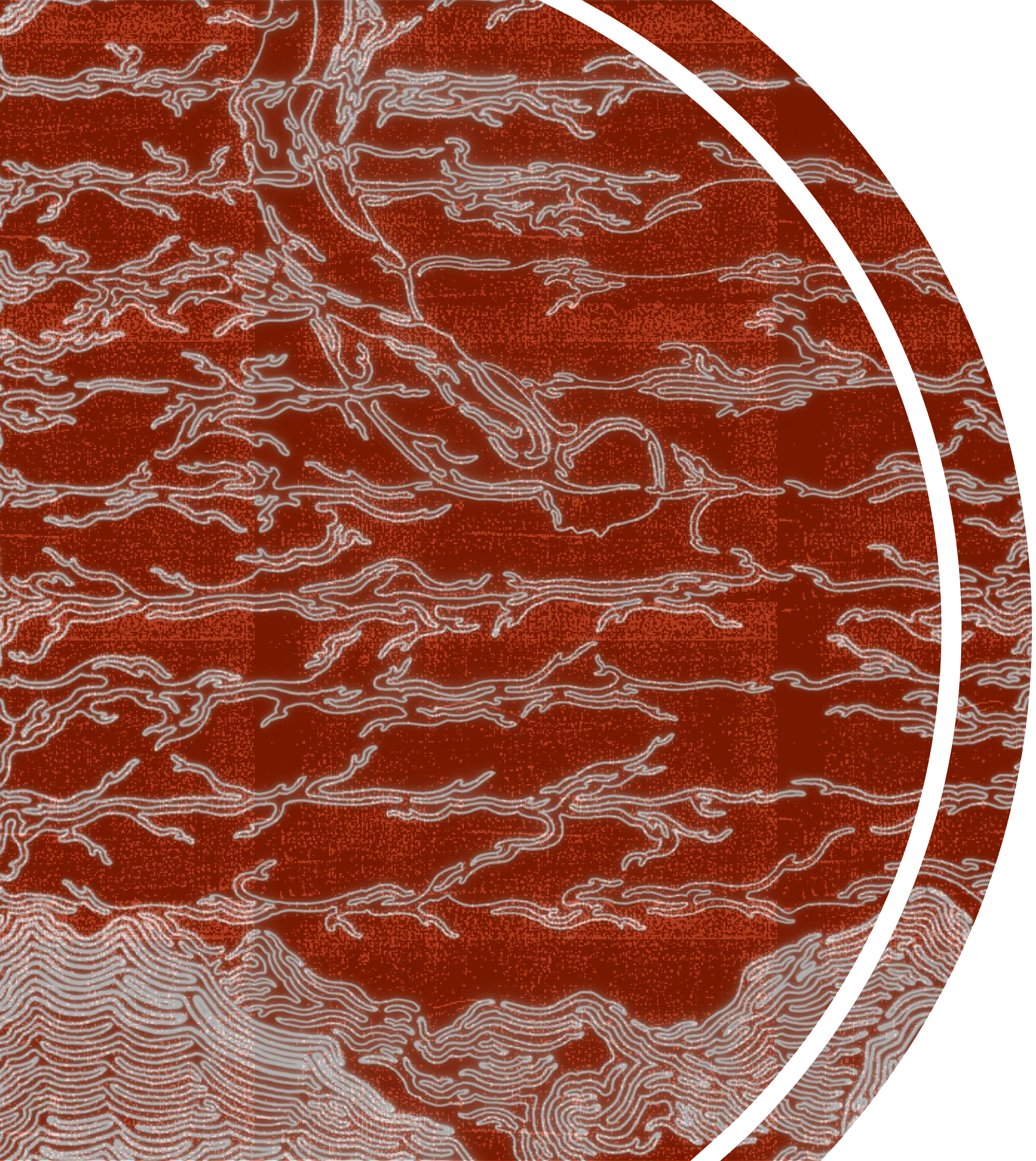


Introduction to

Computational Topology

Hsien-Chih Chang

Lecture 5, September 28, 2021

Administrivia

- Homework 2 is out, due 10/4 (next Monday)
- Is the pace and workload okay?

Geometric Folding Algorithms

6.849: Geometric Folding Algorithms: Linkages, Origami, Polyhedra (Fall 2020)

Prof. Erik Demaine; Martin Demaine; TAs Yevhenii Diomidov & Klara Mundilova

[Home] [Lectures] [Problem Sets] [Project] [Coauthor] [Github] [Accessibility]

