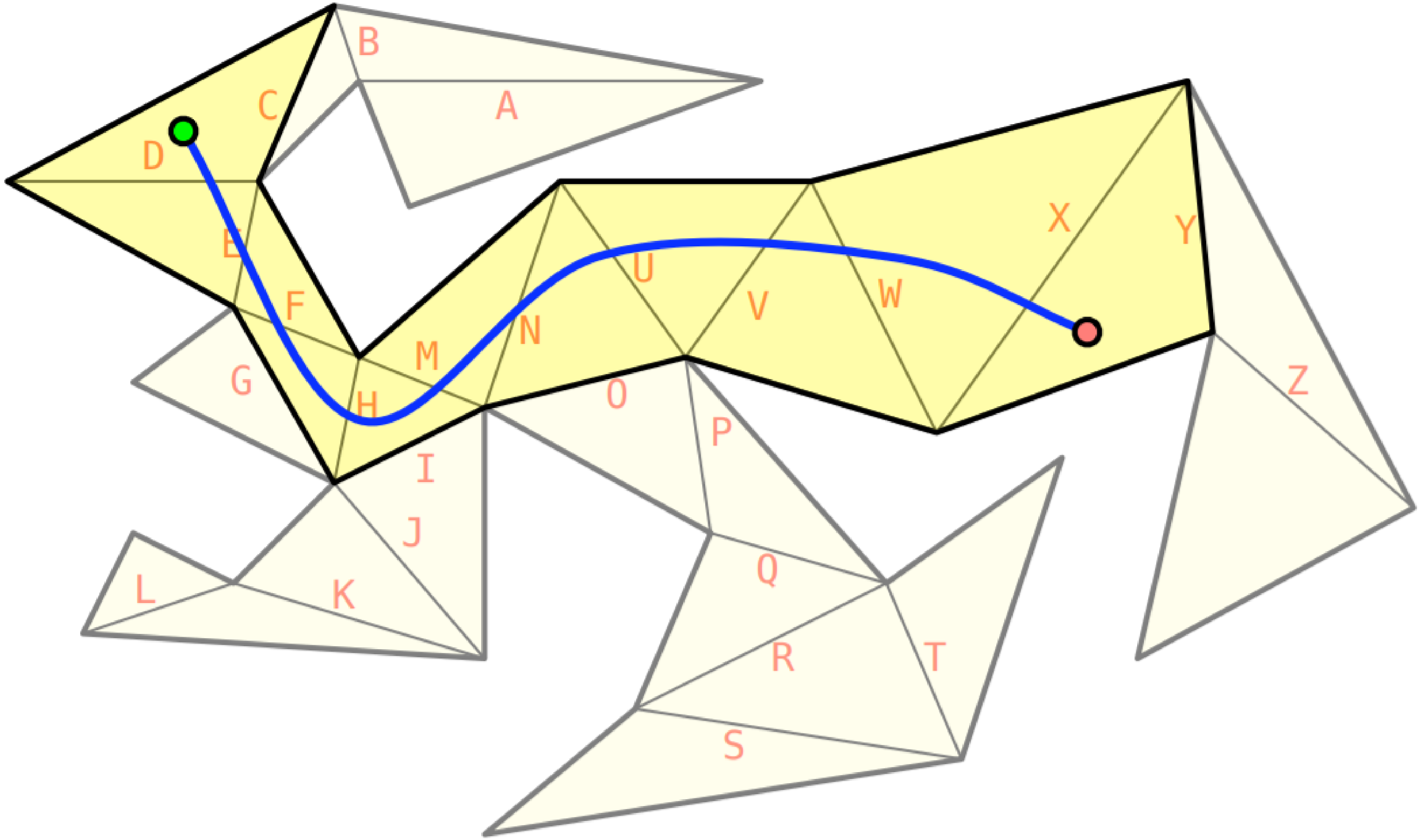
Shortest Homotopic Path?



Funnel Algorithm

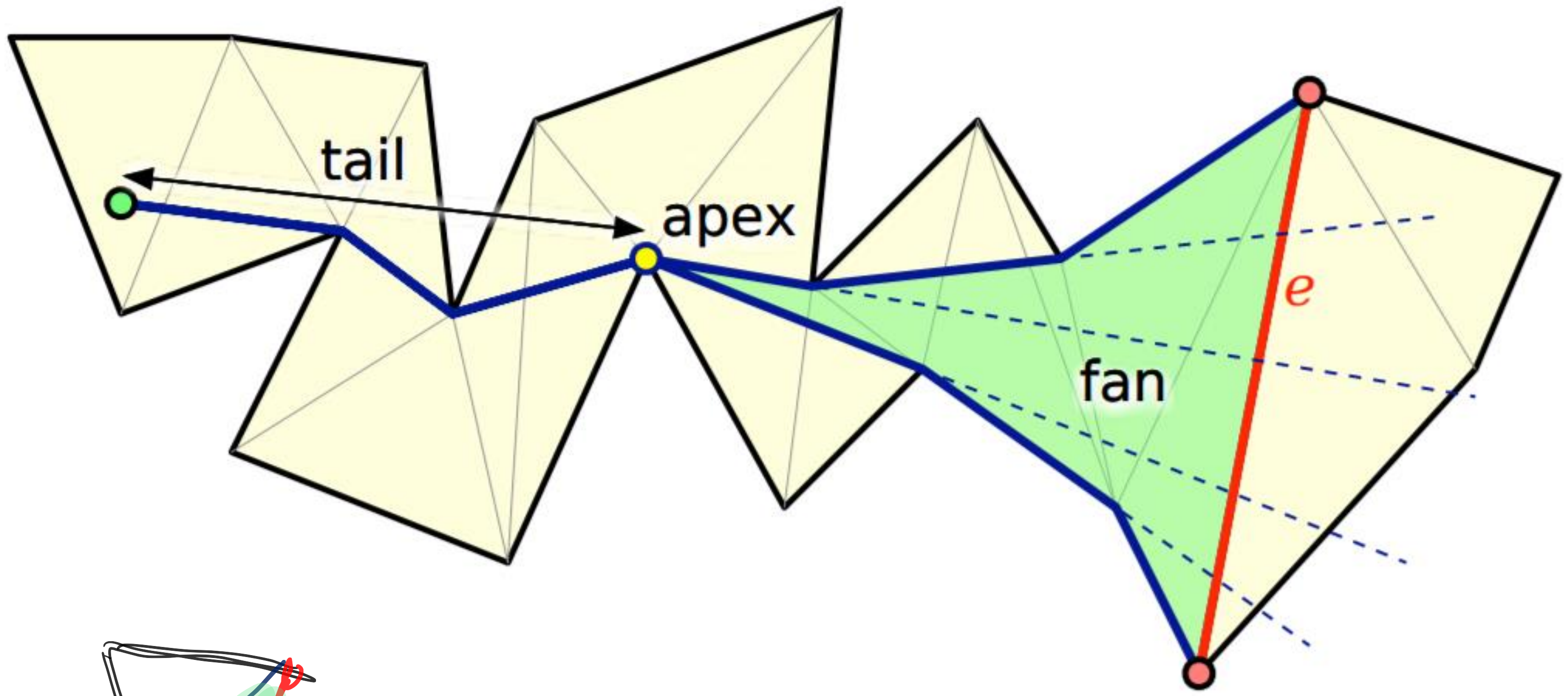
[Tompa 1981] [Chazelle 1982] [Lee-Preparata 1984]

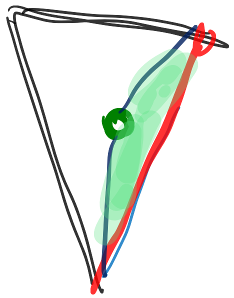
Given a k-edge path π in a simple polygon, find the shortest path homotopic to π takes $O(nk)$ time

