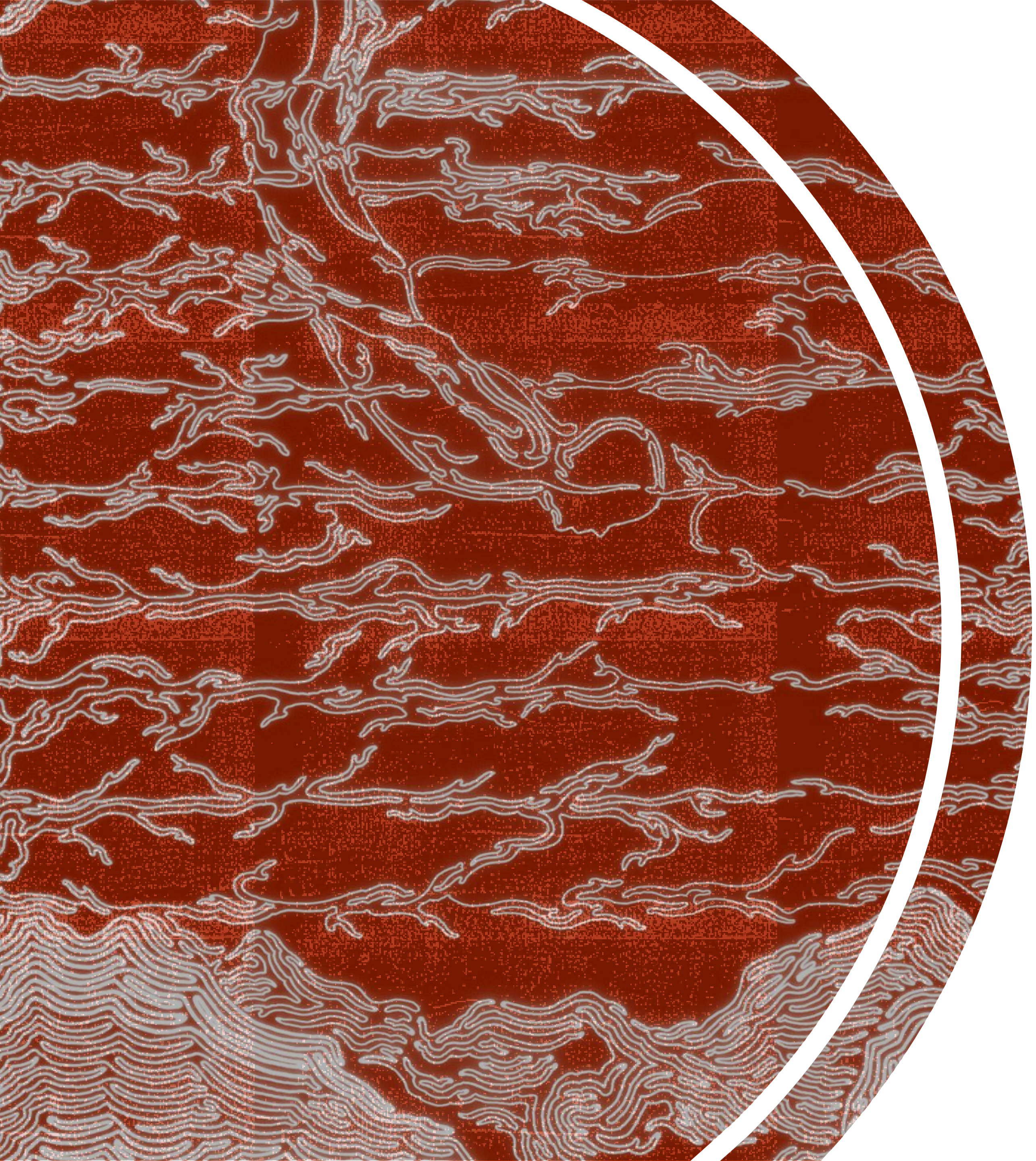


Introduction to

Computational Topology

Hsien-Chih Chang

Lecture 6, September 30, 2021

Administrivia

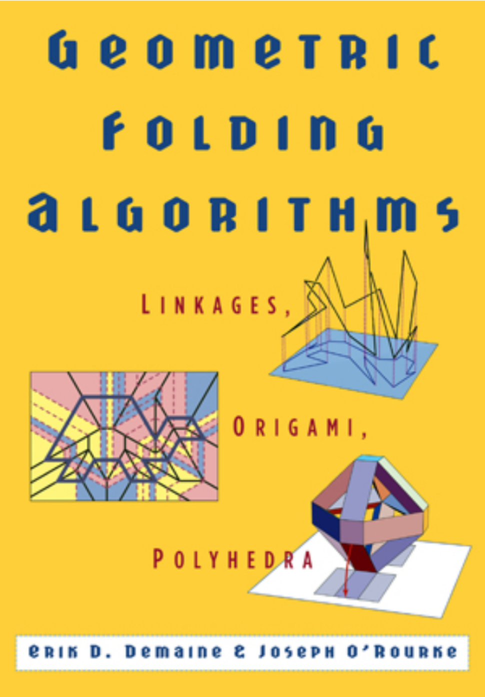
- Thanks for all the feedback!

Geometric Folding Algorithms

6.849: Geometric Folding Algorithms: Linkages, Origami, Polyhedra (Fall 2020)

Prof. Erik Demaine; Martin Demaine; TAs Yevhenii Diomidov & Klara Mundilova

[Home] [Lectures] [Problem Sets] [Project] [Coauthor] [Github] [Accessibility]



textbook

Overview

*the algorithms behind building **TRANSFORMERS** and designing **ORIGAMI***

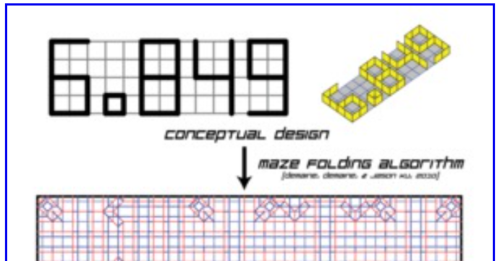
Whenever you have a physical object to be reconfigured, **geometric folding** often comes into play. This class is about algorithms for analyzing and designing such folds. Motivating applications include

- automated design of new and complex origami
- using 2D fabrication technology to manufacture complex 3D objects
- transforming robots by self-folding sheets or chains
- how to fold robotic arms without collision
- how to bend sheet metal into desired 3D shapes
- understanding how proteins fold

Major progress have been made in recent years in many of these directions, thanks to a growing understanding of the mathematics and algorithms underlying folding. Nonetheless, many fundamental questions remain tantalizingly unsolved. This class covers the state-of-the-art in folding research, including a variety of open problems, enabling the student to do research and advance the field.

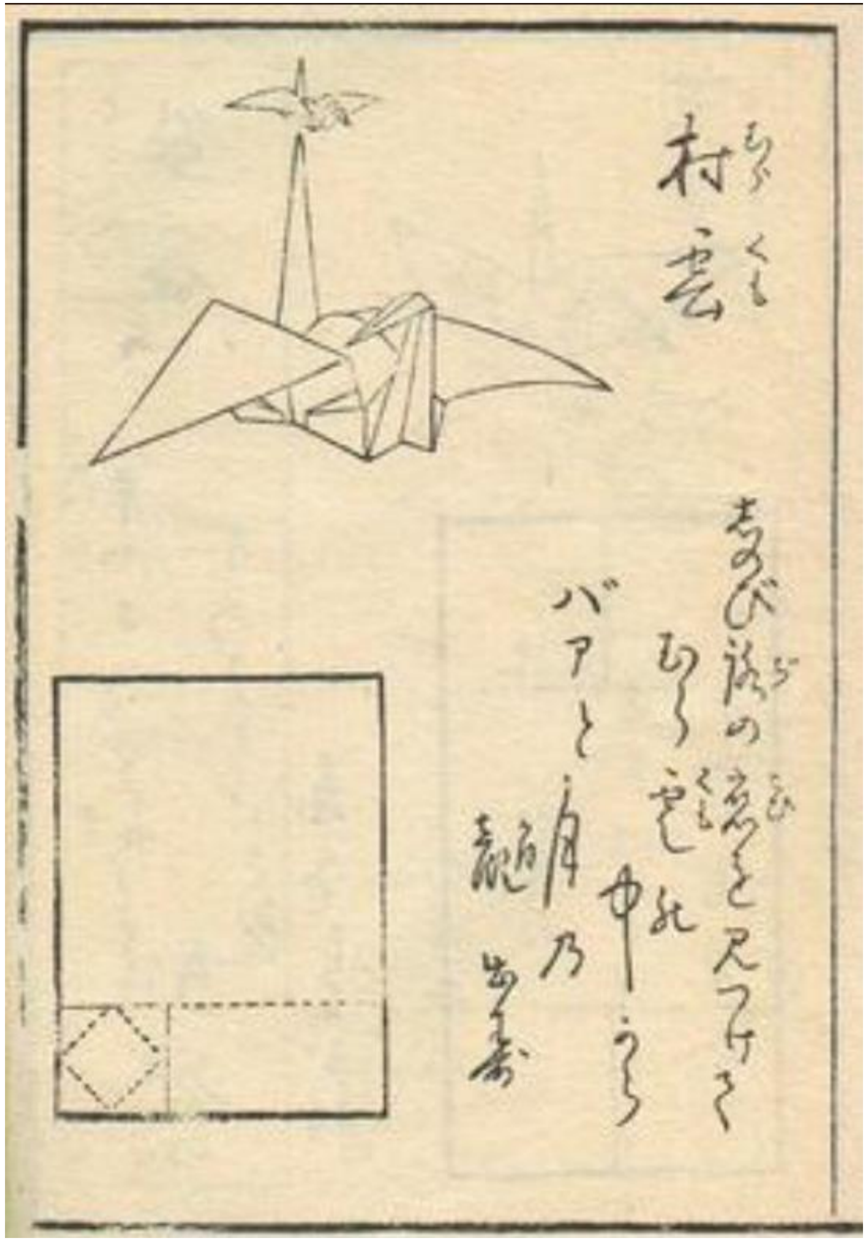
Fully Online Format

Most course material is covered in video lectures recorded in 2010 (already watched by over 19,000 people), which you can conveniently play at faster speed than real time. There may also be some **new material** presented by the professor and/or guest lecturers, which will be recorded for asynchronous viewing.