Overview

*the algorithms behind building **TRANSFORMERS** and designing **ORIGAMI***

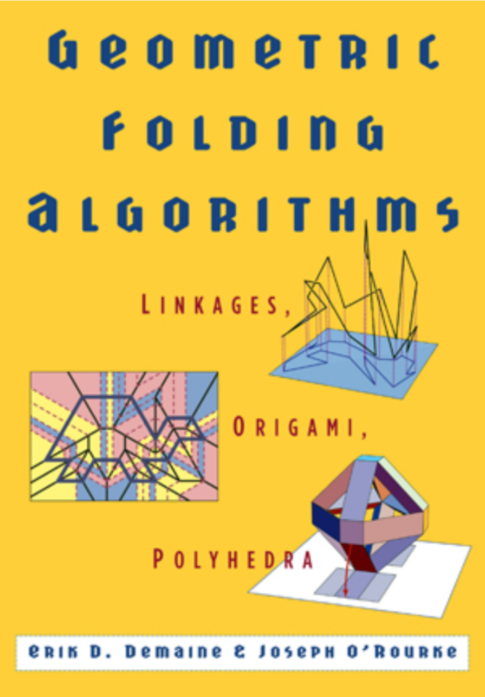
Whenever you have a physical object to be reconfigured, **geometric folding** often comes into play. This class is about algorithms for analyzing and designing such folds. Motivating applications include

- automated design of new and complex origami
- using 2D fabrication technology to manufacture complex 3D objects
- transforming robots by self-folding sheets or chains
- how to fold robotic arms without collision
- how to bend sheet metal into desired 3D shapes
- understanding how proteins fold

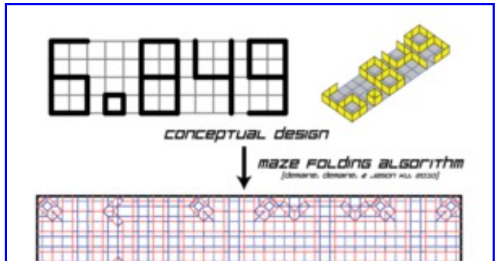
Major progress have been made in recent years in many of these directions, thanks to a growing understanding of the mathematics and algorithms underlying folding. Nonetheless, many fundamental questions remain tantalizingly unsolved. This class covers the state-of-the-art in folding research, including a variety of open problems, enabling the student to do research and advance the field.

Fully Online Format

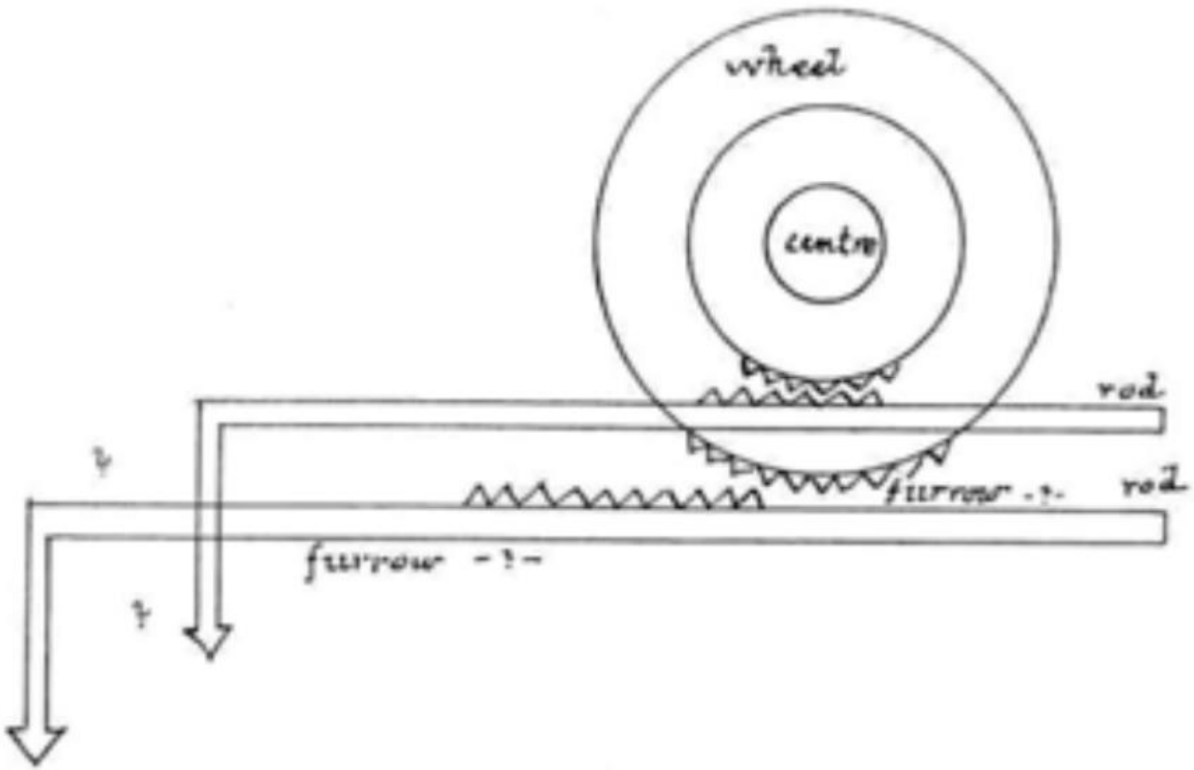
Most course material is covered in video lectures recorded in 2010 (already watched by over 19,000 people), which you can conveniently play at faster speed than real time. There may also be some **new material** presented by the professor and/or guest lecturers, which will be recorded for asynchronous viewing.