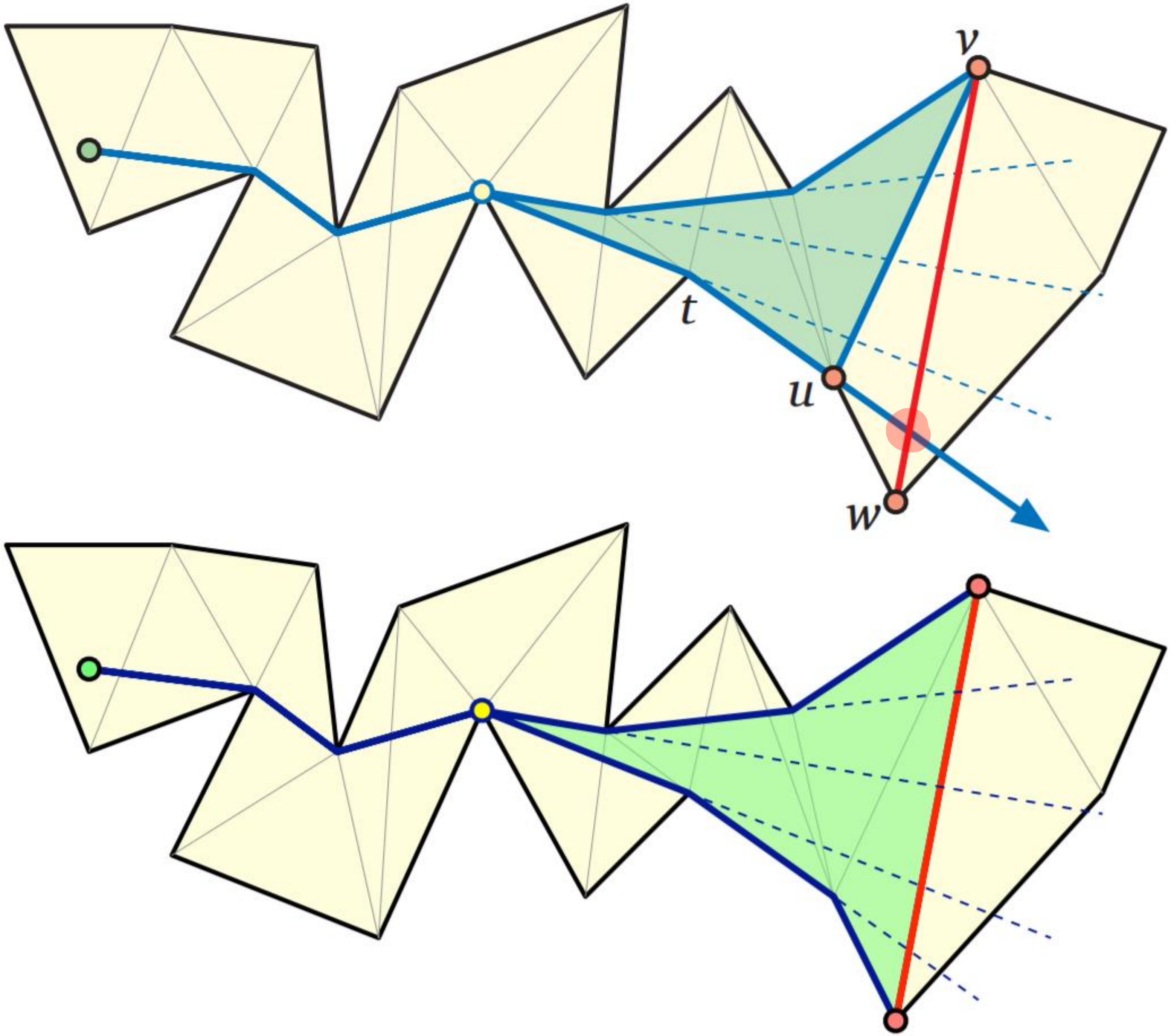


Sleeve

Funnel

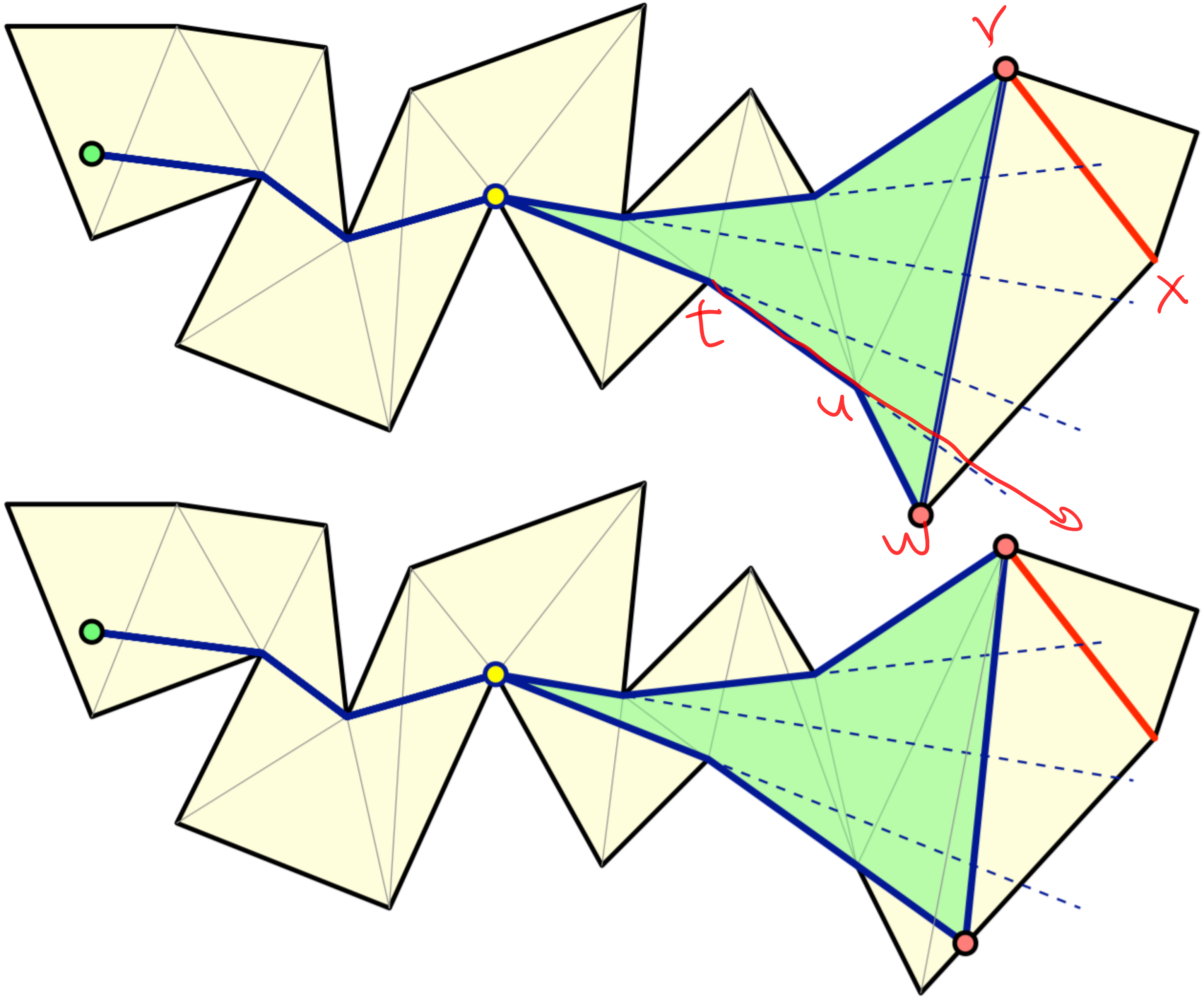






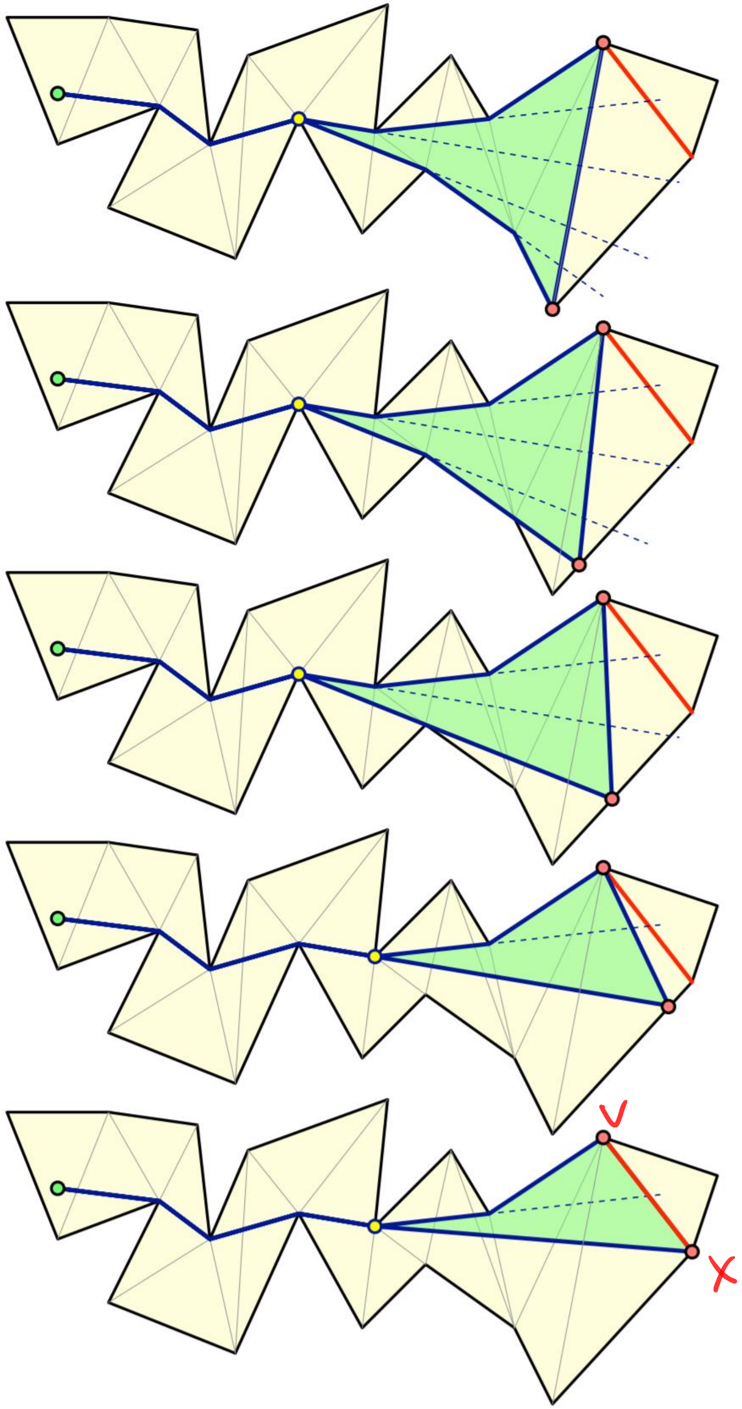
Extending Funnel

$\overrightarrow{tu} \cap \overline{vw} \neq \phi$



Narrowing Funnel

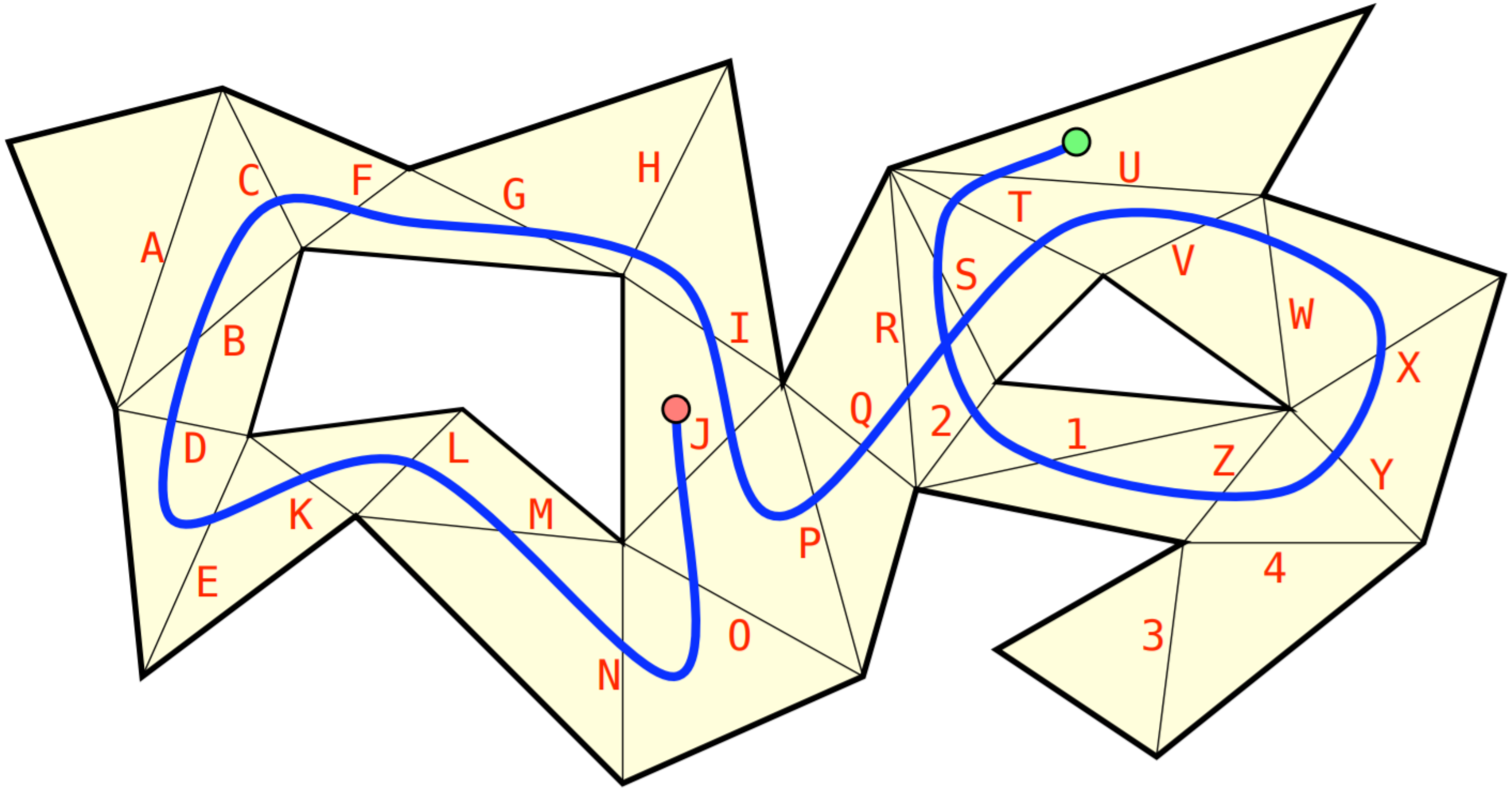