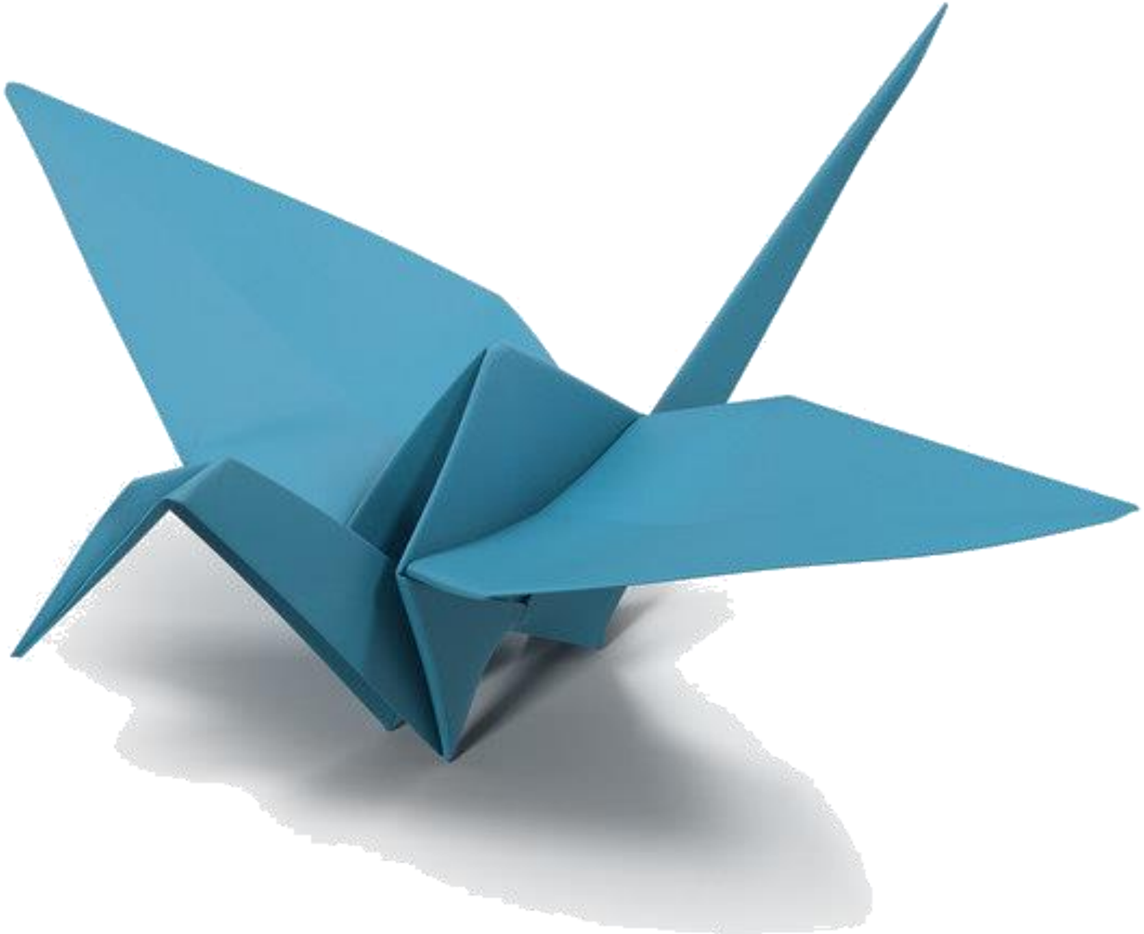
Paper Folding

折り紙
ORIGAMI

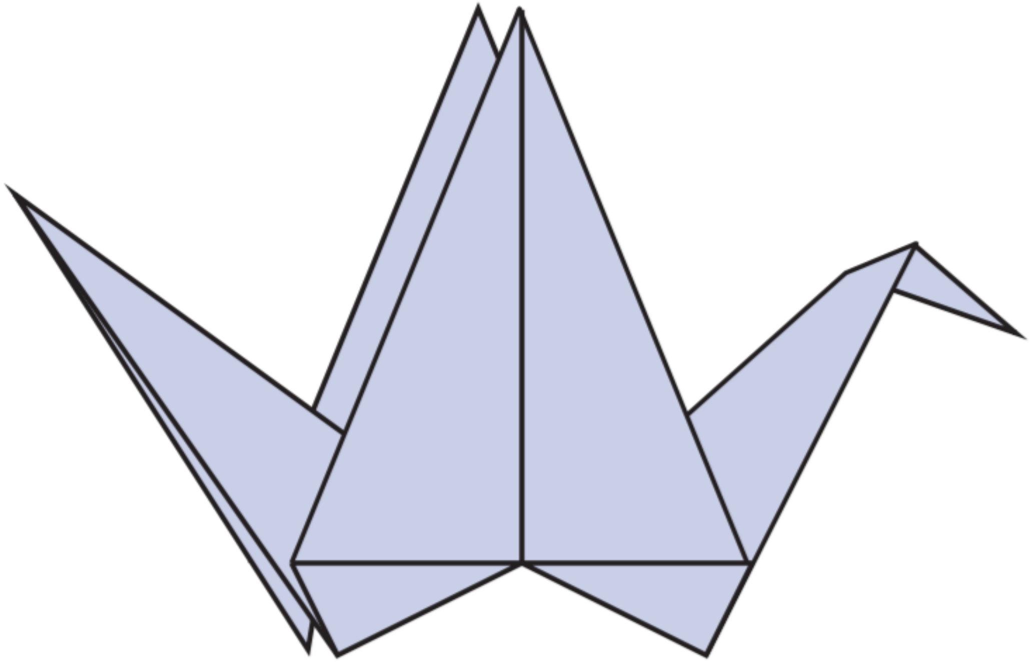


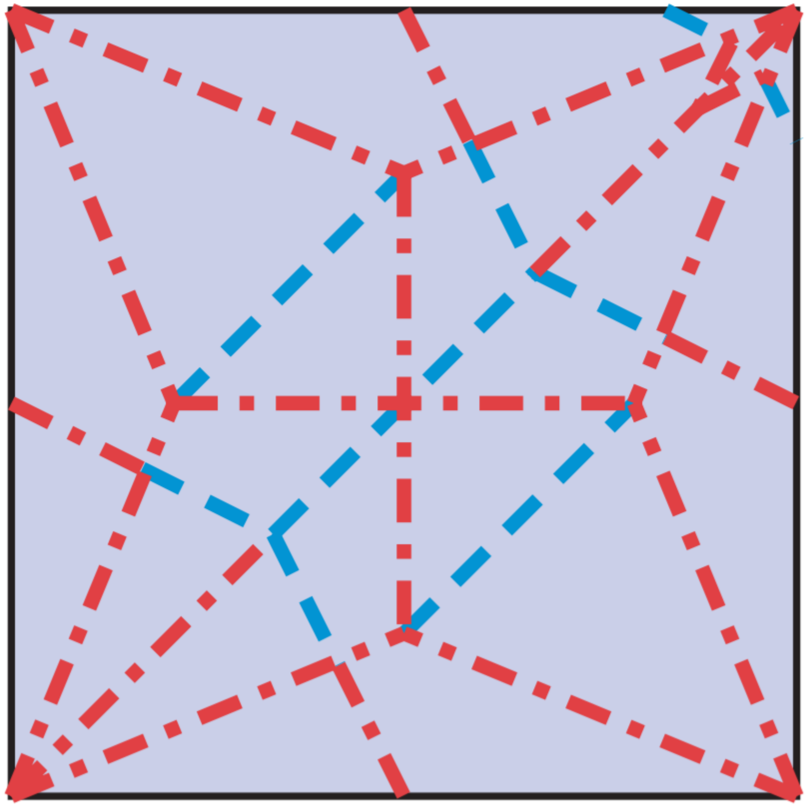
[秋里籬島 Akisato Ritō, 1798]