textbook



Linkage (1d Paper Folding)



Pantograph

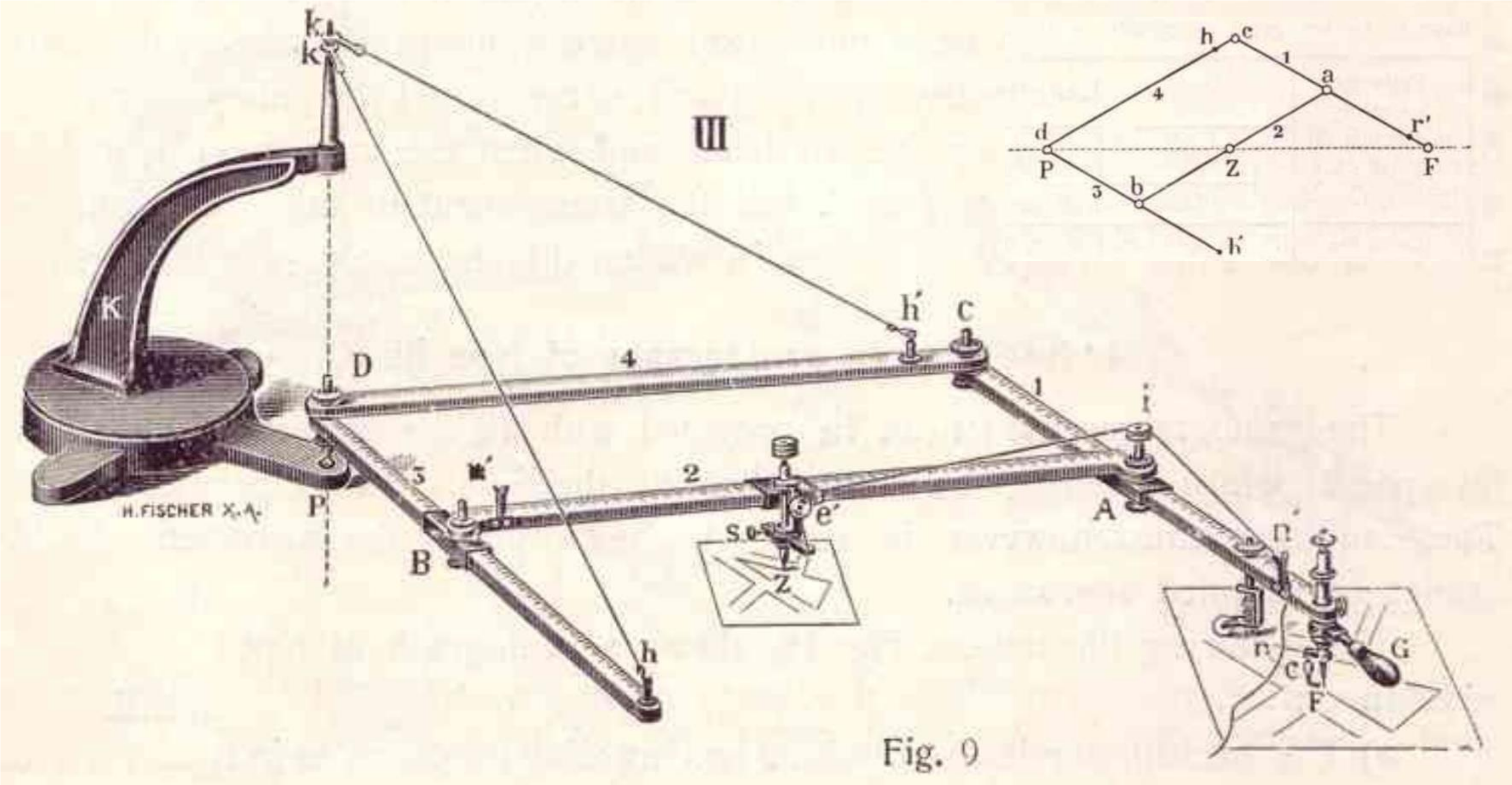
[Ήρων ο Αλεξανδρεύς, 10-85 AD]

And let us now prove how to make a figure similar to the known plane figure in the given ratio by means of an instrument.