$\overrightarrow{tu} \cap \overline{vx} = \phi$



Contracting Funnel

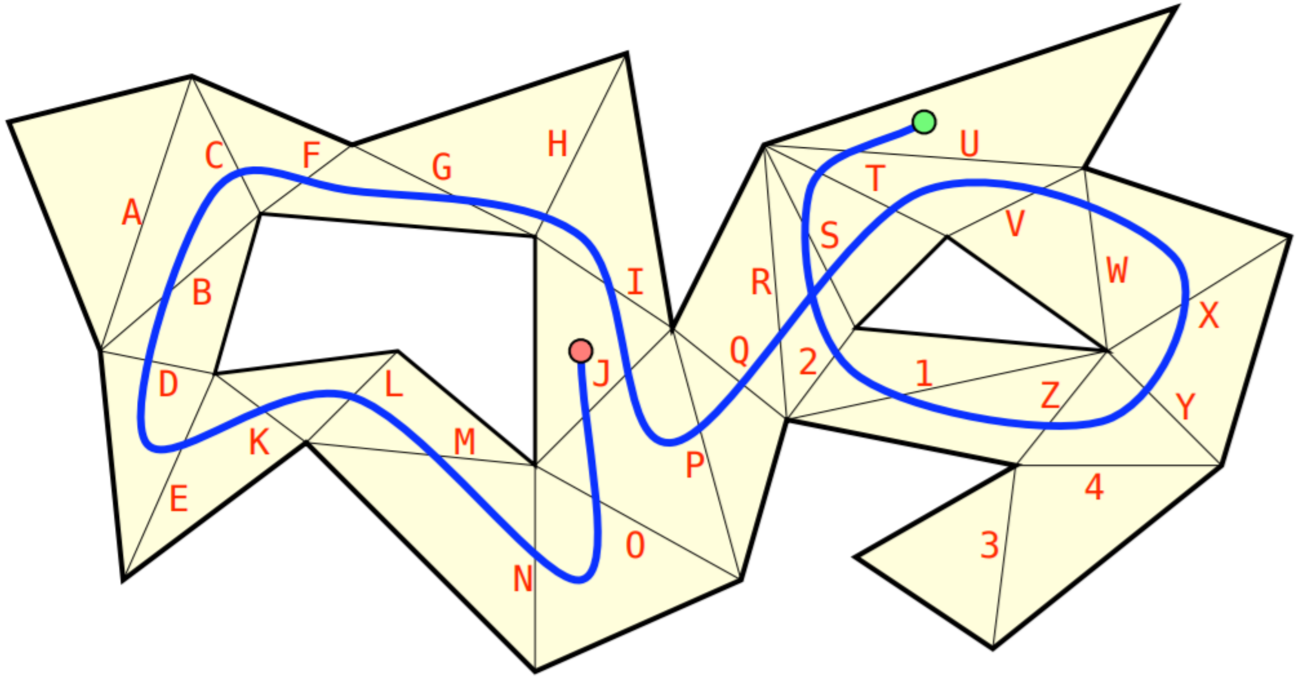
O(1) amortized time per deletion,

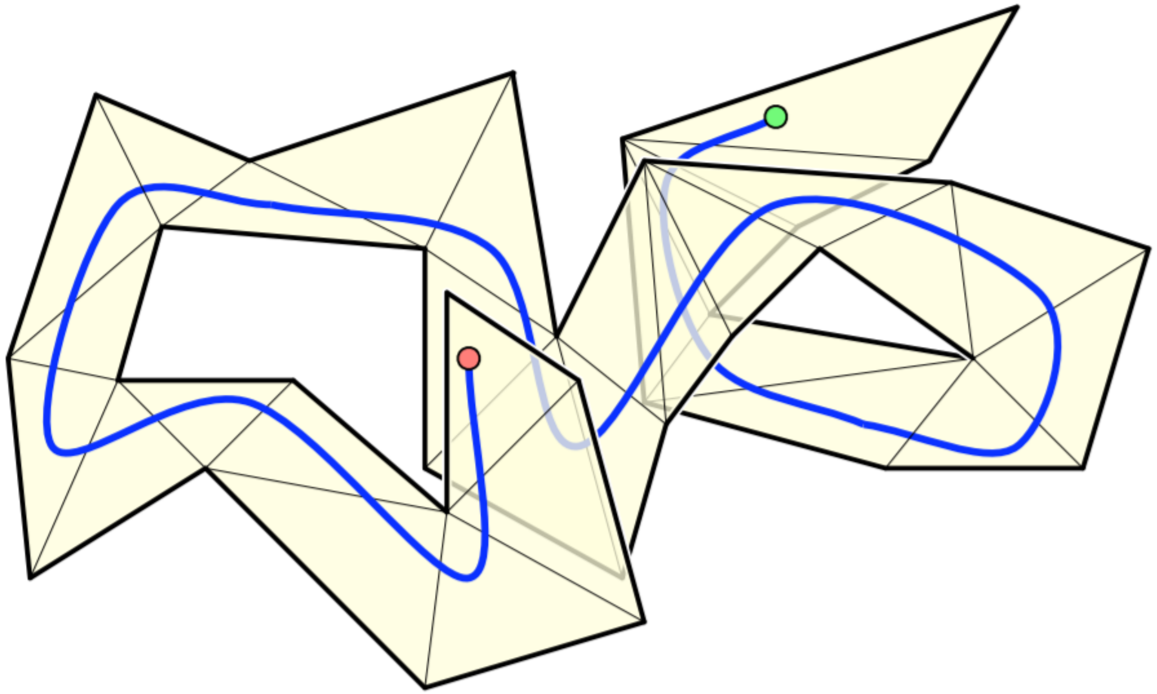
⇒ O(n·k) time algorithm,



Funnel Algorithm [Leiserson-Maley 1985] [Hershberger-Snoeyink 1994]