Folded State

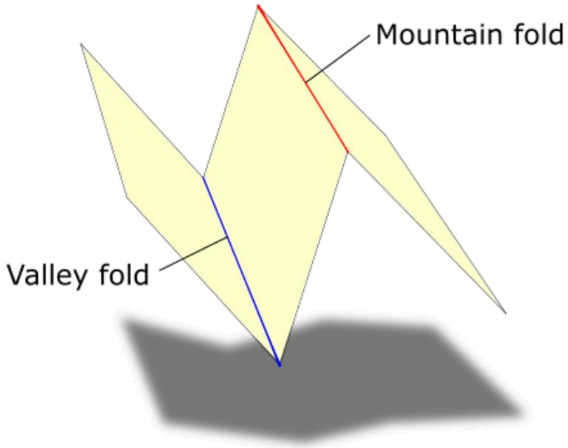


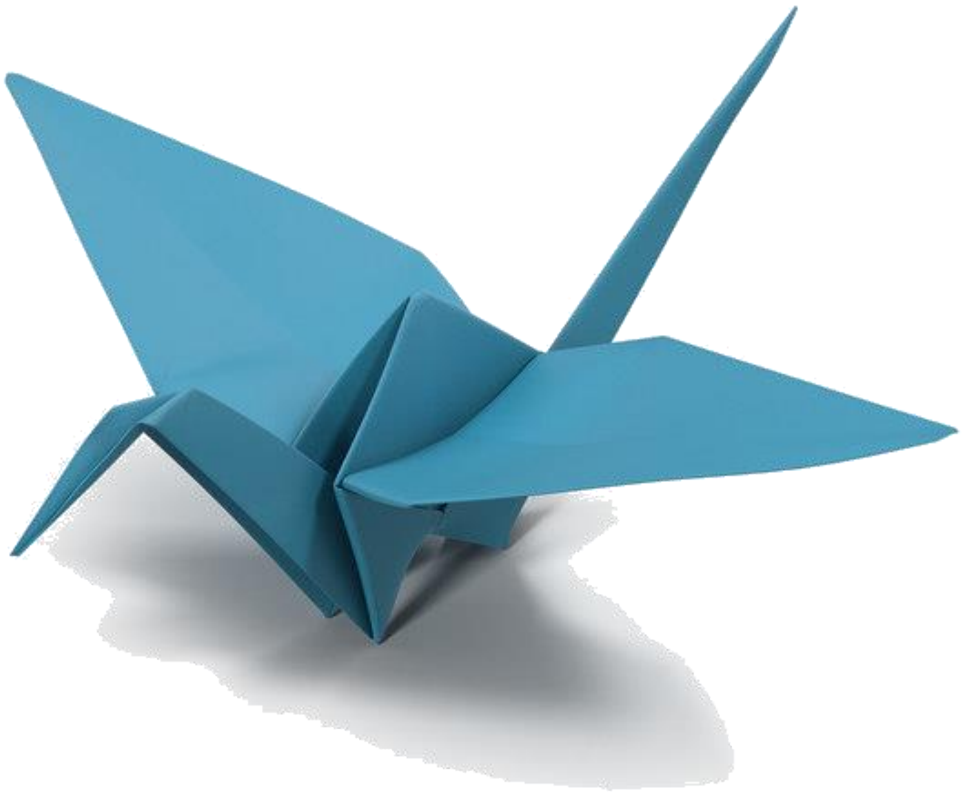
Crease Pattern

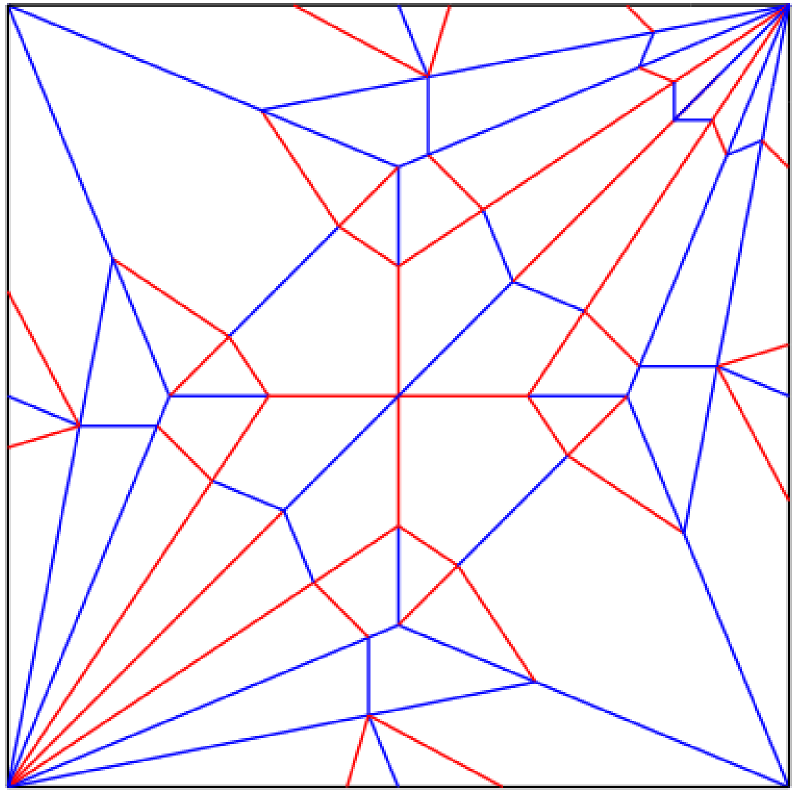




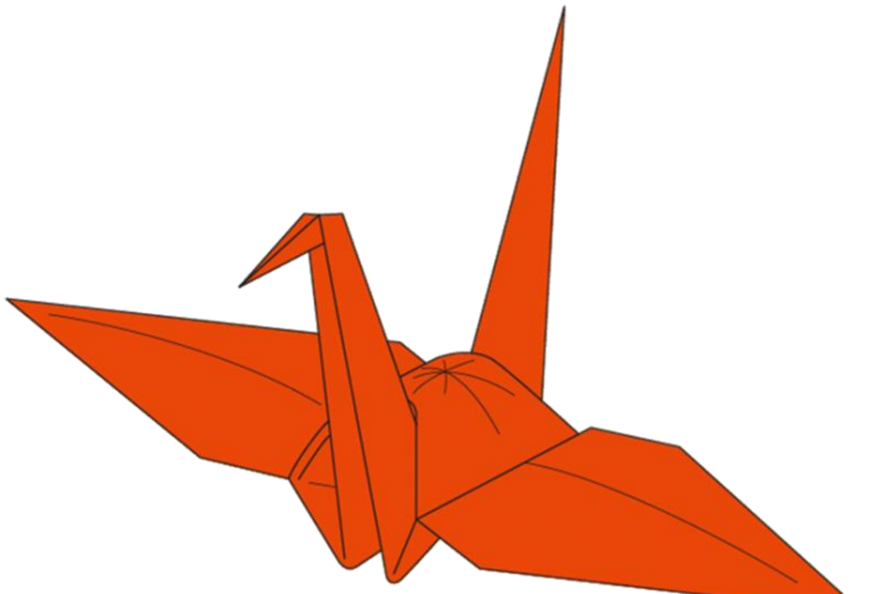
Crease Pattern

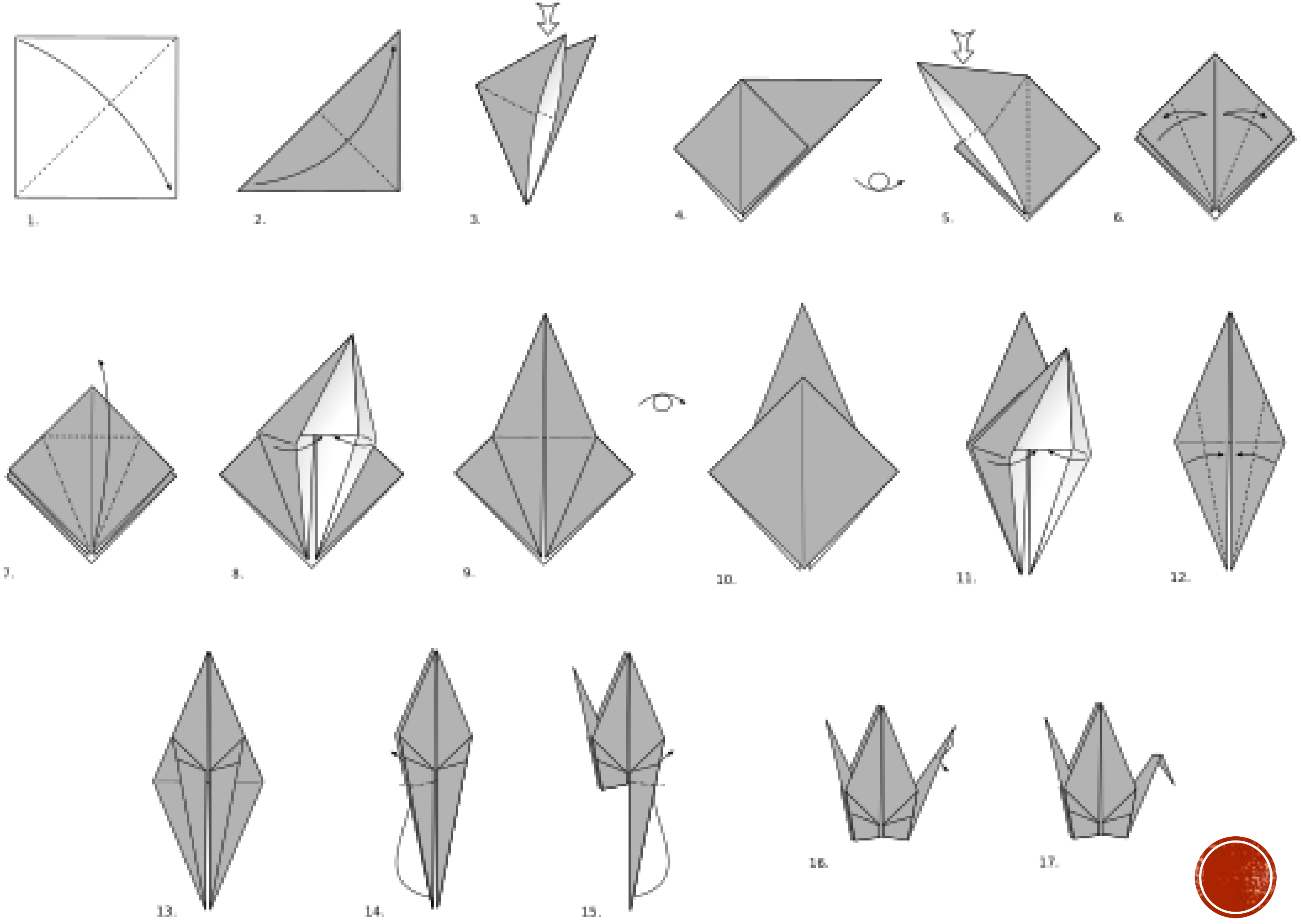






Folding Motion



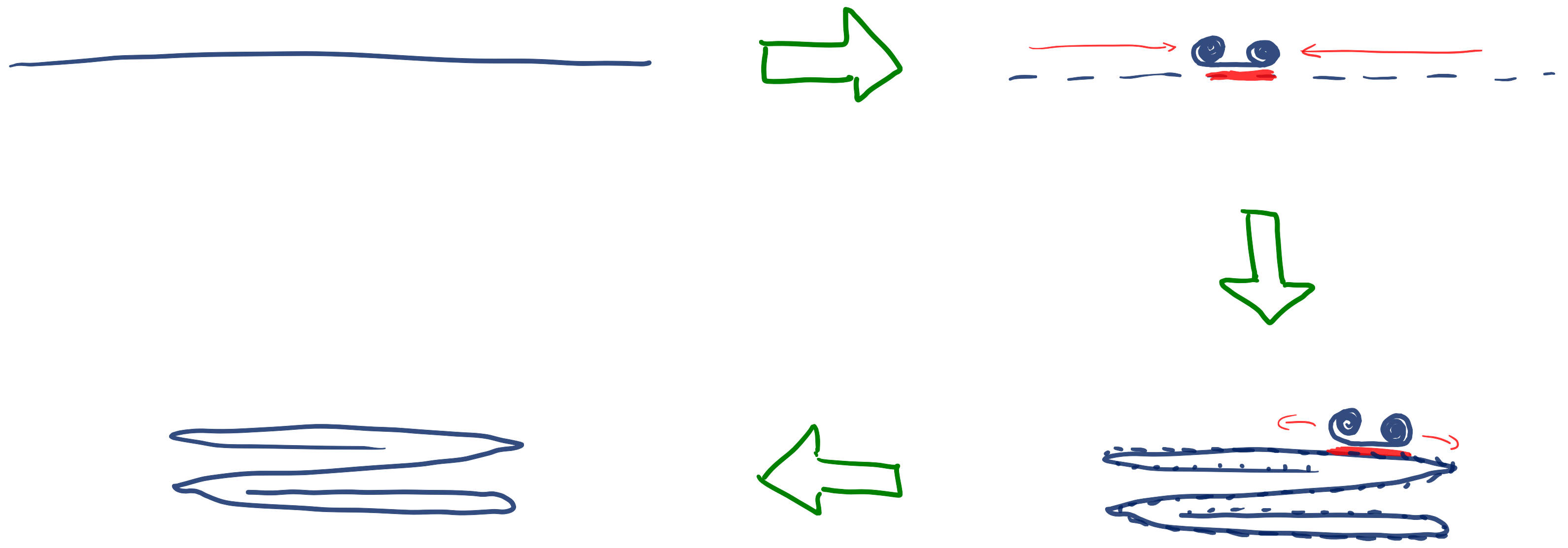


Folding Problems

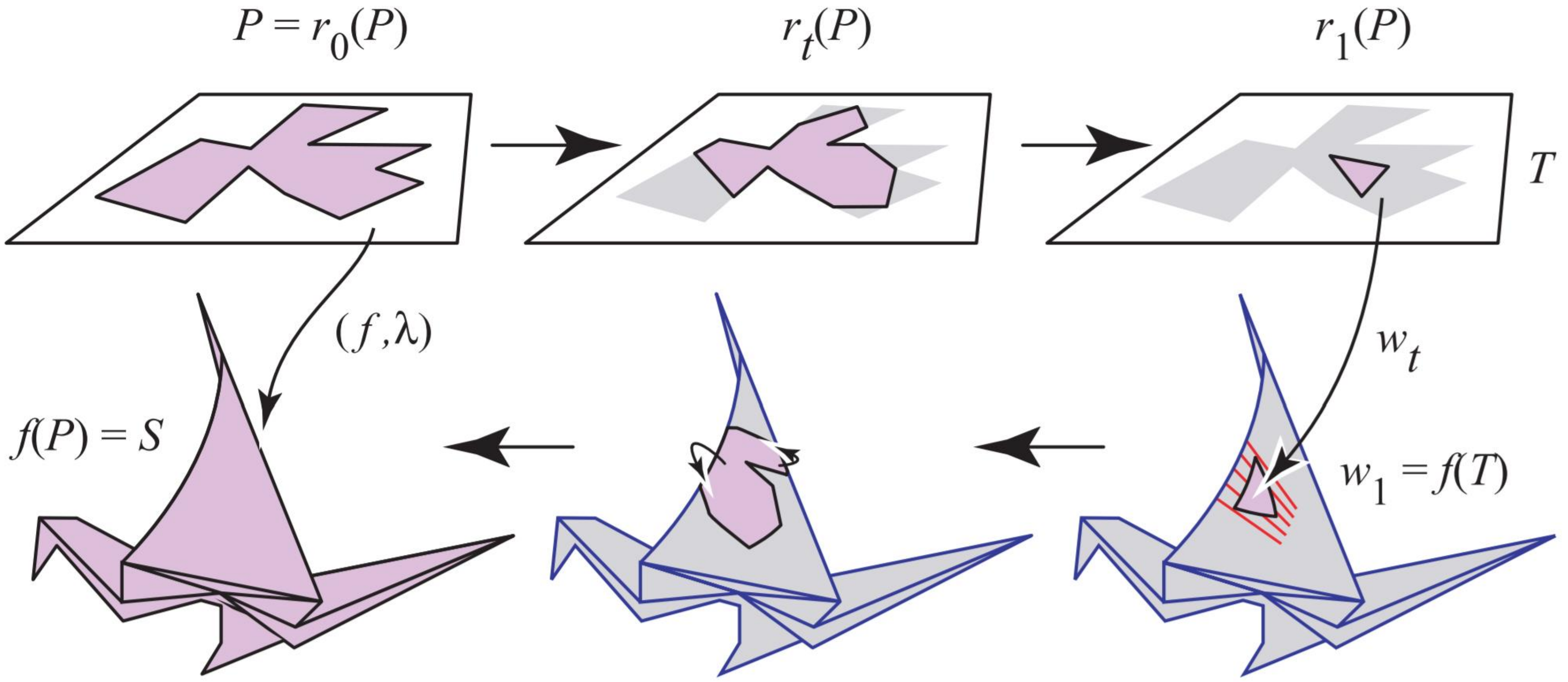
- Given a crease pattern, can it be flat-folded?
- Given a flat-foldable mountain-valley pattern, what is the final state?

- Folding into arbitrary 2D/3D shape?

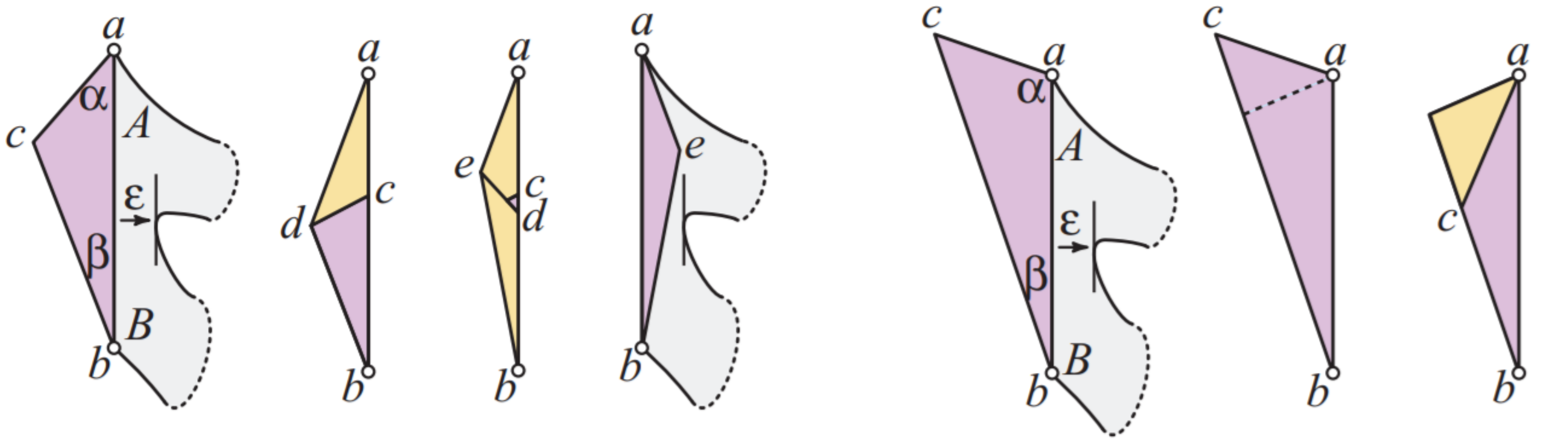
Proposition. There is always a folding motion from the unfolded state to any folded state.

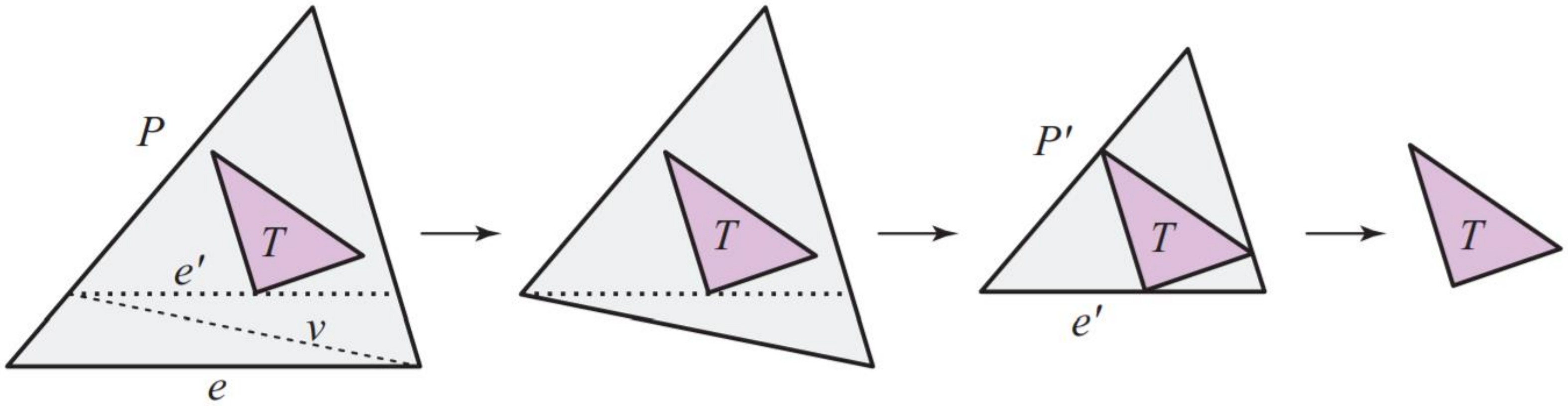


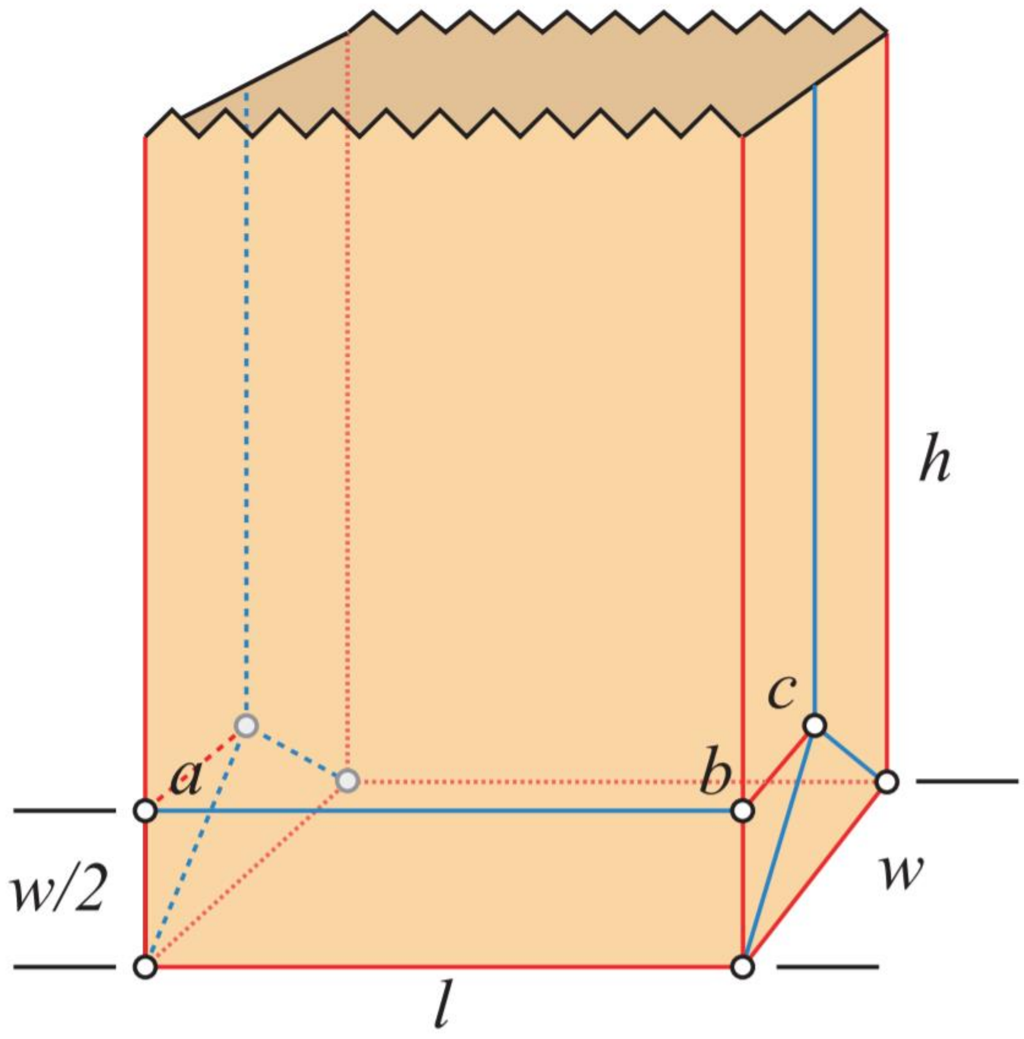
Proposition. There is always a folding motion from the unfolded state to any folded state for flat papers.