We make two wheels on the same centre, fixed to it, and provided with teeth, moving on a single axle in the plane where is the figure we want to copy; and the ratio of one wheel to the other should be the given ratio …

Pantograph

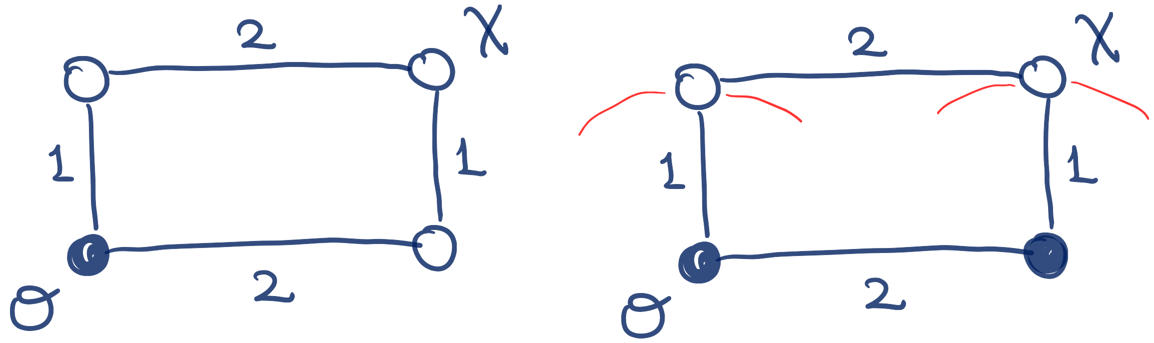
[Ήρων ο Αλεξανδρεύς, 10-85 AD]



[Coradi, 1914]

What is a Linkage?

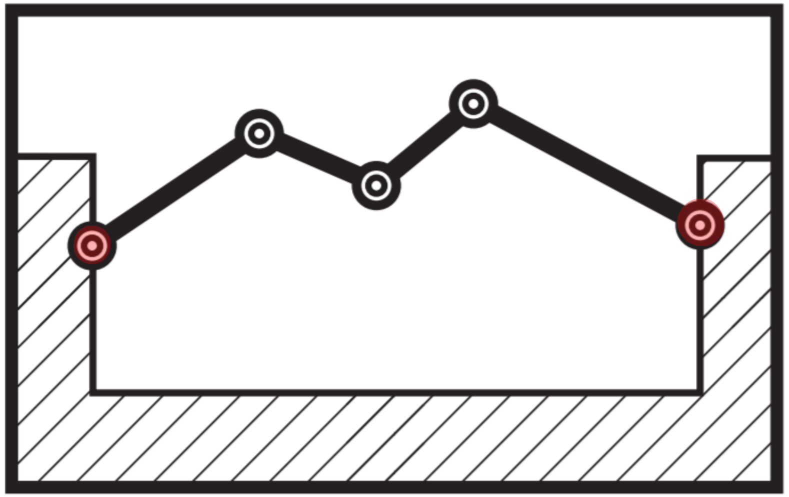
- Graph with rigid bars



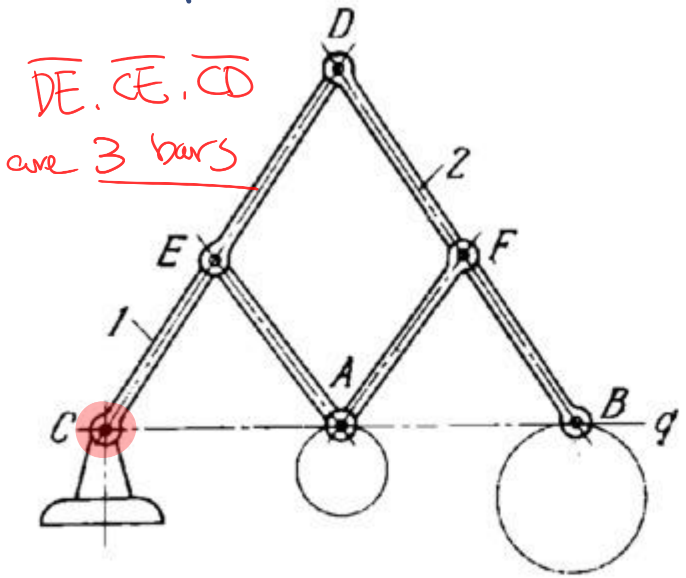
- Self-intersect?