Given a k-edge path π in a polygon with obstacles, find the shortest path homotopic to π takes O(nk) time





Without Modification

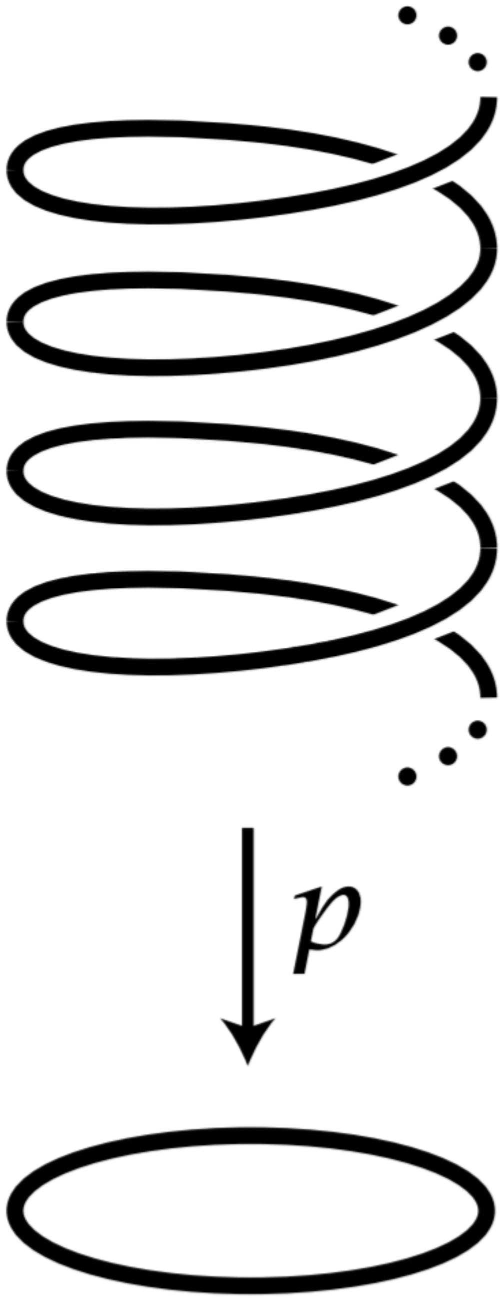
Intermission

Food for Thought.

Can the "lifted space" have non-trivial topology?

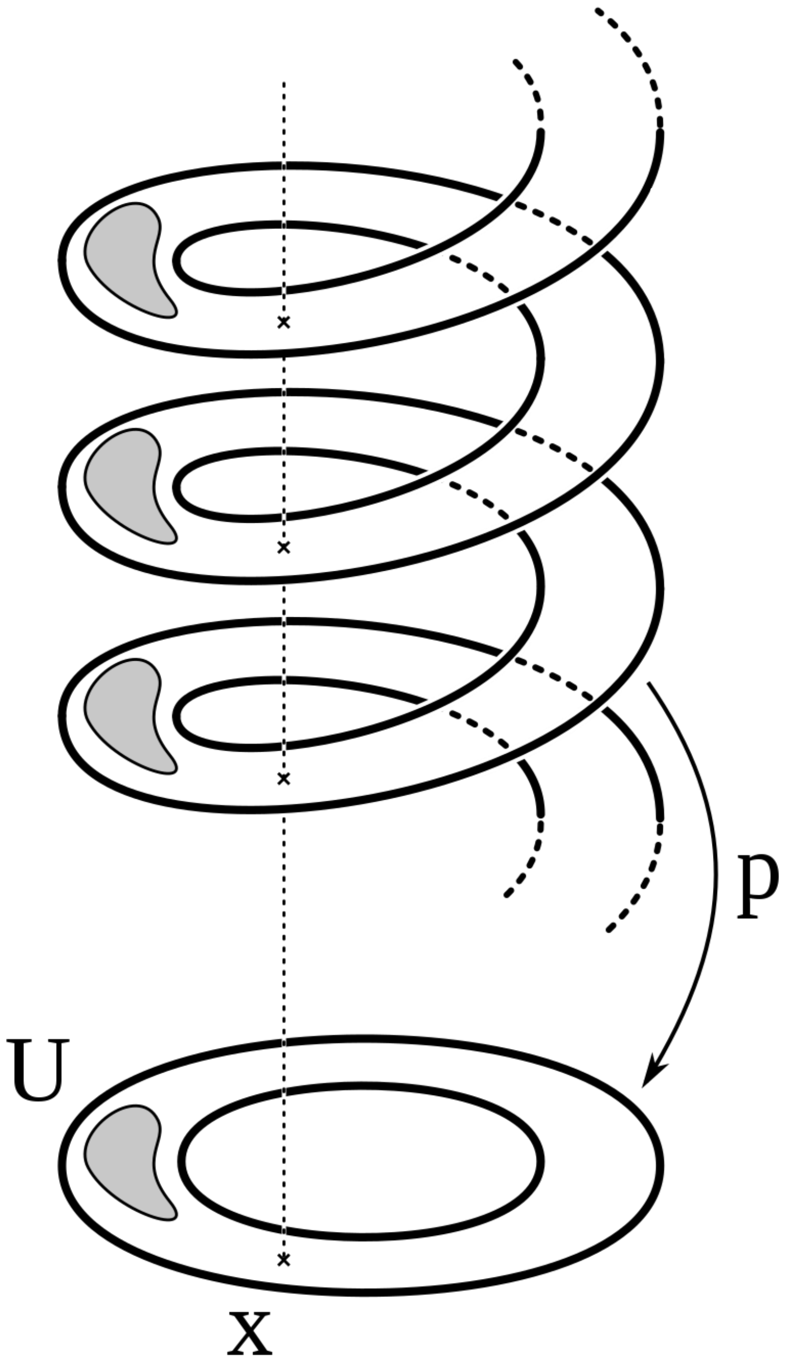
Covering Space and Fundamental Group

VISUALIZING WINDING NUM.



Covering Space

- Space Z' and local homeomorphism p: Z' $\rightarrow$ Z
  - For every point x in Z, there's an open disk U_x such that $p^{-1}(U_x)$ is a union of disjoint open disks, each maps homeomorphically unto U_x by p



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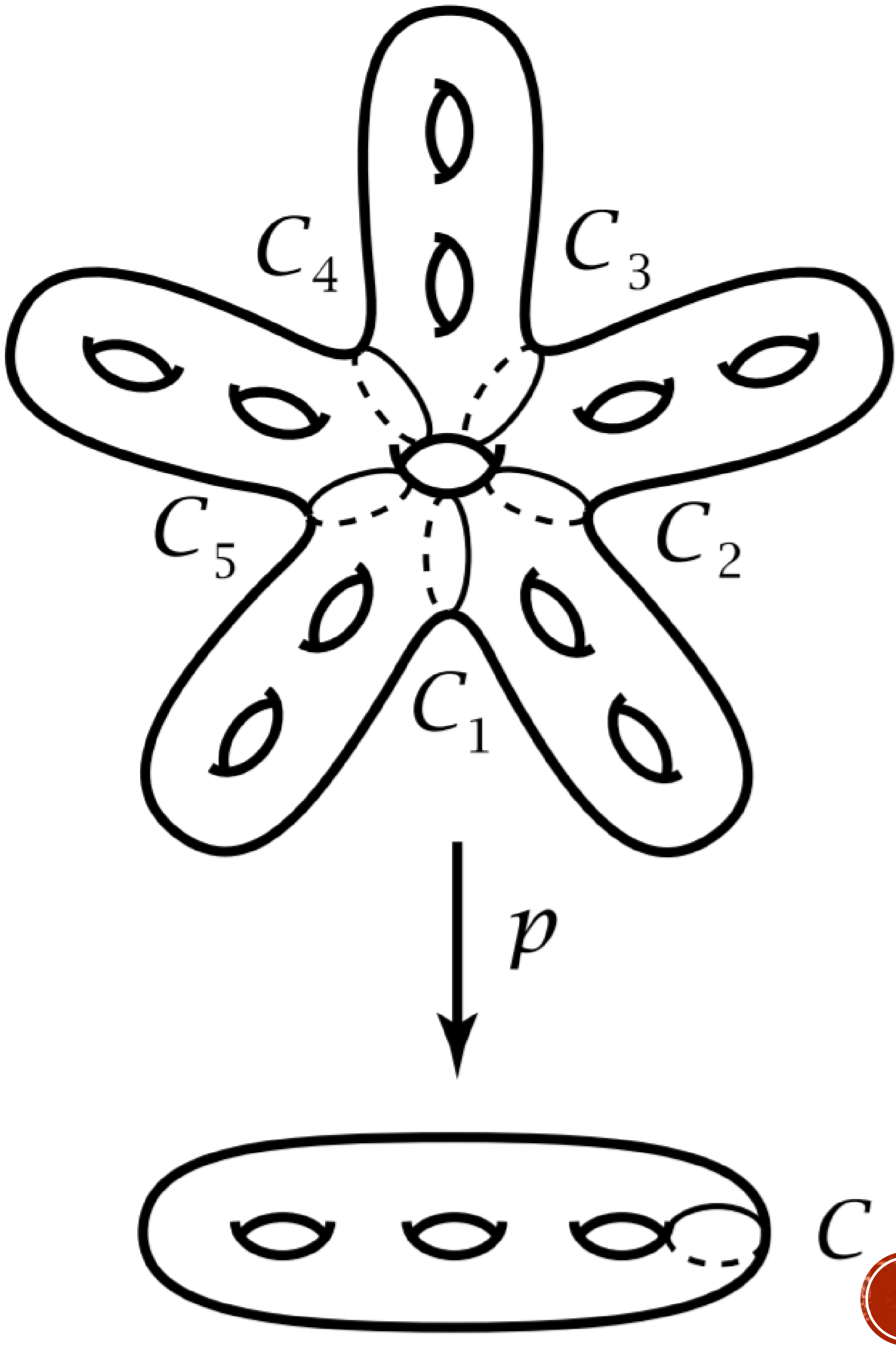