PROPOSITION. There is always a folding motion from the unfolded state to any folded state.

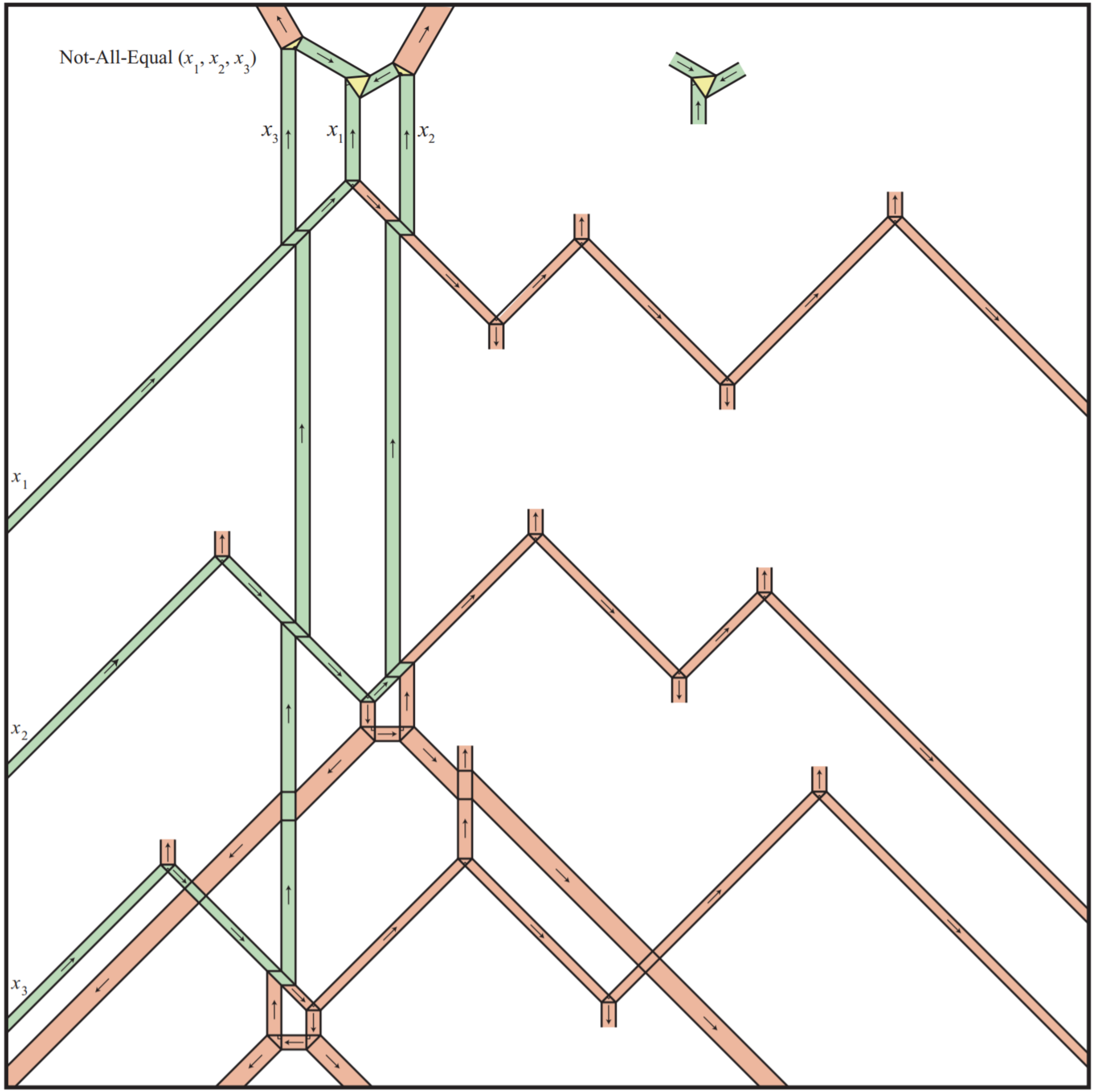






Bending is Necessary!

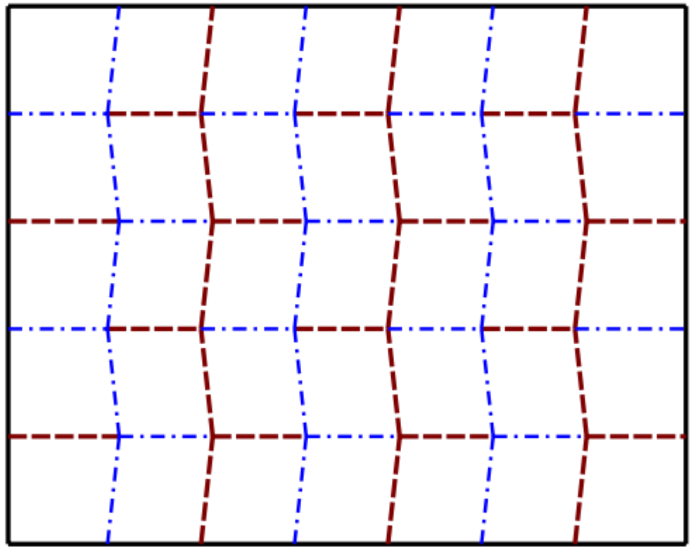
- **Open:**
  Does folding motion exist when the original paper is not flat?

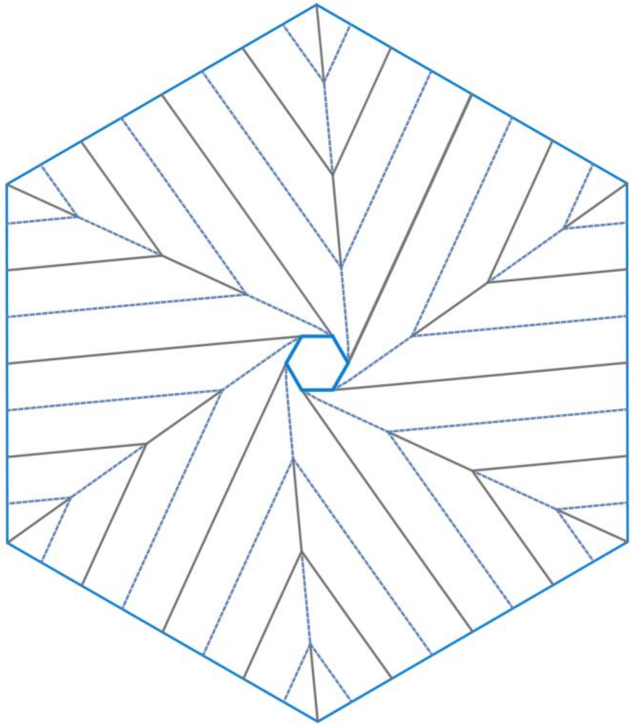


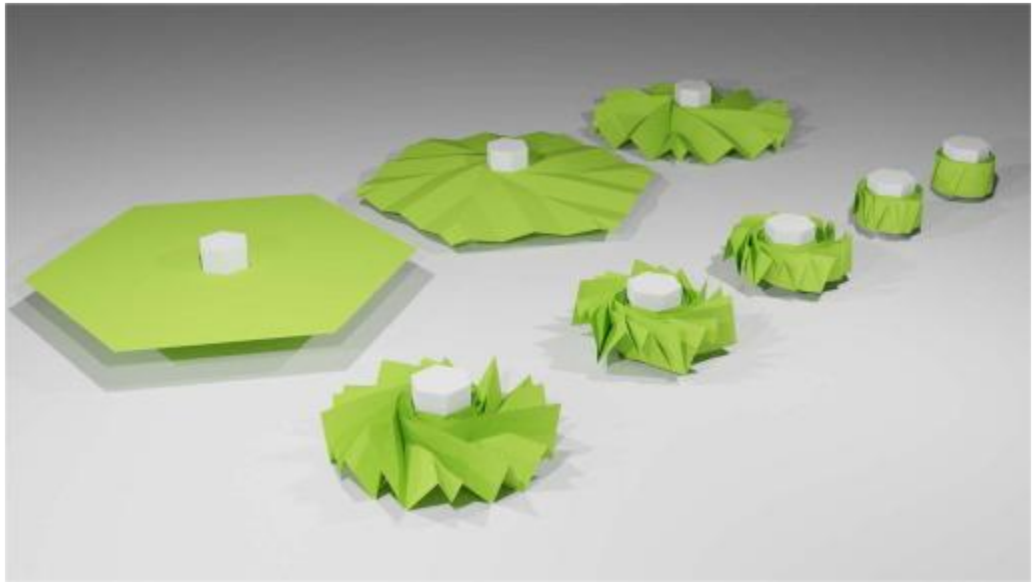
Folding is Hard

- Given a crease pattern, it is NP-hard to decide if can be flat-folded.
- Given a flat-foldable mountain-valley pattern, it is NP-hard to compute the flat-folded state.

[Bern-Hayes 1996]

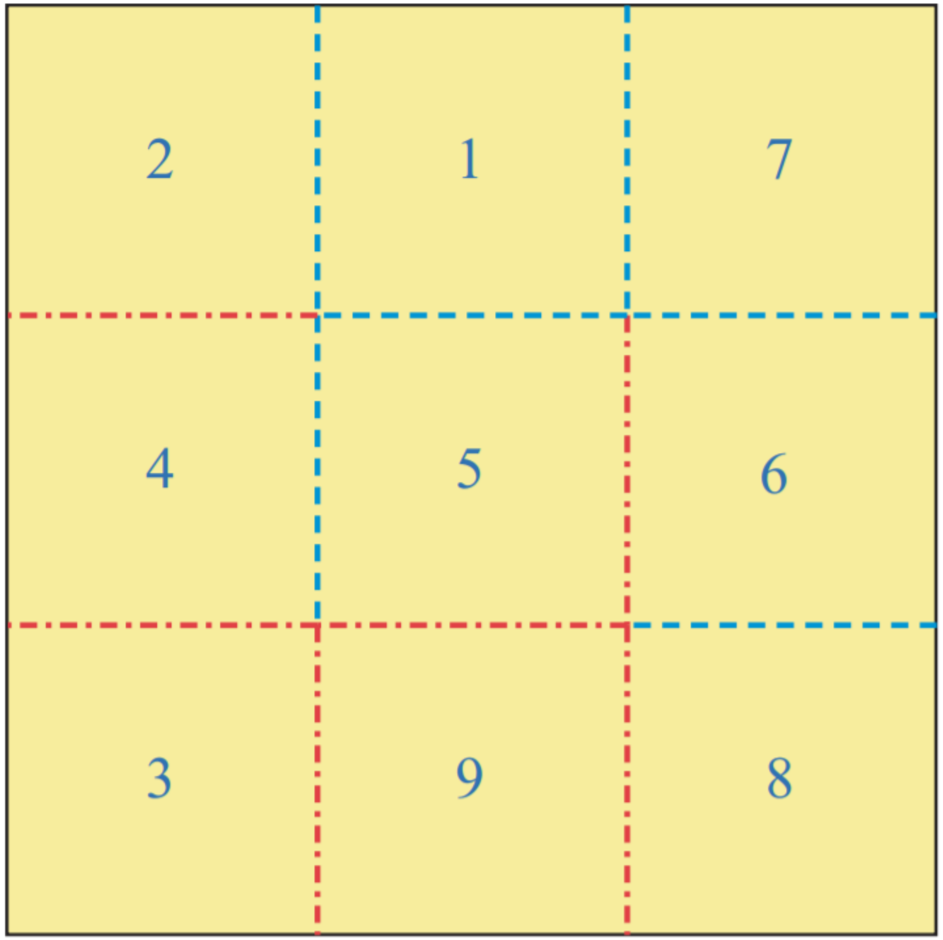






Folding is Hard

- Not all folding motions are simple.