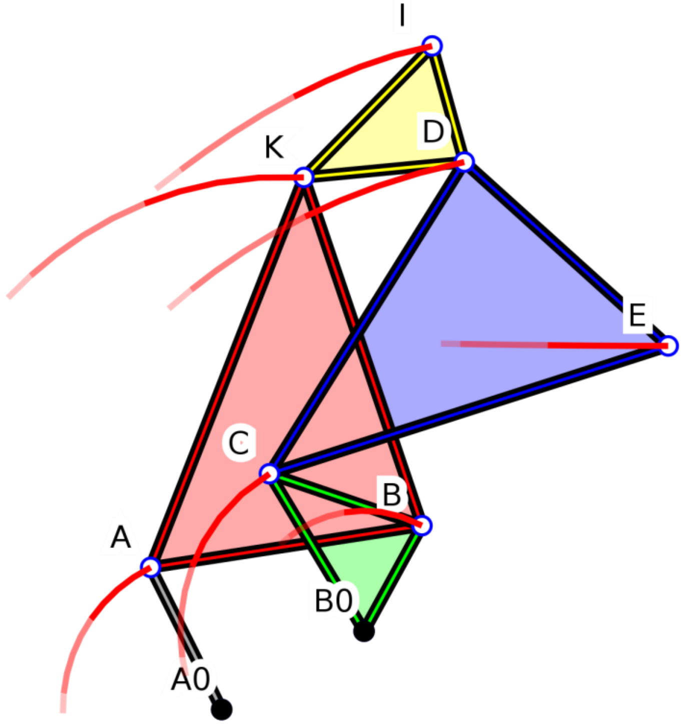
Degree-of-Freedom

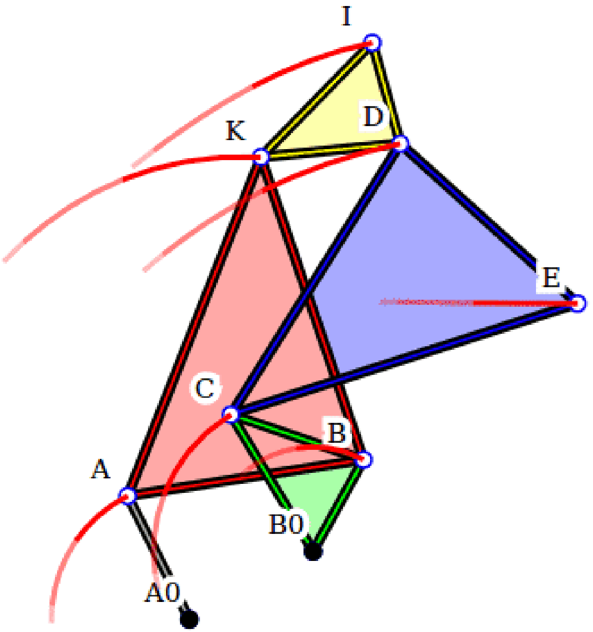
$DoF = 2 \cdot 3 - 4 = 2$



$DoF = 2 \cdot 5 - 8 = 2$







$DoF = 2 \cdot \#\text{free joints} - \#\text{bars}$

Linkage Problems

- Reconfiguration vs Reachability
- Obstacles?
- Allowing self-intersection?

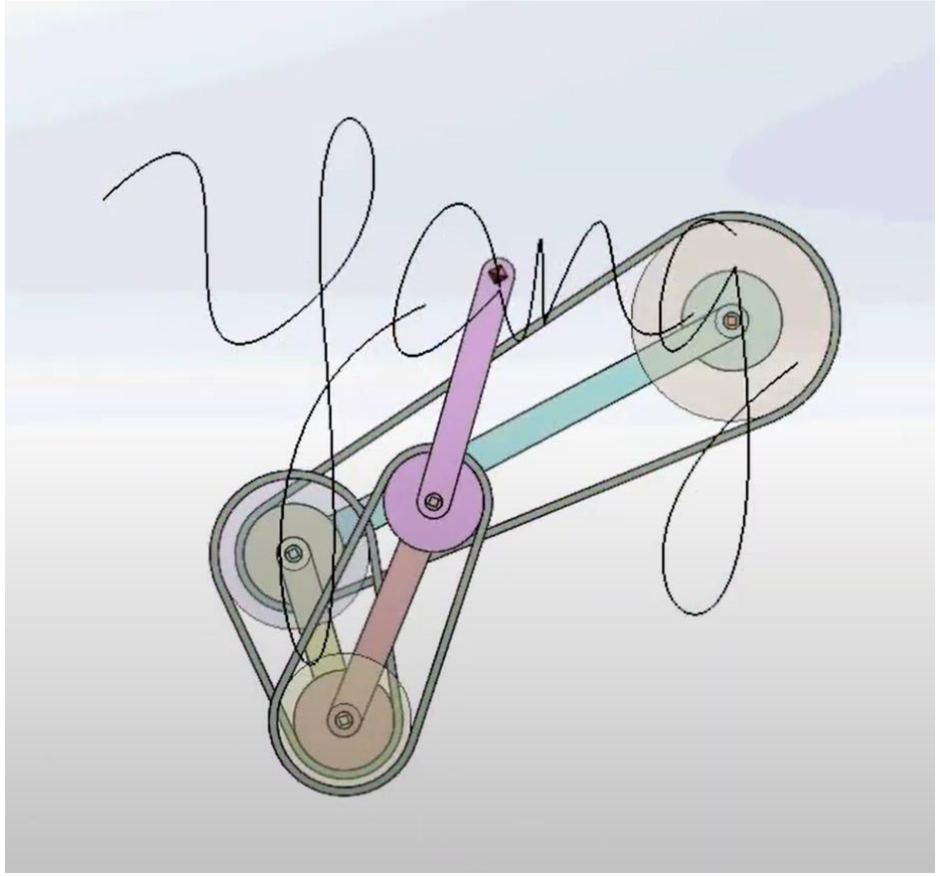
Is the Configuration Space Connected?