Covering Space

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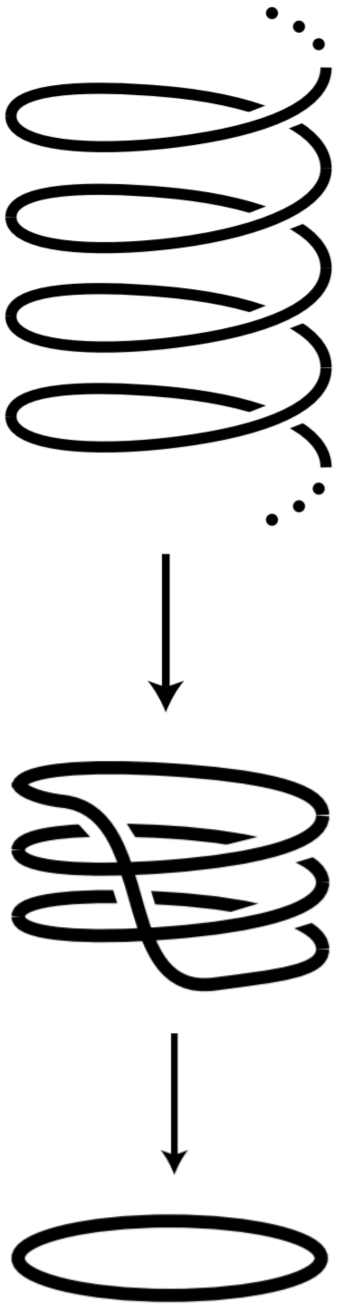
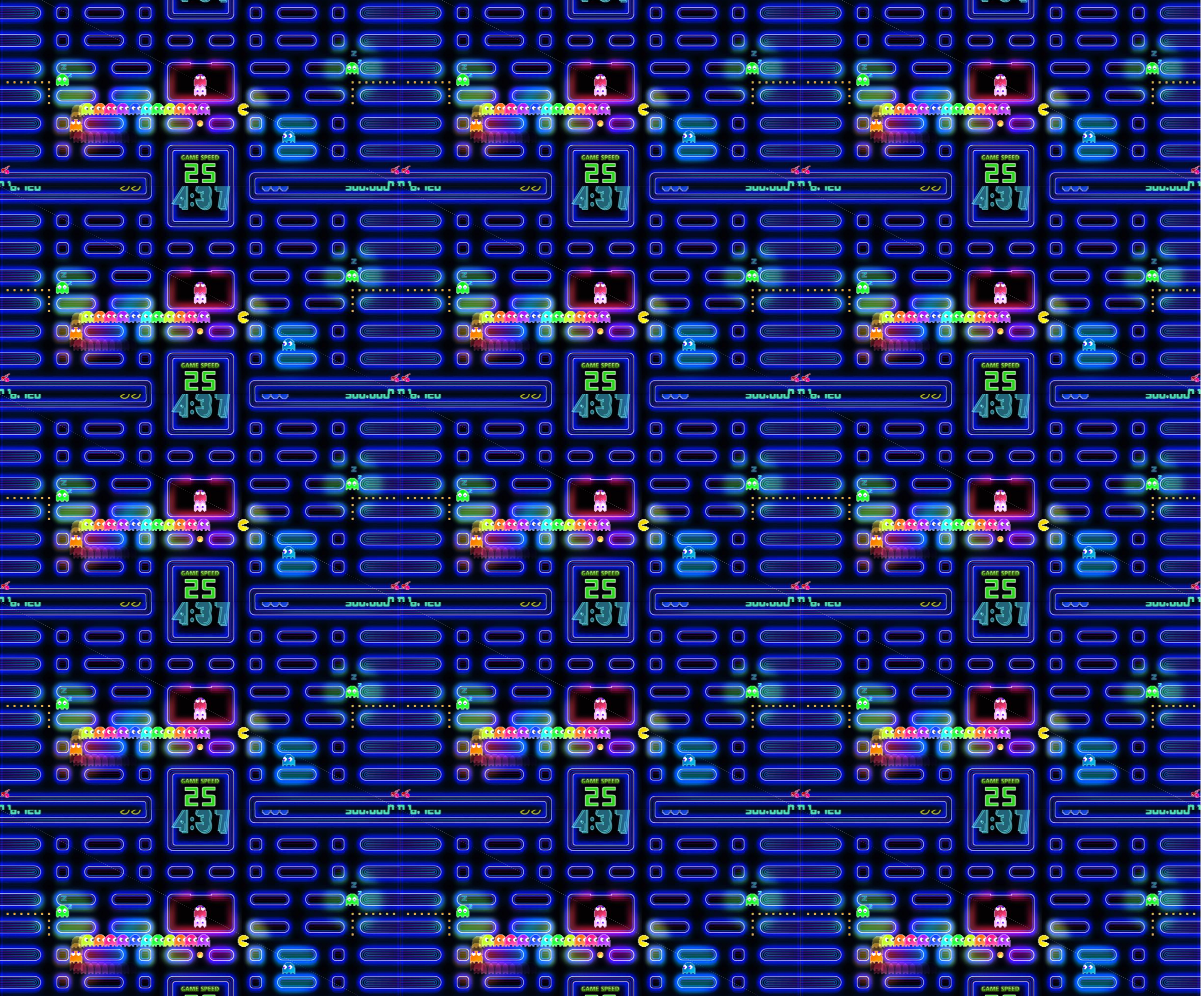
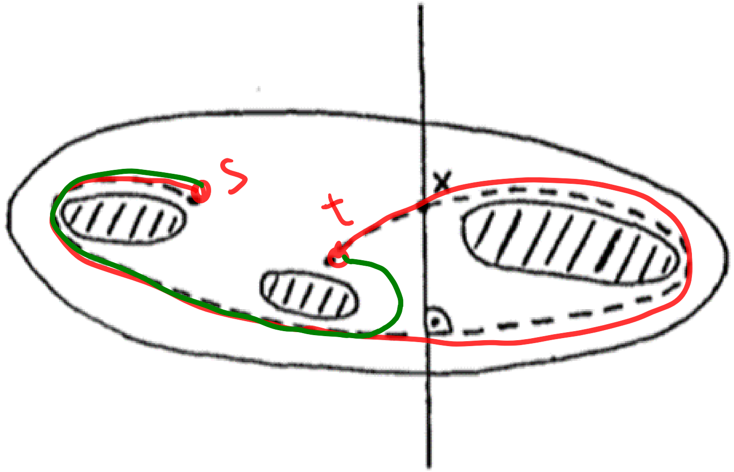
- Universal cover Ž