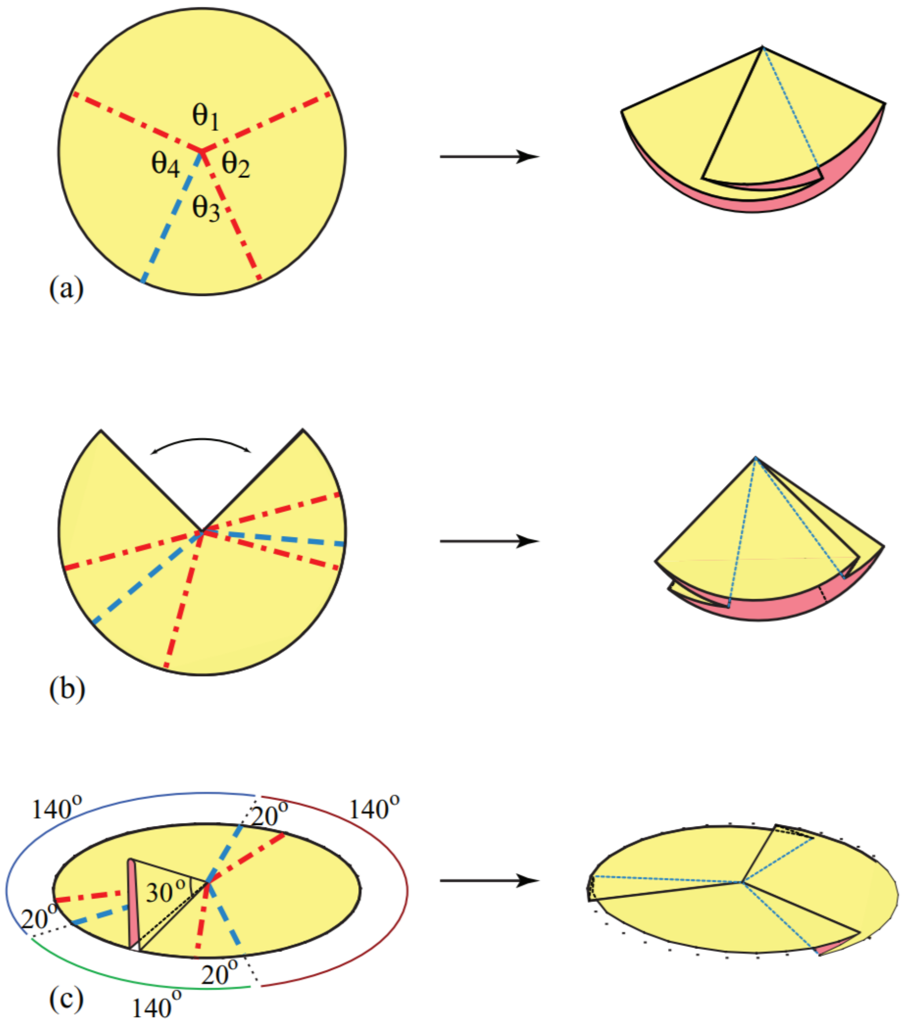
Folding is Hard

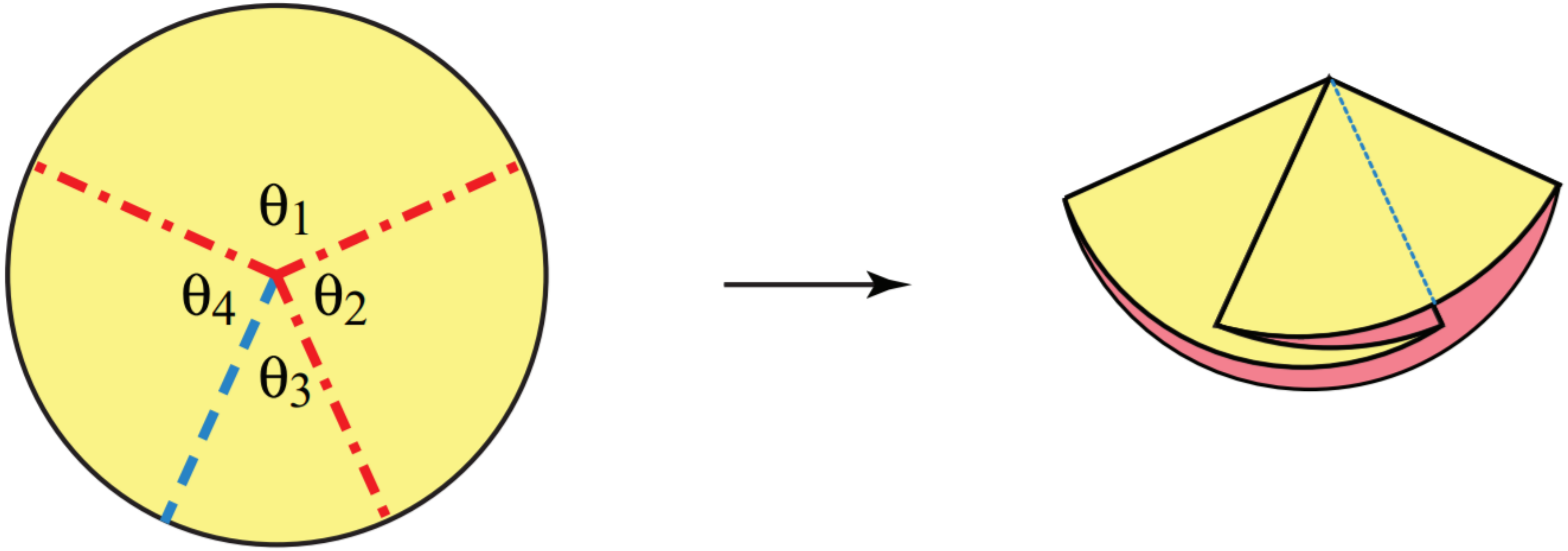
- **Open:**
  Given a 2-by-n grid with mountain-valley pattern, can it fold flat?

Let's Look at Simple Folding at a Single Vertex

Single-vertex Crease Pattern

- Disk of paper
- Exactly n creases
- Angles $\theta_1, \theta_2, \ldots, \theta_n$





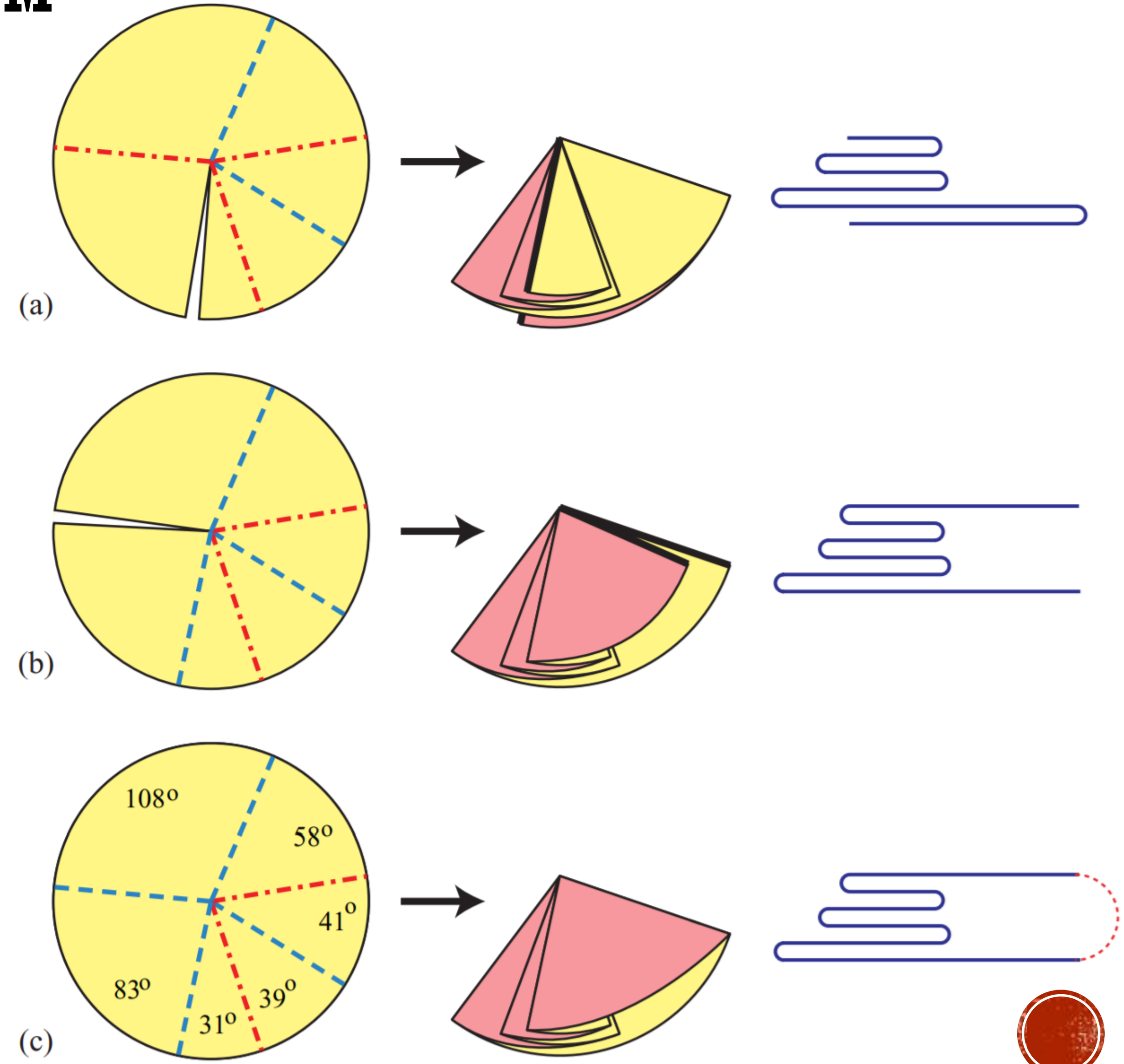
Kawasaki's Theorem

[Kawasaki 1989, Justin 1989]

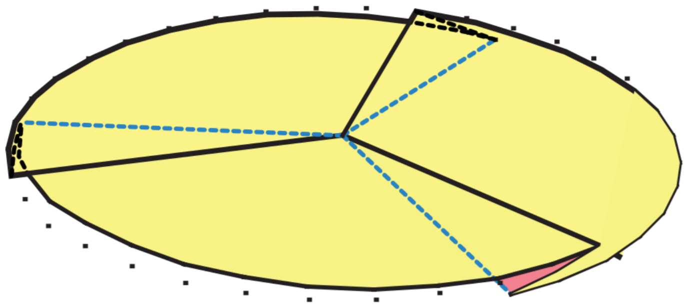
A single-vertex crease pattern is flat-foldable if

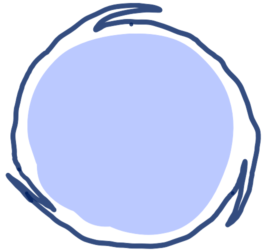
$$(\theta_1+\theta_3+\ldots+\theta_{n-1}) - (\theta_2+\theta_4+\ldots+\theta_n) \in \{0, \pm 2\pi\}$$

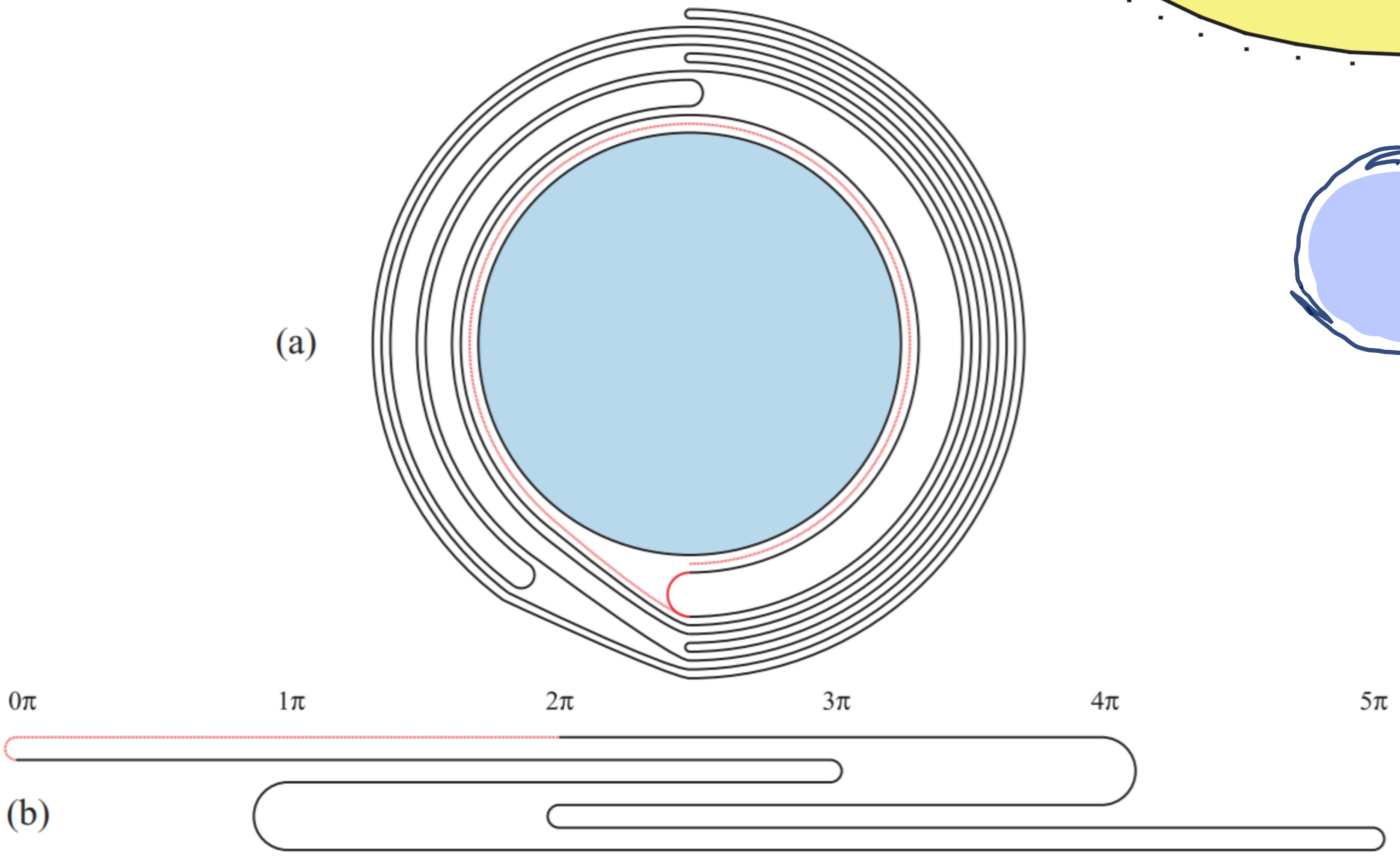
Proof of Kawasaki's Theorem



Proof of Kawasaki's Theorem







Intermission

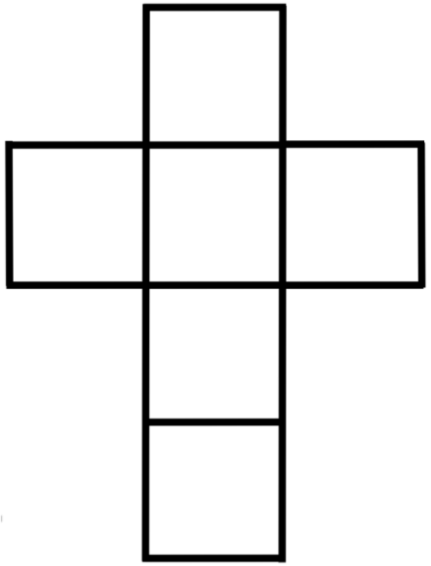
Paper Unfolding

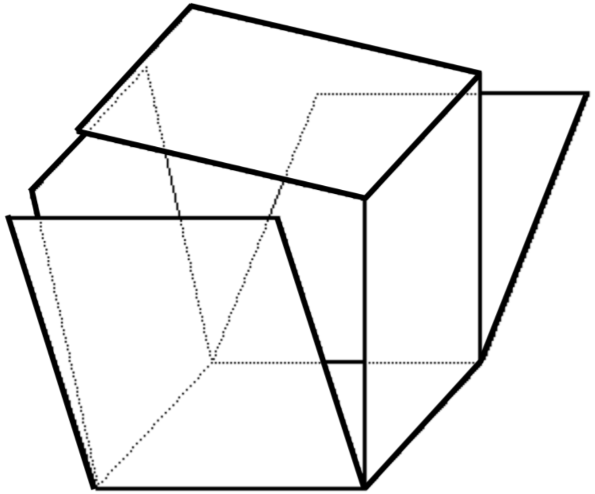
Unfolding Problems

- Given a 3D object, can it be unfolded into something flat?