Known Results

Table 2.1: Lower bounds[a]

Dim	Link graph	Linkage intersection	Obstacles	Question	Complexity
3D	Tree	Simple	Polyhedra	Reachability	PSPACE-complete (Reif 1987)
3D	Chain	Simple	None	Reconfiguration	PSPACE-complete (Alt et al. 2004)
2D	Graph	Permitted	None	Reachability	PSPACE-complete (Hopcroft et al. 1984)
2D	Chain	Permitted	4 segments	Reachability	NP-hard (Hopcroft et al. 1985)
2D	Chain	Permitted	Polygons	Reachability	PSPACE-complete (Joseph and Plantinga 1985)
2D	Tree	Simple	None	Reconfiguration	PSPACE-complete (Alt et al. 2004)

[a] "Simple" means non-self-intersecting.



Universality Theorem

[Kempe 1876] [Kapovich-Millson 2002]

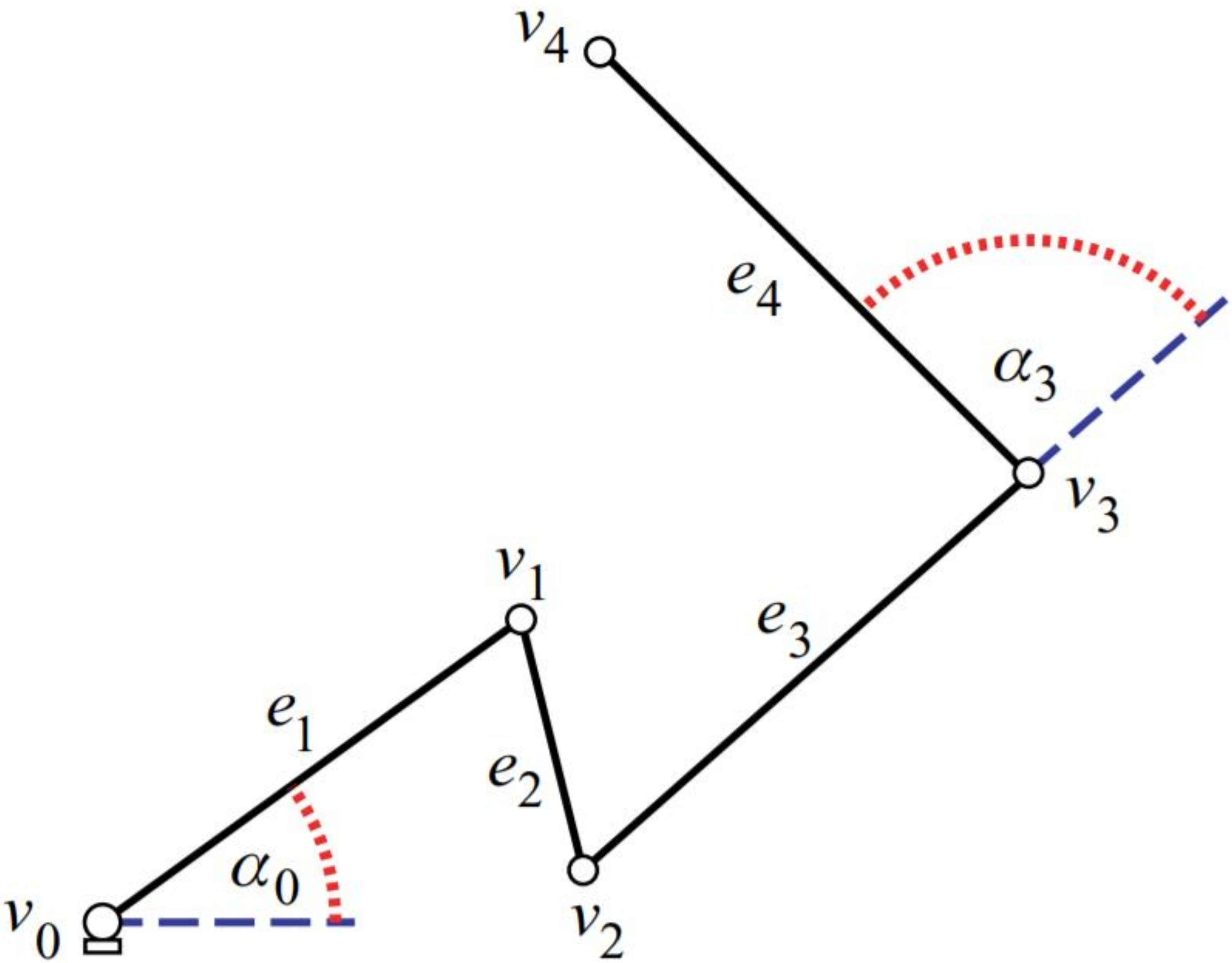
There is a planar linkage whose orbit is
a bounded portion of any algebraic curve.