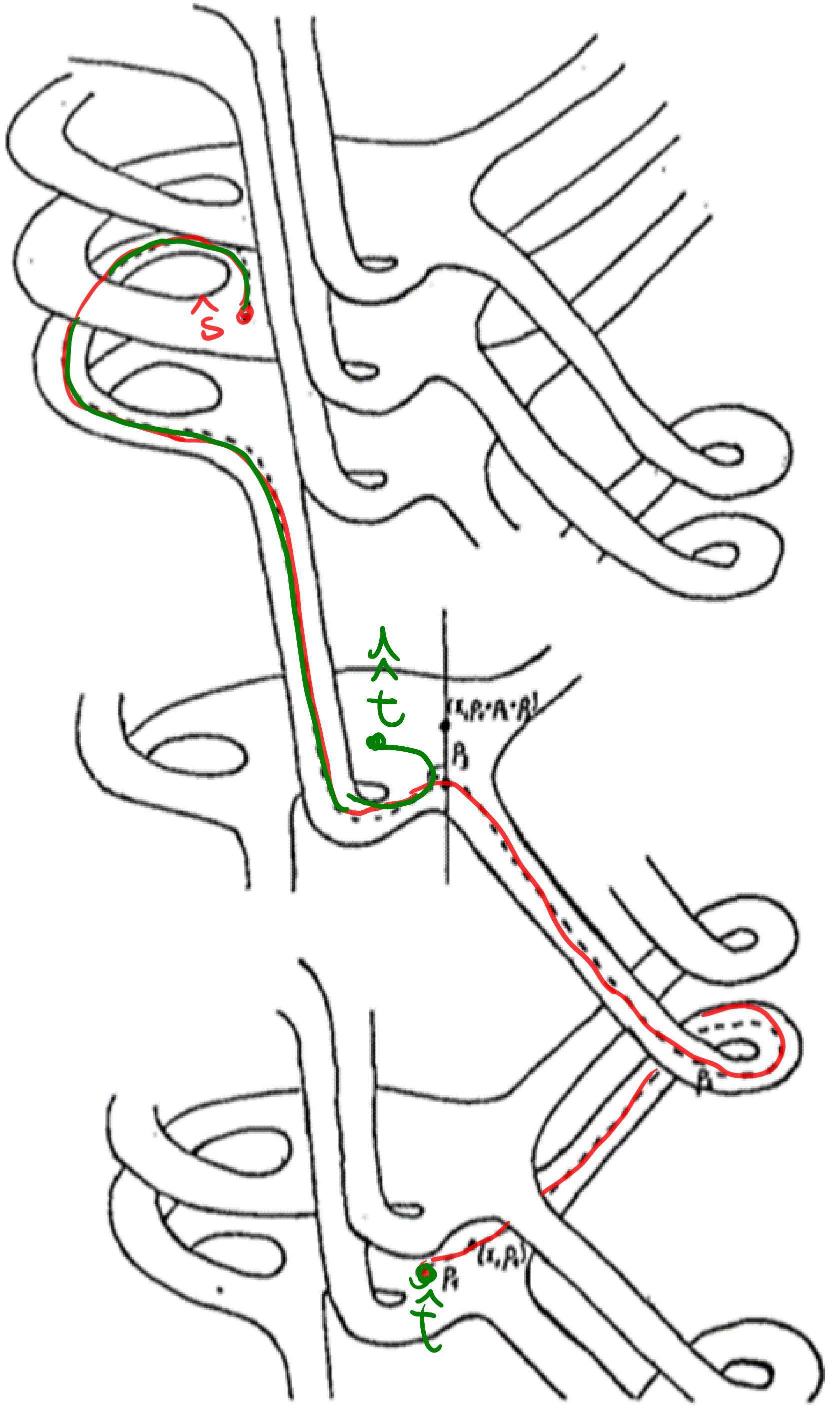


Covering of Pacman Space



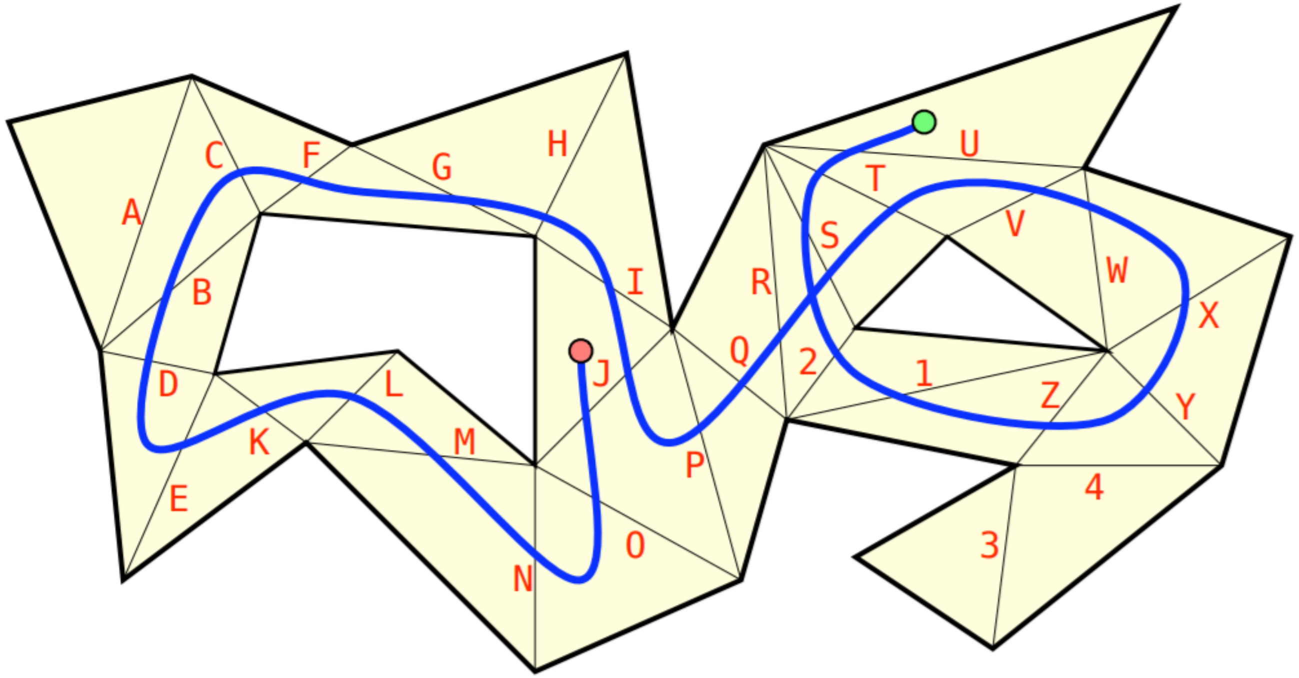
Covering of Pacman Space

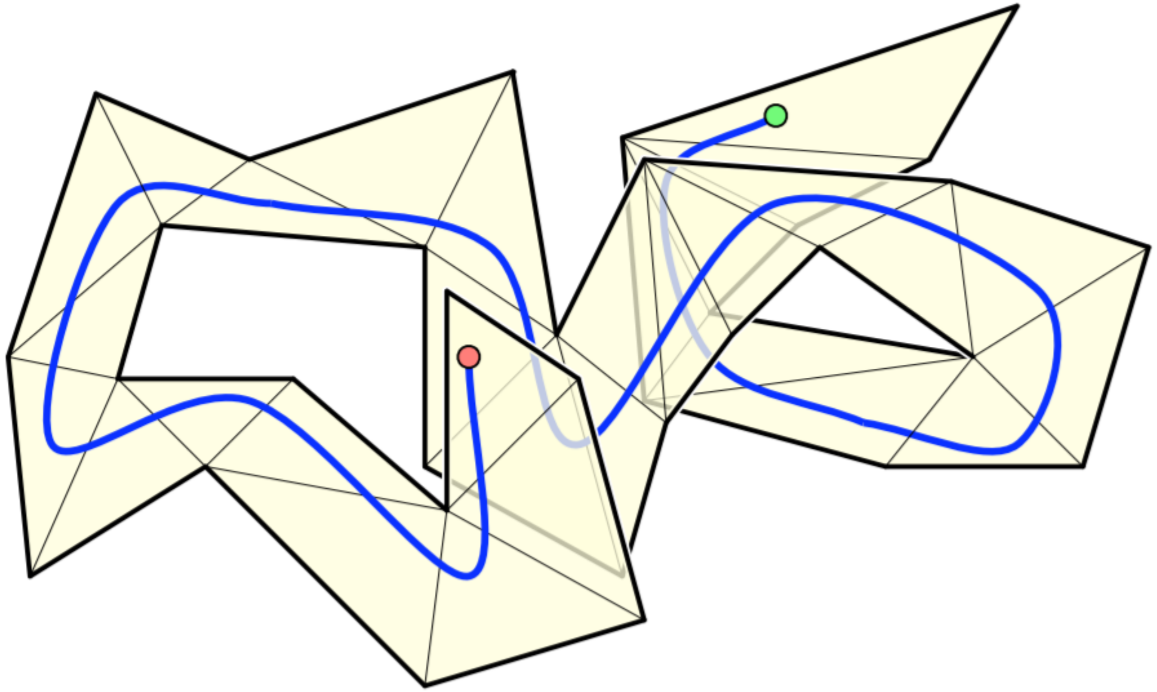




Lifting a Path

Lifting a Path





Proposition. Two paths are homotopic if and only if their lifts start and end at the same endpoints in Ž.

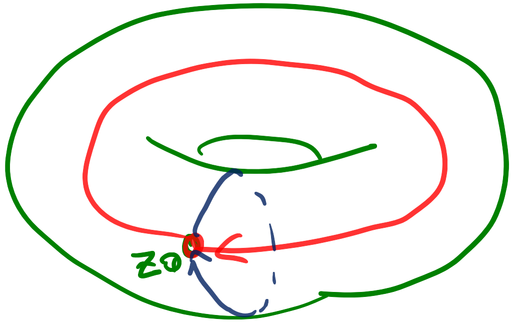
Fundamental Group

- [γ] is the class of closed paths homotopic to γ in space Z

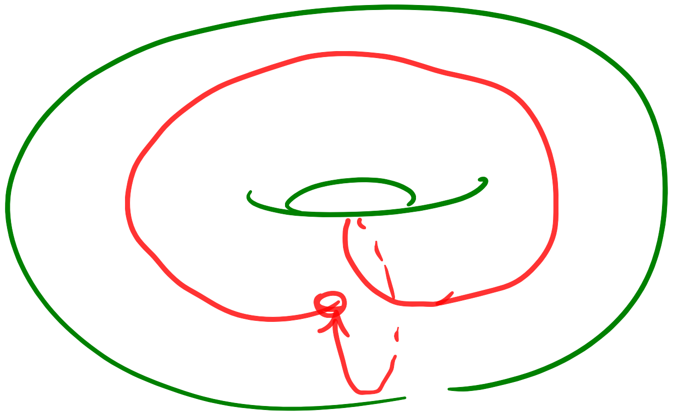
- $\pi_1(Z, z_0) =$

$$\{[\gamma] : \text{closed path } \gamma \text{ in } Z \text{ starting and ending at } z_0\}$$

PROPOSITION. $\pi_1(Z, z_0)$ is a group.