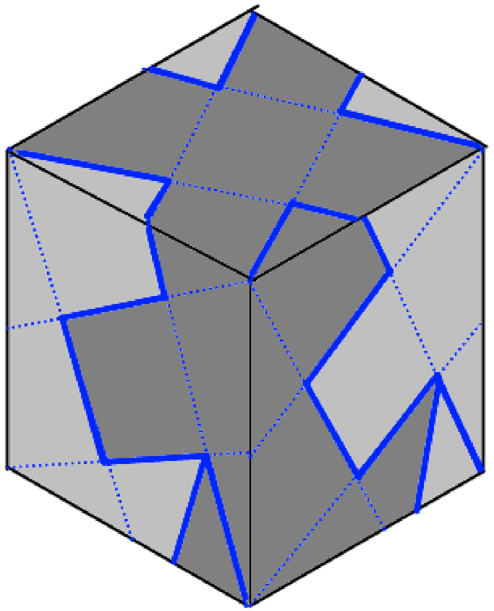
Two Different Unfolding Rules

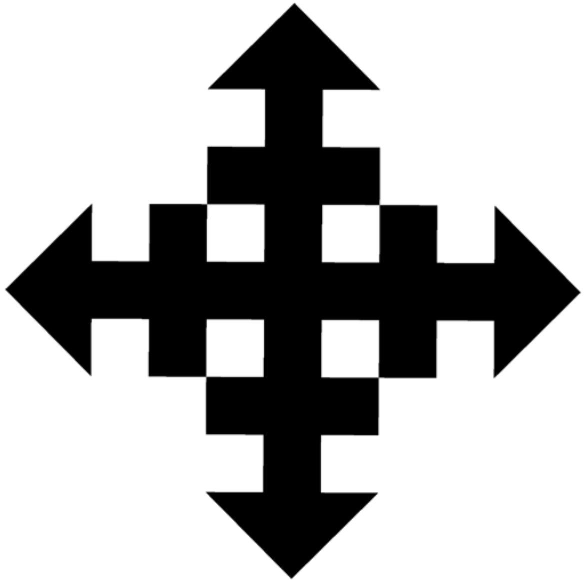
- Edge-unfolding





- General-unfolding

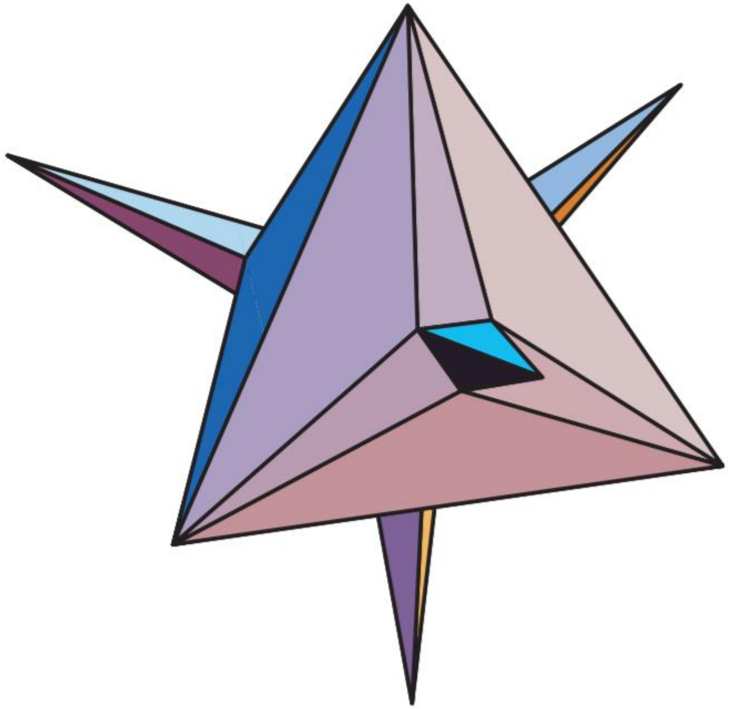




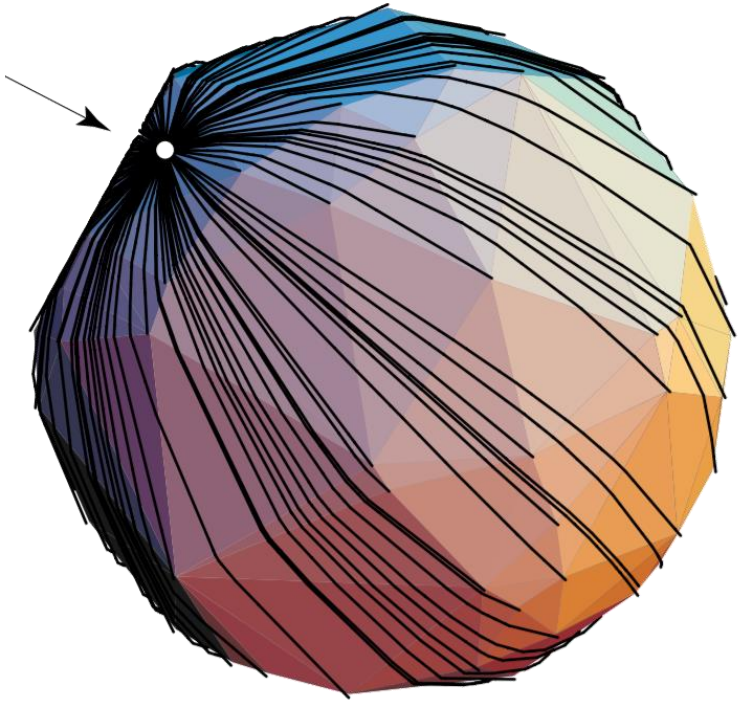
Known Results

Table 22.1: Status of main questions concerning nonoverlapping unfoldings

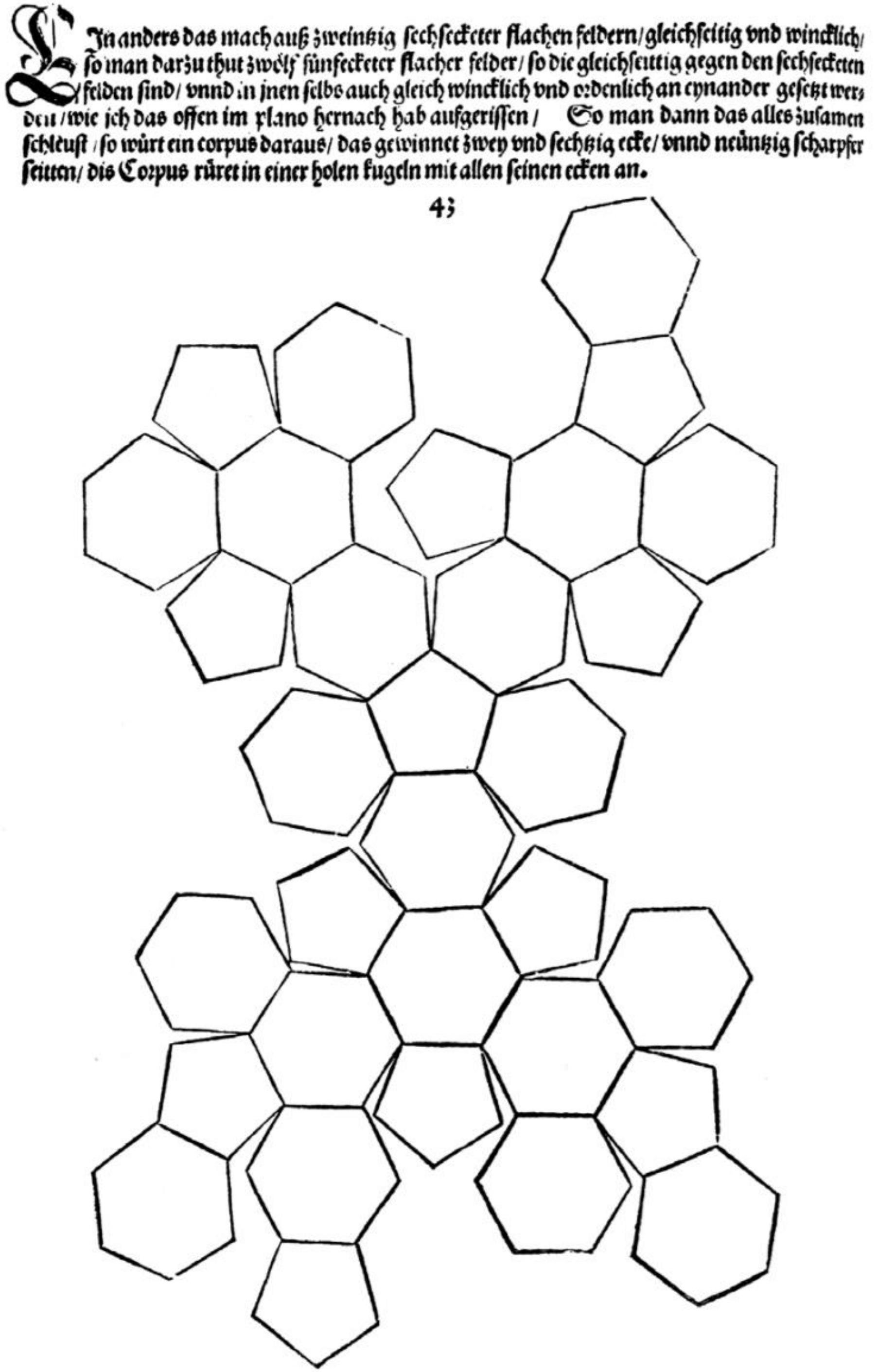
Shapes	Edge unfolding?	General unfolding?
Convex polyhedra	Open	YES
Nonconvex polyhedra	NO	Open



[Bern-Demaine-Eppstein-Kuo-Mantler-Snoeyink 2003]



[Xu 1996]



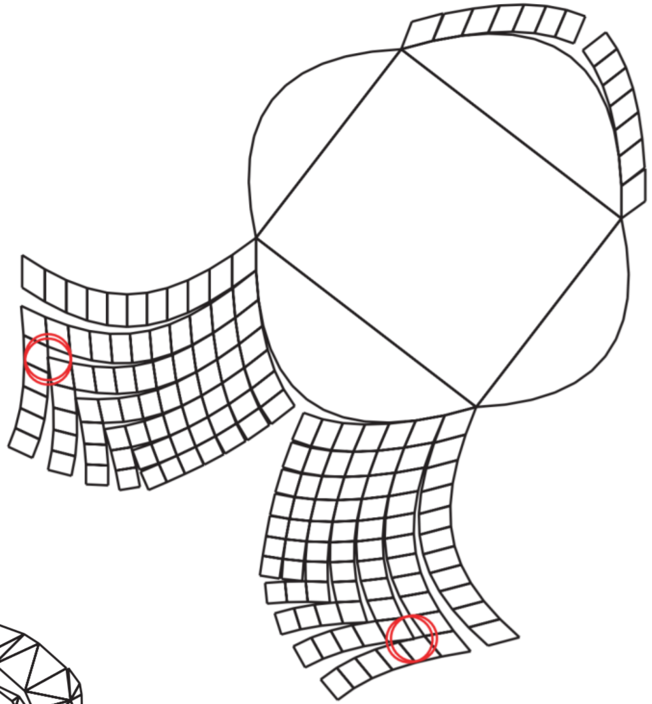
Ein anders das mach auß zweintzig ſechſeckter flachen feldern/ gleichſeitig vnd wincklich/ ſo man darzu thut zwölf fünfeckter flacher felder/ ſo die gleichſeittig gegen den ſechſeckten felden ſind/ vnnd in jnen ſelbs auch gleich wincklich vnd ordenlich an eynander geſetzt werden/ wie ich das offen im plano hernach hab aufgeriſſen/ So man dann das alles zuſamen ſchleuſt/ ſo würt ein corpus daraus/ das gewinnet zwey vnd ſechtzig ecke/ vnnd neüntzig ſcharpfer ſeitten/ dis Corpus rüret in einer holen kugeln mit allen ſeinen ecken an.