"There is a planar linkage that signs your name"

Next Important Questions

- Reconfigure 2D chain with/without self-intersection, with no obstacles?

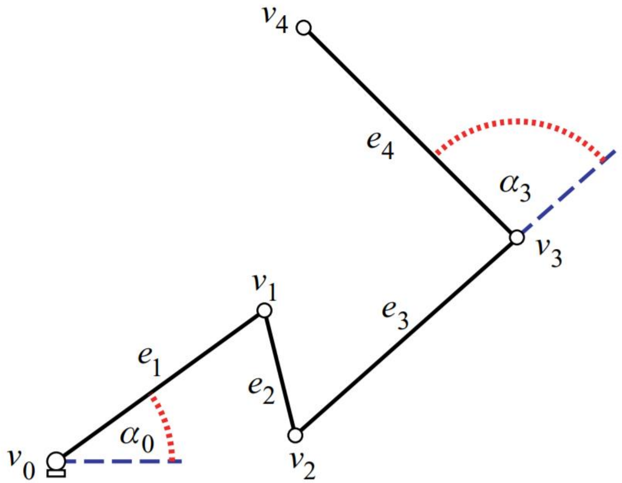
2D Open Chains

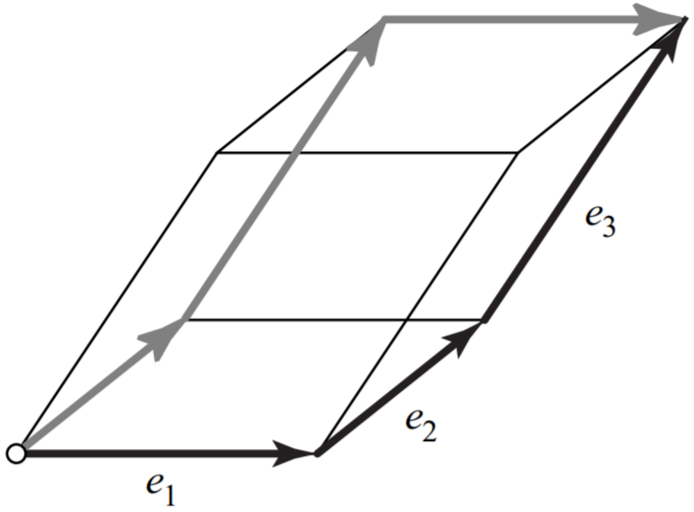


Open Chain

- What are the set of points reachable by v_n?

(Self-intersections allowed)





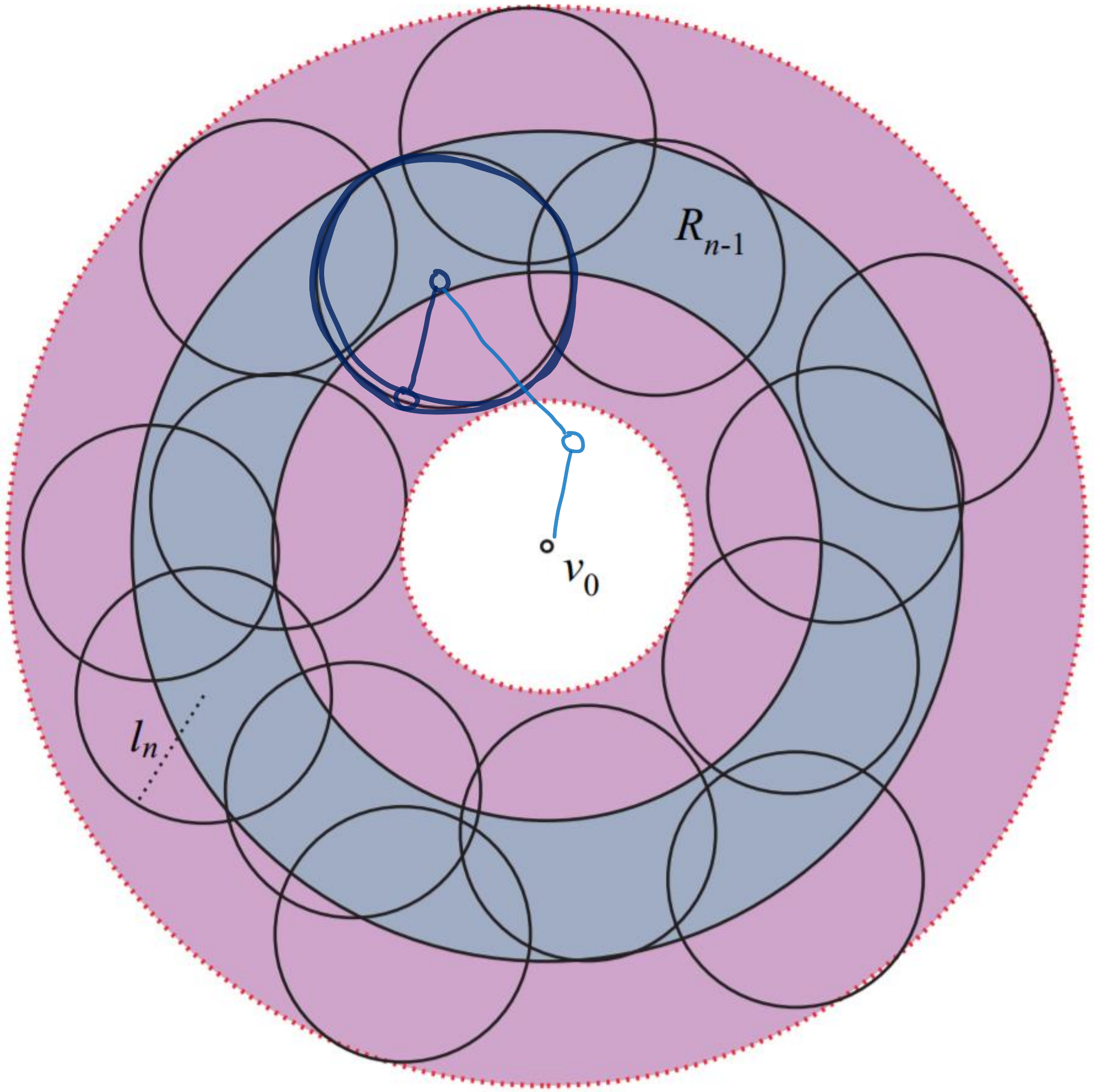
Claim. Rearranging bars doesn't change reachability region.