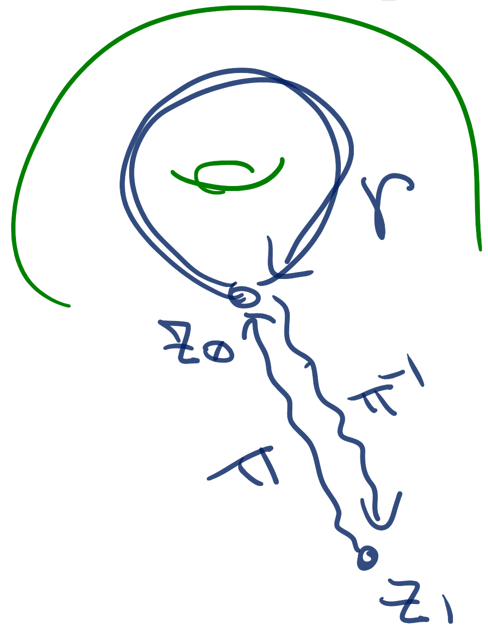


$$[\gamma_1]\cdot[\gamma_2] := [\gamma_1\cdot\gamma_2]$$



Proposition. $\pi_1(Z, z_0) \cong \pi_1(Z, z_1)$ as groups.

Pf.



$[\gamma] \longmapsto [\pi^{-1} \circ \gamma \circ \pi]$

Relation between two notions

- $\pi_1(Z, z_0) =$

$$\{[\gamma] : \text{closed path } \gamma \text{ in } Z \text{ starting and ending at } z_0\}$$

$$\check{Z} \xrightarrow{p} Z$$

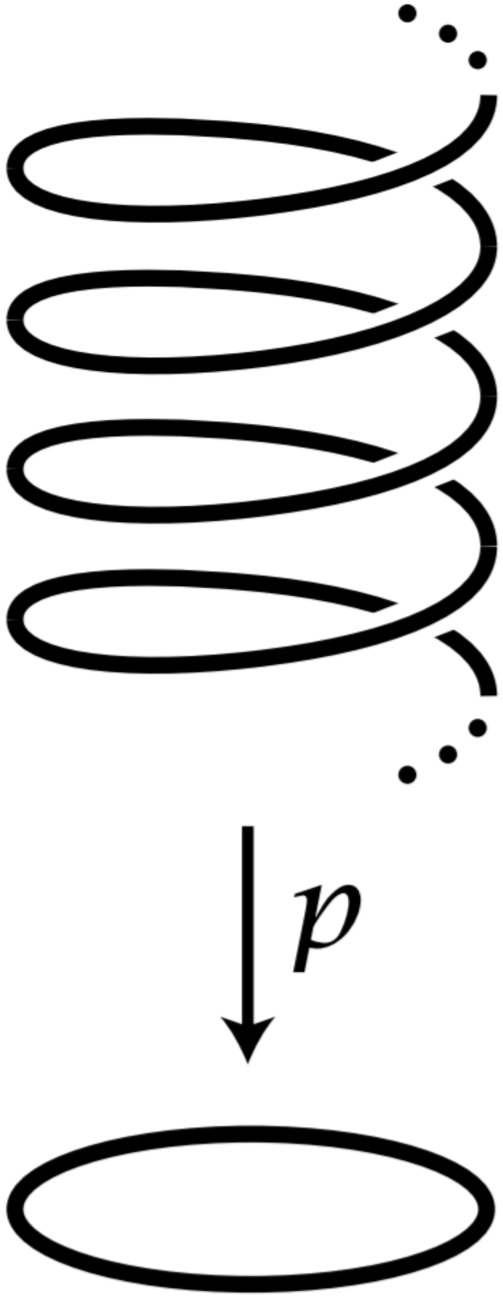
- $\check{Z} =$

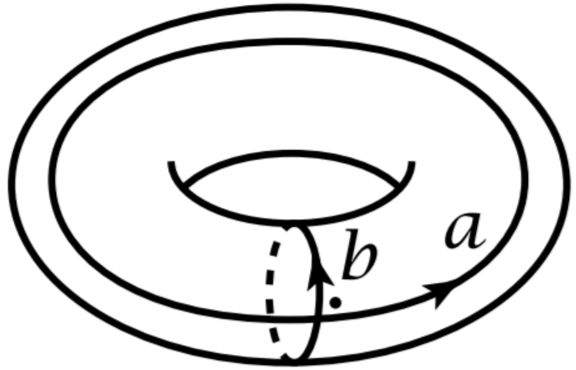
$$\{[\gamma] : \text{path } \gamma \text{ in } Z \text{ starting at } z_0\}$$

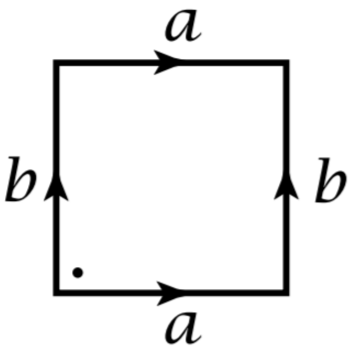
lifts of z_0. $p^{-1}(z_0) = \{[\gamma] :$ closed path γ in Z, starting & ending at $z_0\}$

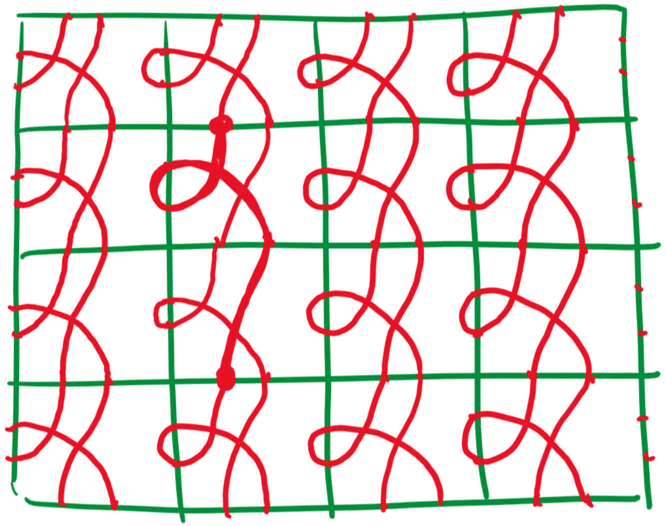
THEOREM. $\pi_1(S^1) \cong \mathbb{Z}$

This is just saying winding number exists on S^1.

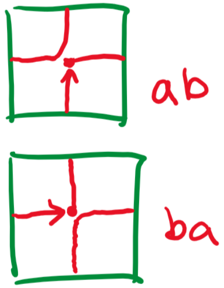


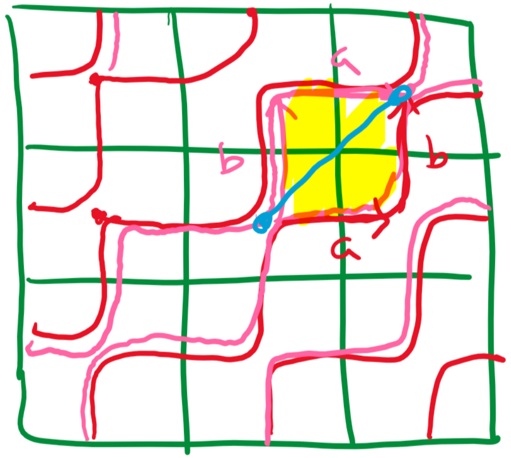


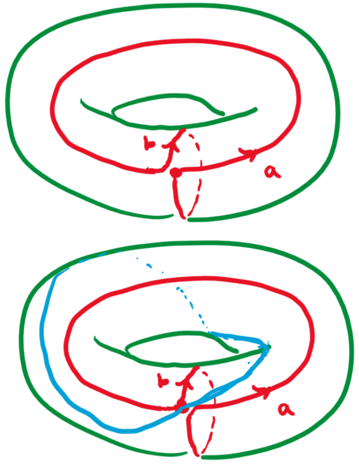




Exercise: π_1(Pacman)