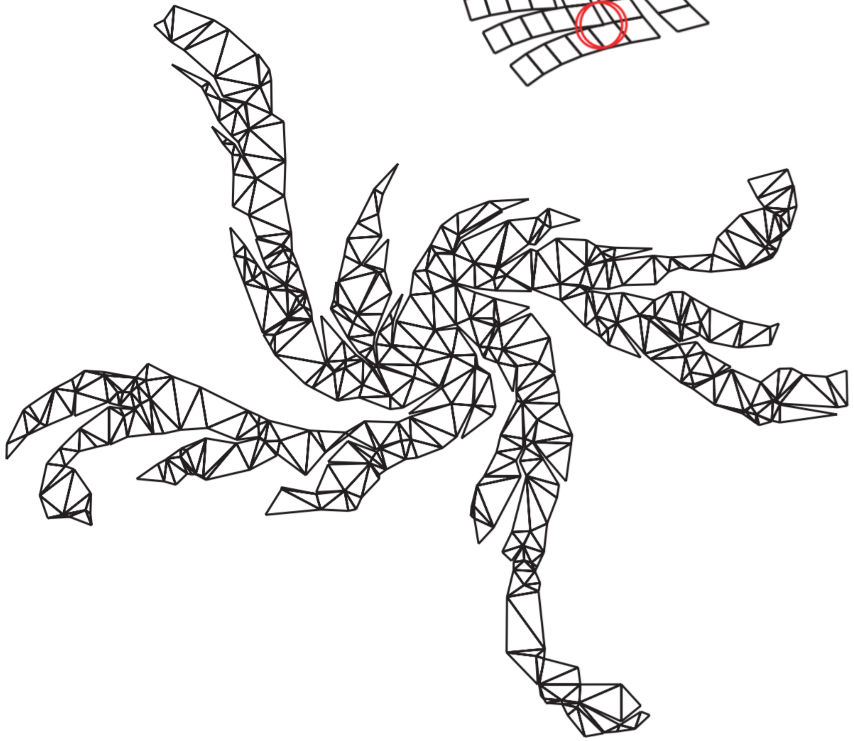
43

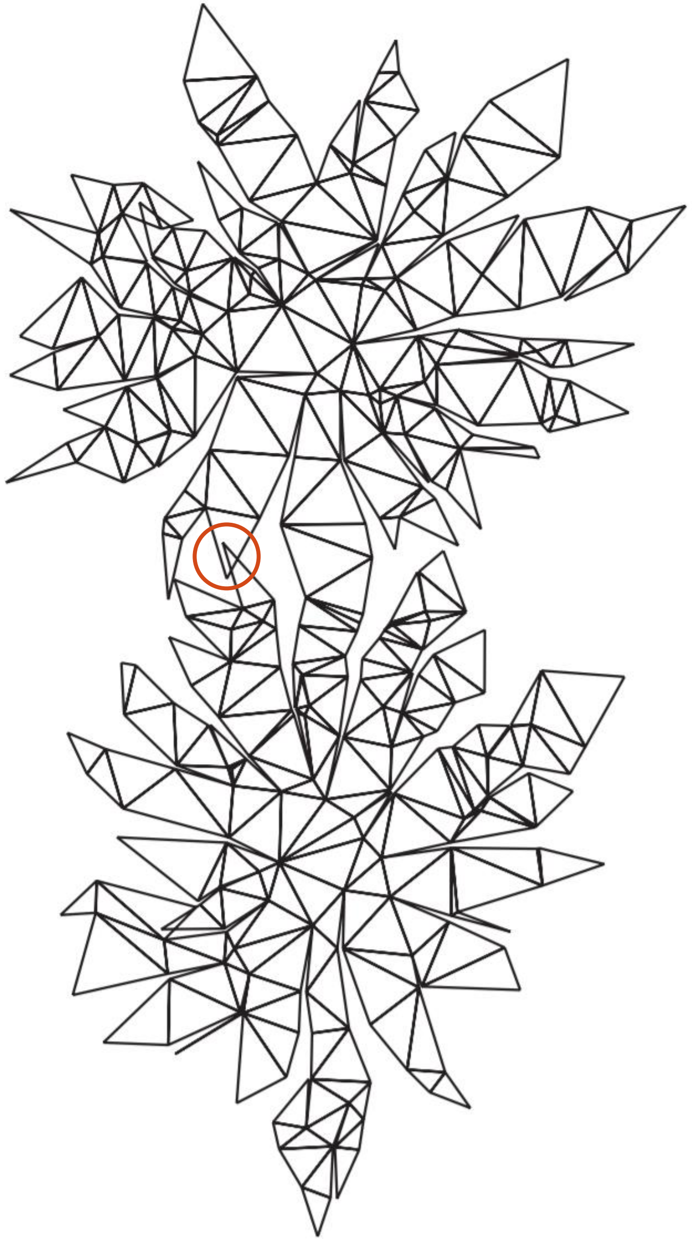
Unfolding Polyhedron

- **Open:**
  Does every convex polyhedron have an edge-unfolding?

[Albrecht Dürer 1525]
[Shephard 1975]
[Schlickenrieder 1997]





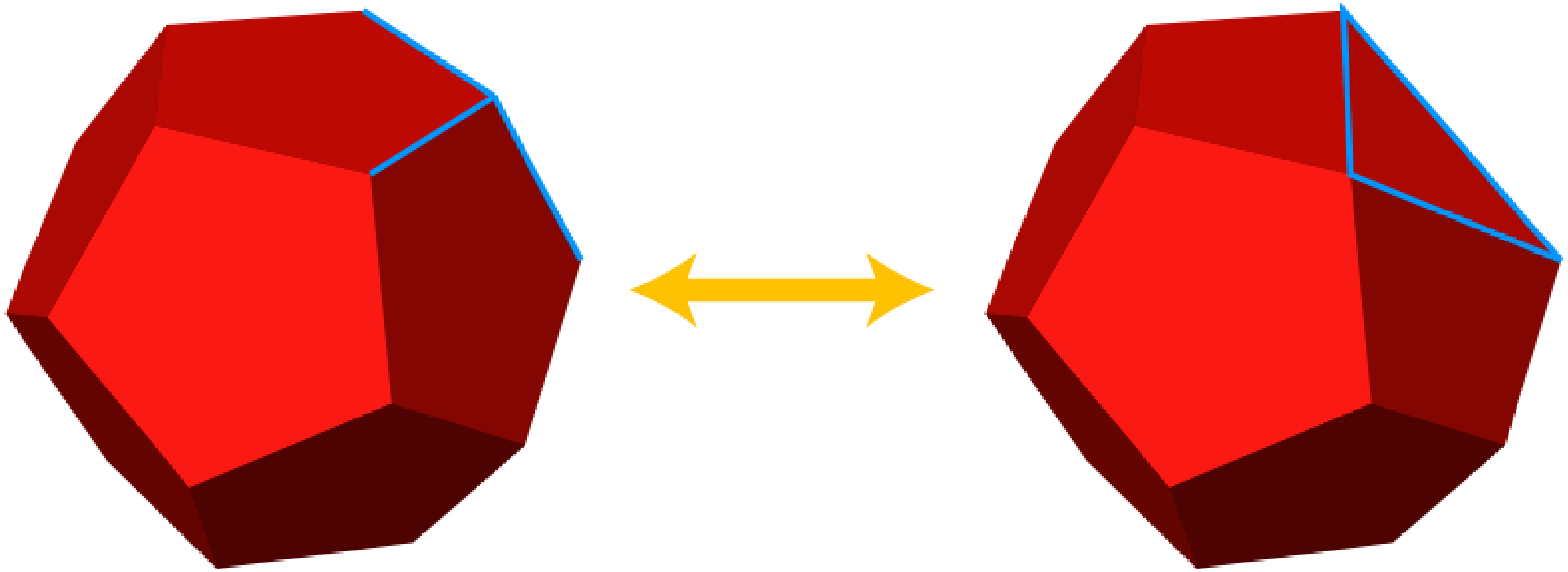


UNFOLDING POLYHEDRON

- **OPEN:**
  Does every convex polyhedron have an edge-unfolding?

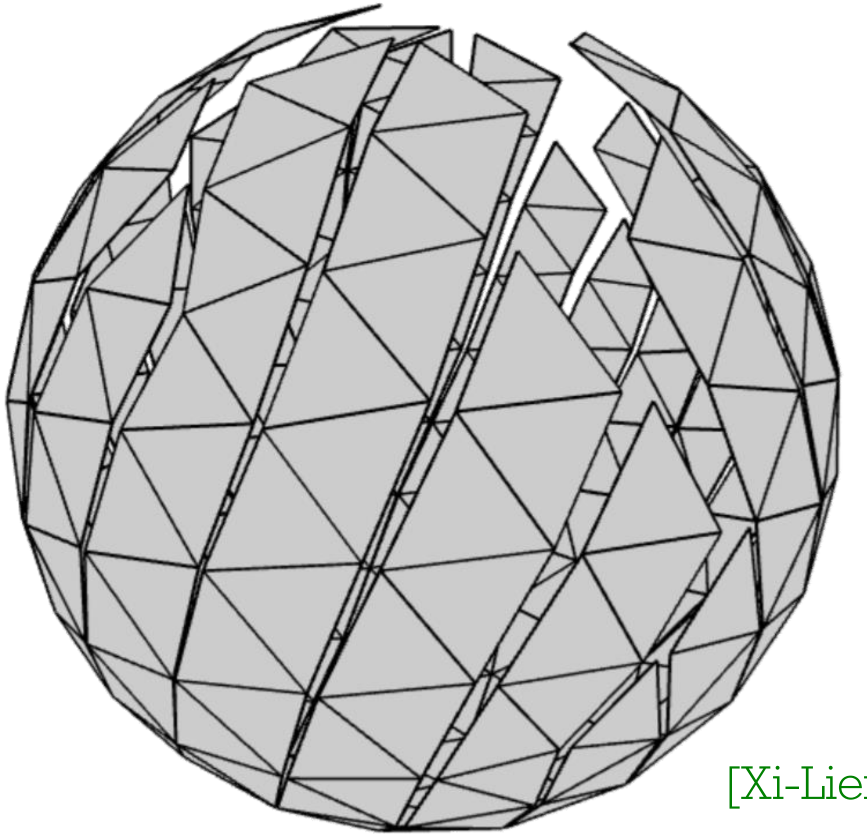
[Albrecht Dürer 1525]
[Shephard 1975]
[Schlickenrieder 1997]

When does an Unfolding Coming from a Polyhedron?



Steinitz's Theorem [Steinitz 1916]

A graph G forms a convex polyhedron if and only if G is simple, planar, and 3-connected

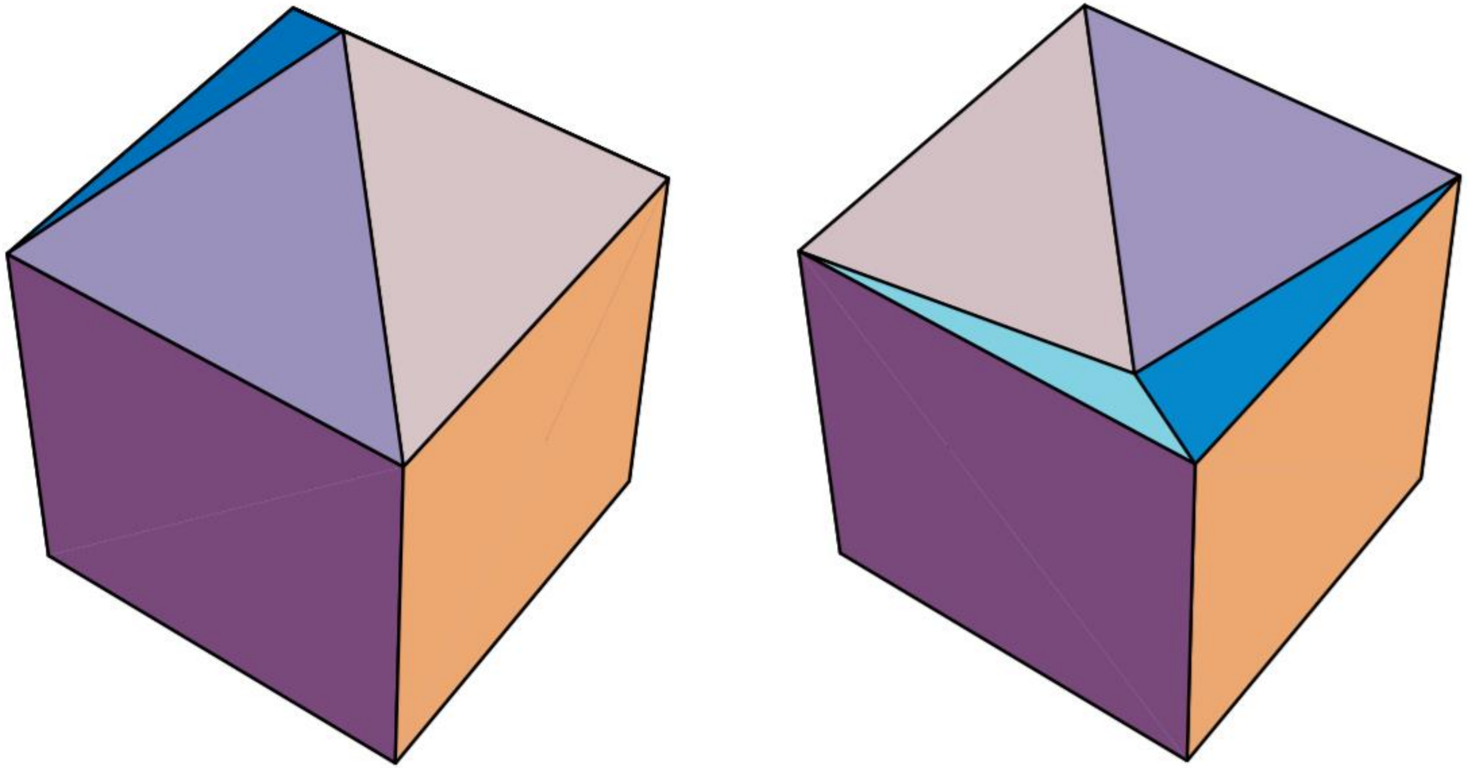


[Xi-Lien 2015]

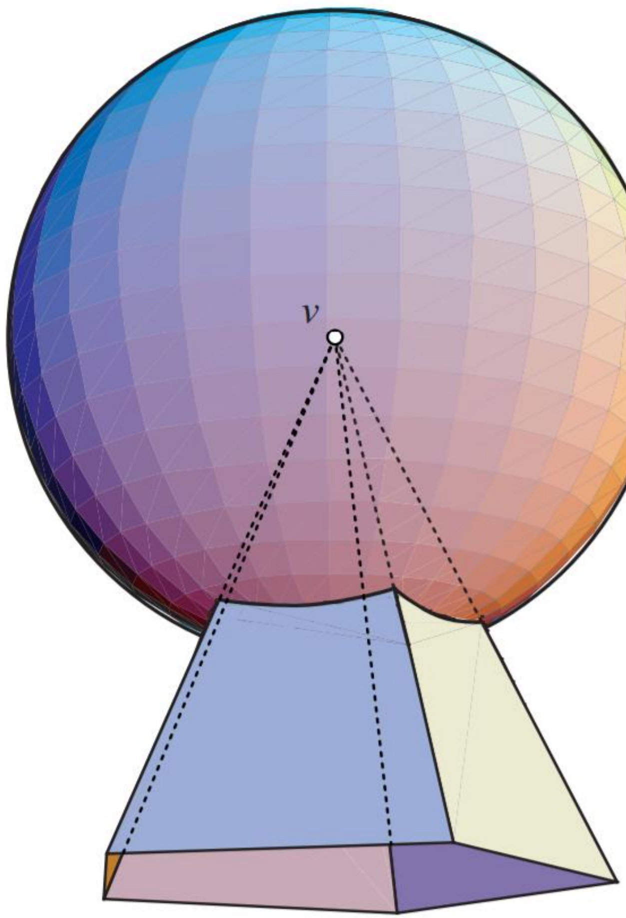
Cauchy Rigidity Theorem [Cauchy 1813] [Steinitz-Rademacher 1934]

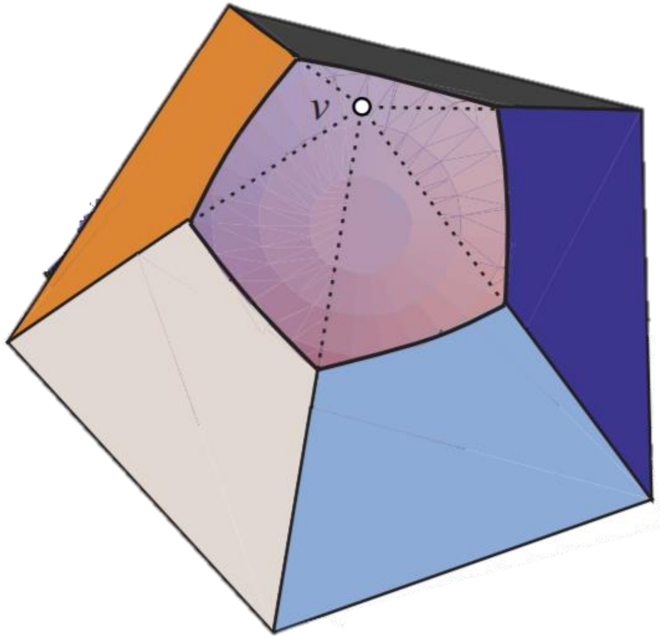
Two combinatorially equivalent, convex polyhedral must be congruent

Not True if Non-Convex

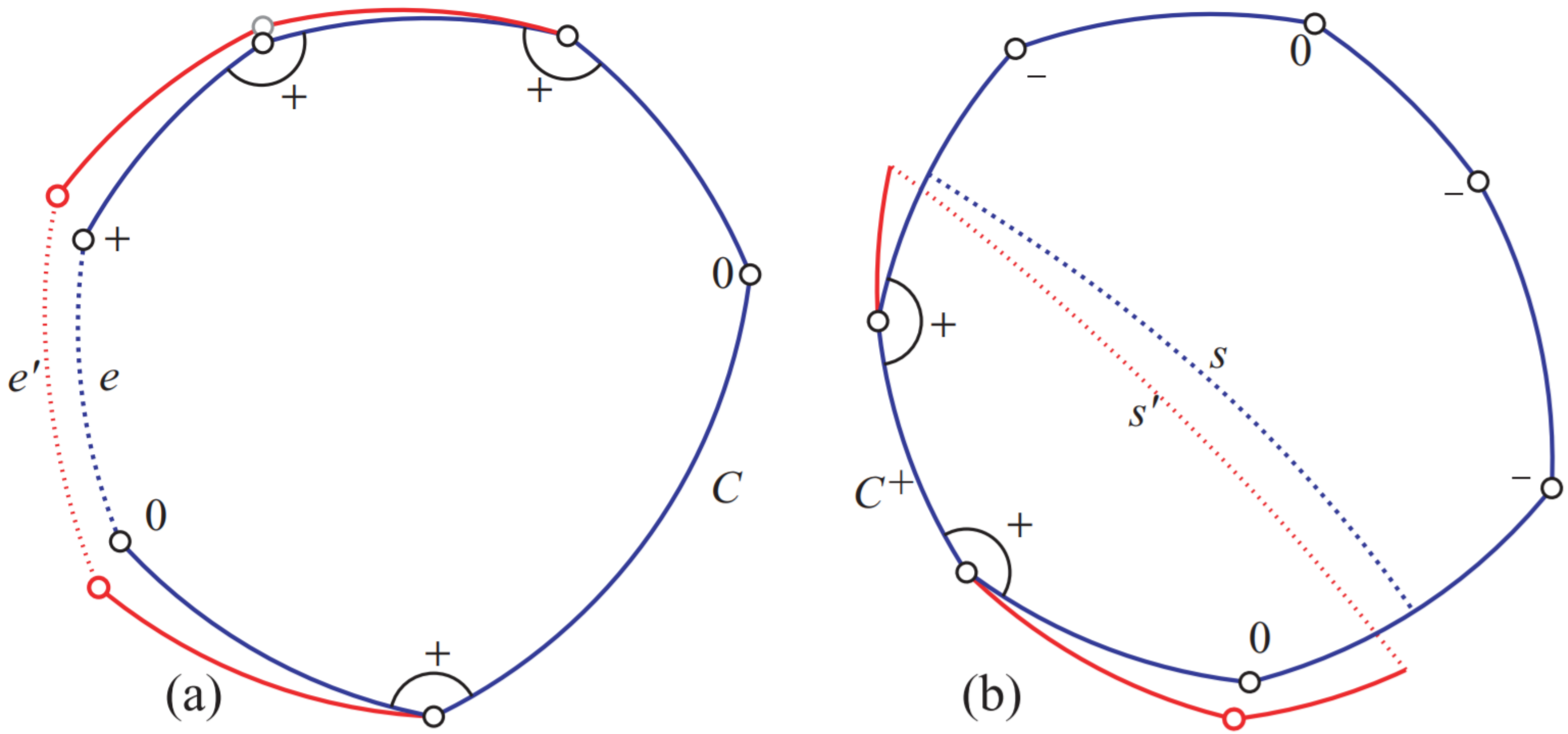


Proof of Rigidity Theorem



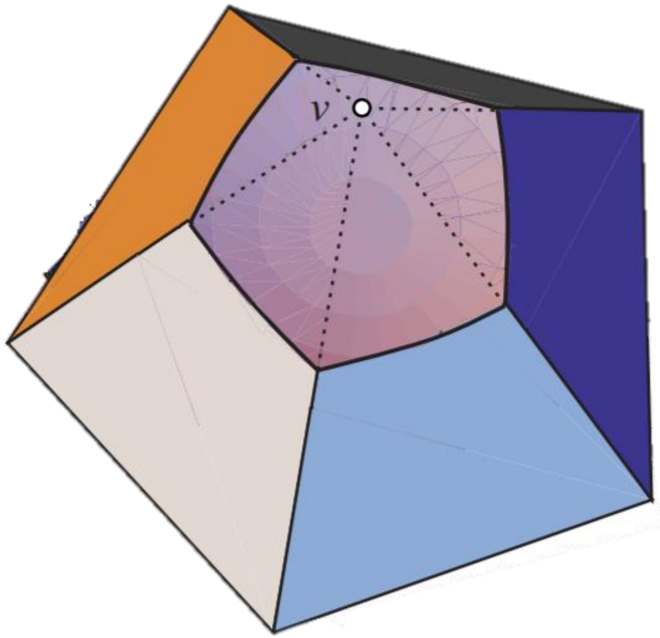


Cauchy-Steinitz Lemma. Any ±-labeling on the corners of convex polygon must have 4 alternations.



Proof of Rigidity Theorem

C-S Lemma. ≥4 alternations.



$V - E + F = 2$ $\quad 4V = (2 + E - F)\cdot 4$

$\Sigma := \#$ total alternations $= \sum_{v \in P} \mathrm{alt}(v)$

$\Sigma \geq 4\cdot V$ (by C-S Lemma)

$\boxed{4V > \Sigma^{+} \geq \Sigma \geq 4\cdot V}$ $\longrightarrow$ ✗

$< 4V$

$\Sigma^{+} \leq \underline{2\cdot f_3 + 4\cdot f_4 + 4\cdot f_5 + 6\cdot f_6 + \dots}$

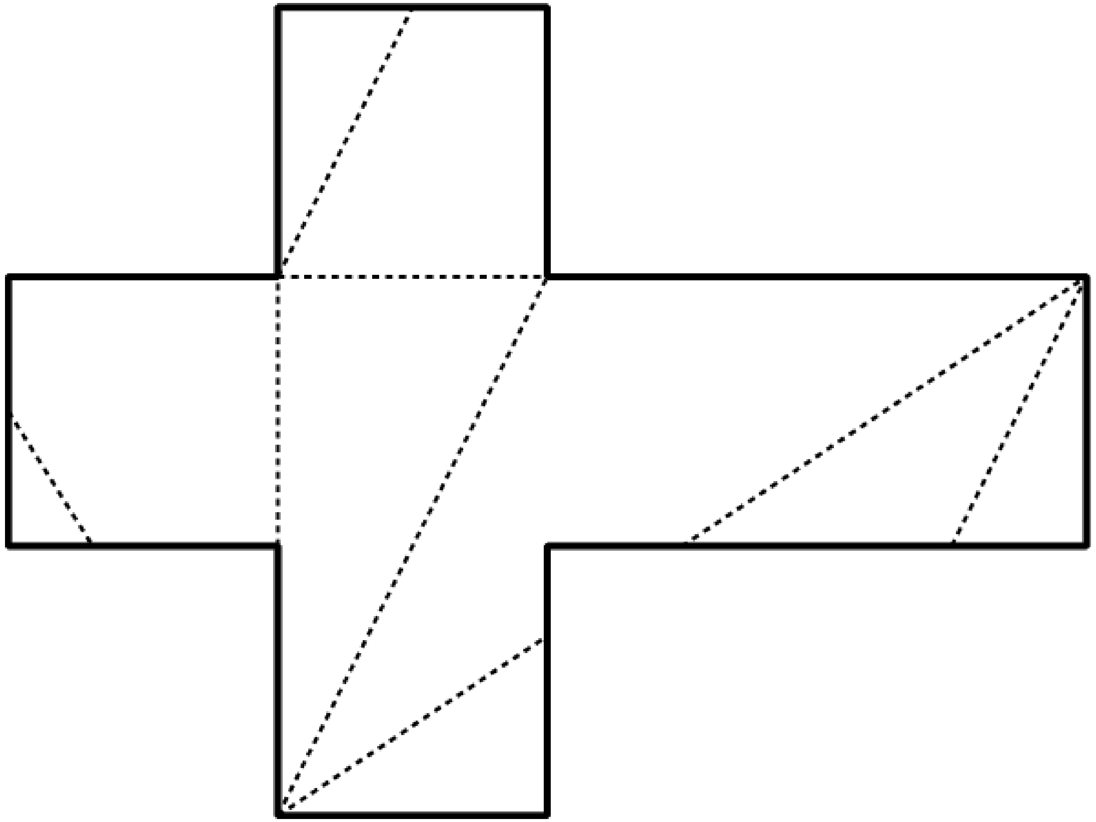
$2E = 3\cdot f_3 + 4\cdot f_4 + 5\cdot f_5 + 6\cdot f_6 + \dots$

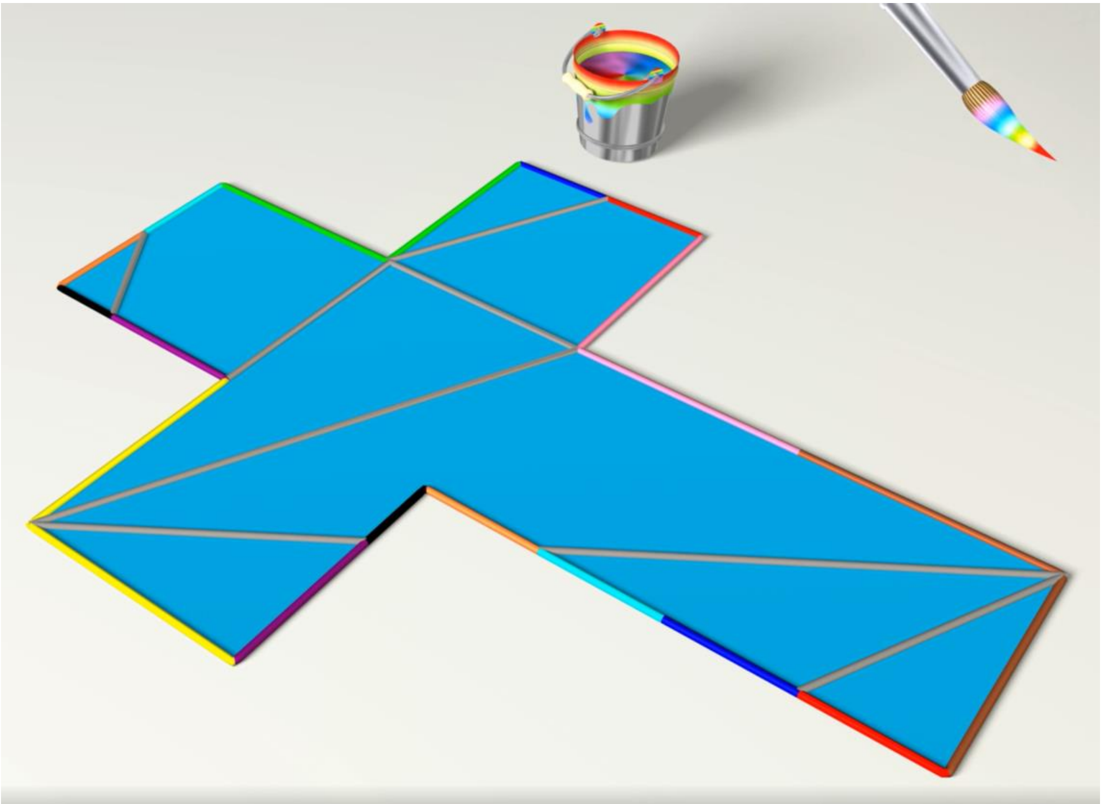
$F = f_3 + f_4 + f_5 + f_6 + \dots$

$4V = 8 + 6f_3 + 8f_4 + 10f_5 + \dots$

$\quad - 4f_3 - 4f_4 - 4f_5 + \dots$

$= 8 + 2f_3 + 4f_4 + 6f_5 + 8f_6 + \dots$





Alexandrov's Theorem [Alexandrov 1941]

Any gluing of polygons that (1) uses ≤2π angle per point
and (2) results in a topological sphere
must uniquely determine a convex polygon

Happy Folding!

Next time.

Behavior of closed curves dictates the shape of space.