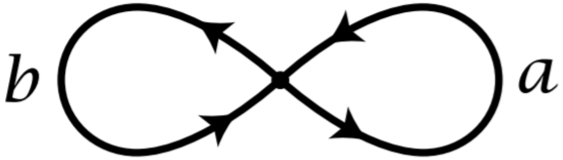


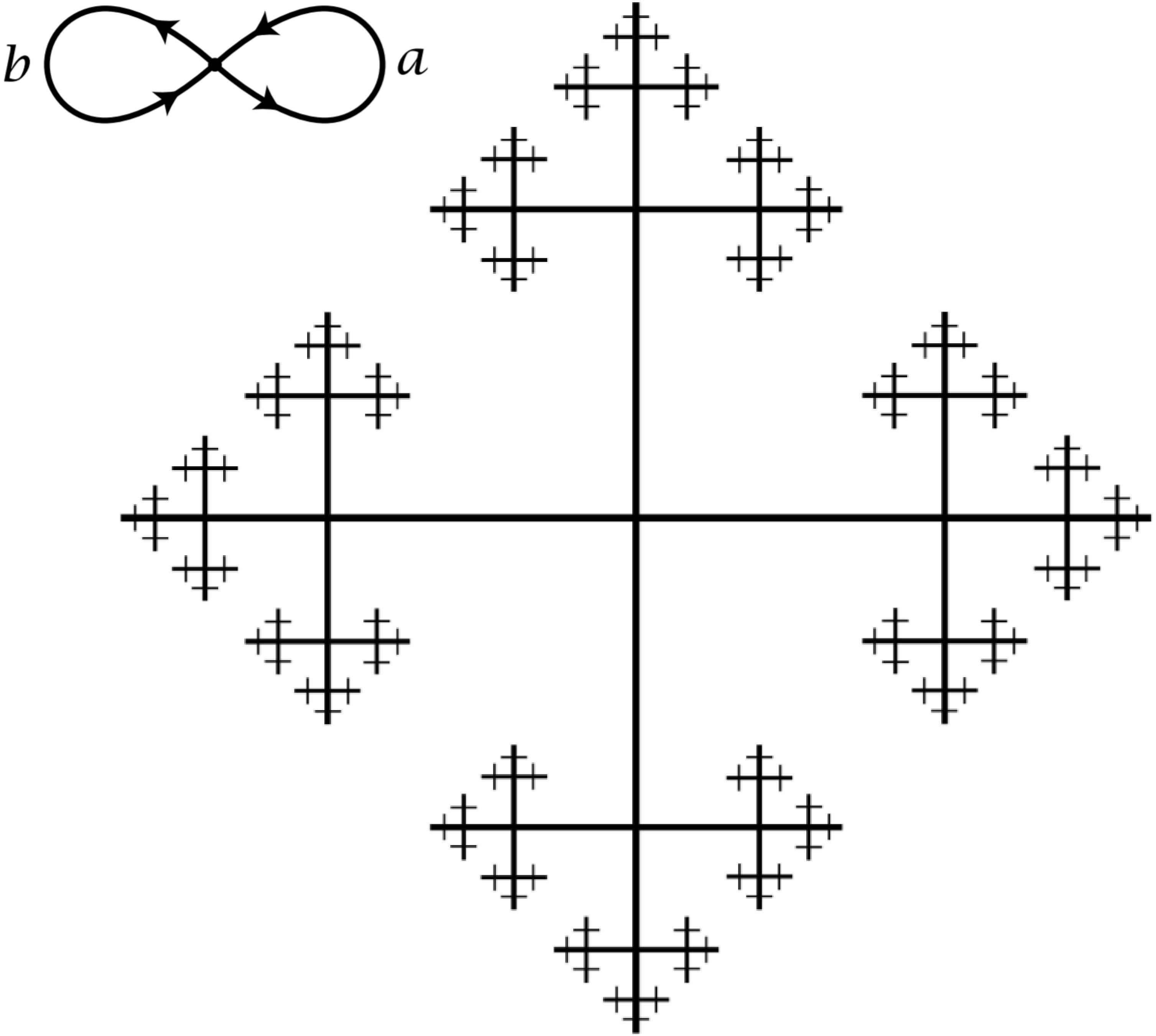




$$\pi_1(\text{Torus}) = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

EXERCISE: π_1(2-LOOPS)





Exercise: π_1(2-Loops)

Things under the rug

- For covering spaces to exist, space Z has to be
  - path-connected
  - locally path-connected
  - semilocally simply-connected

Element : Fundamental Group
::
Lift : Covering Space

Next time.

Fundamental group $\pi_1(X)$ is a homotopy invariant of X.

Induced Homomorphism

- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$

Induced Homomorphism

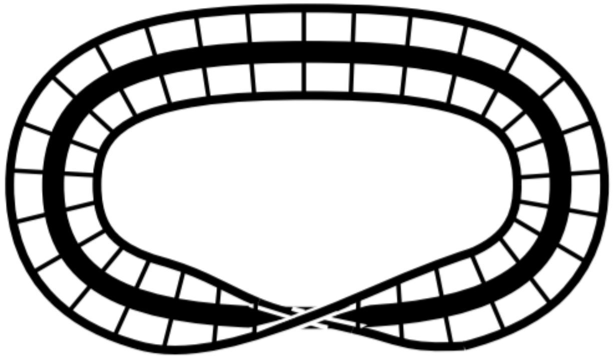
- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$

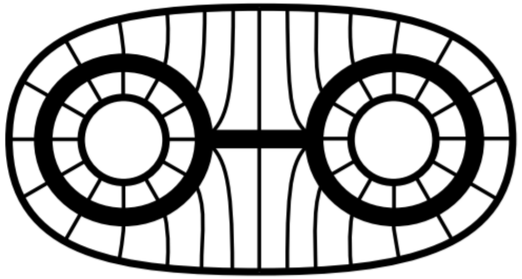
Proposition. ϕ_* is a group homomorphism.

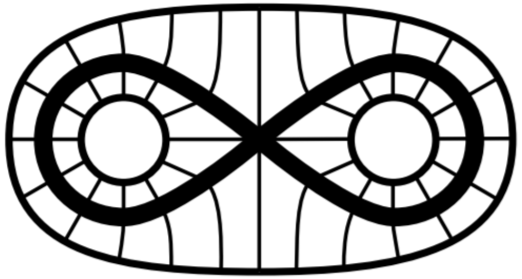
Equivalence

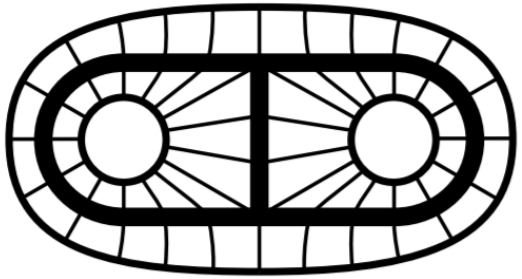
- Homeomorphism
  - $f: X \longrightarrow Y$ continuous bijection
  - $g: Y \longrightarrow X$ continuous bijection
  - $f \bullet g = id_X$
  - $g \bullet f = id_Y$
- Homotopy equivalence
  - $f: X \longrightarrow Y$ continuous bijection
  - $g: Y \longrightarrow X$ continuous bijection
  - $f \bullet g$ homotopic to id_X
  - $g \bullet f$ homotopic to id_Y

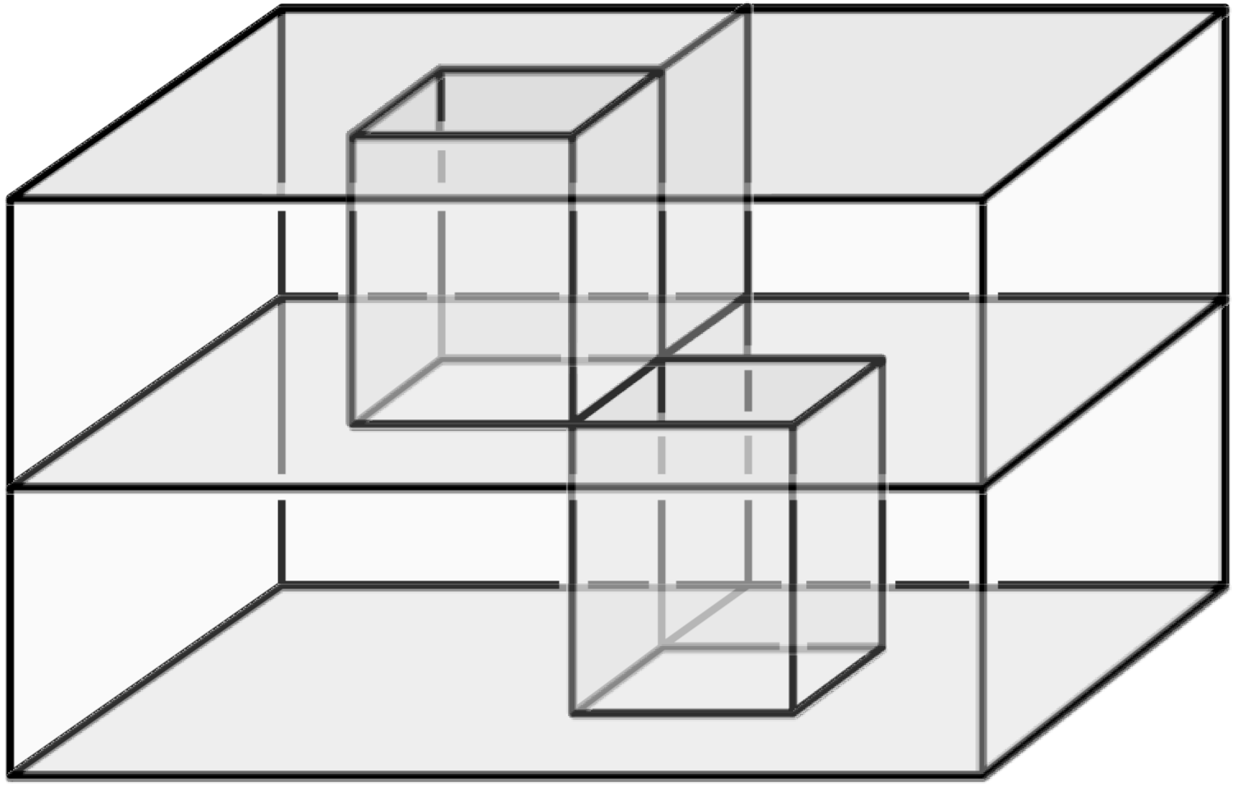
Homotopy Equivalence











Homotopy Equivalence

- House with two rooms

Theorem. Homotopy equivalence induces group isomorphism on π_1.