So, rearrange and make the longest bar e_1.

Open Chain

- What are the set of points reachable by v_n?

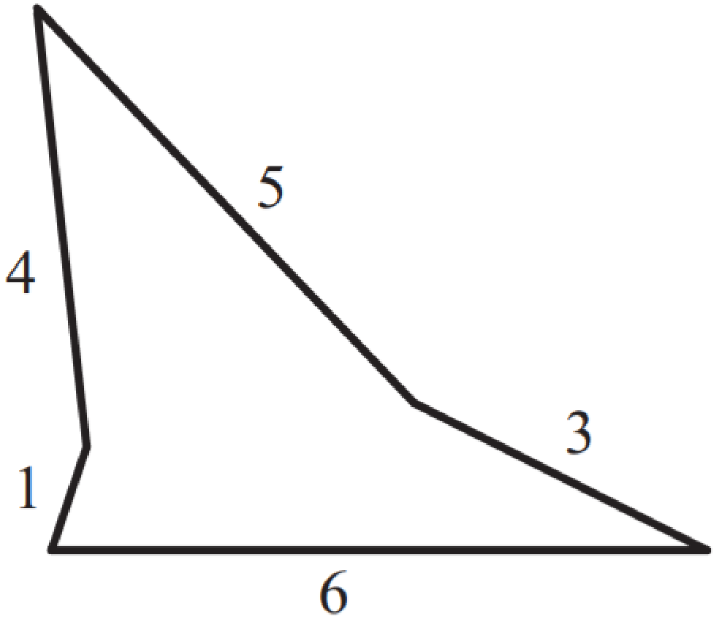
(Self-intersections allowed)

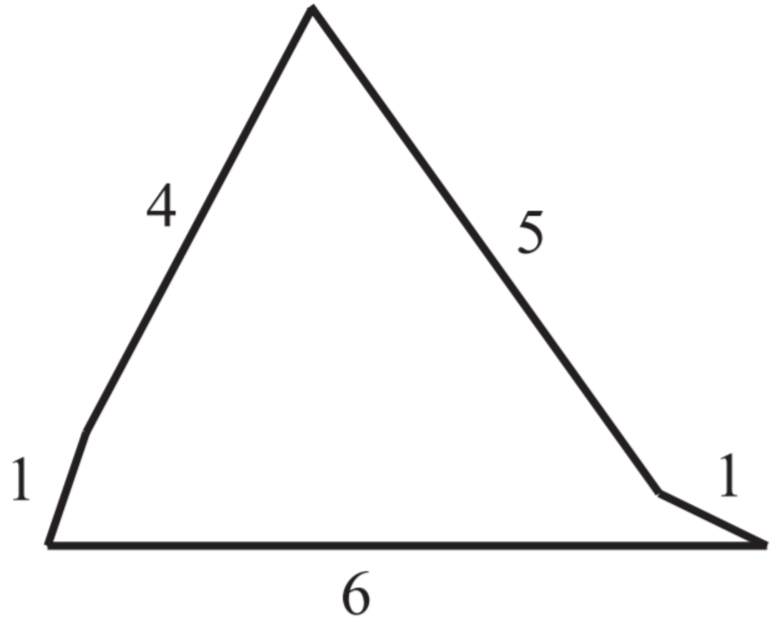


Open Chain

- What are the set of points reachable by v_n?

(Self-intersections allowed)

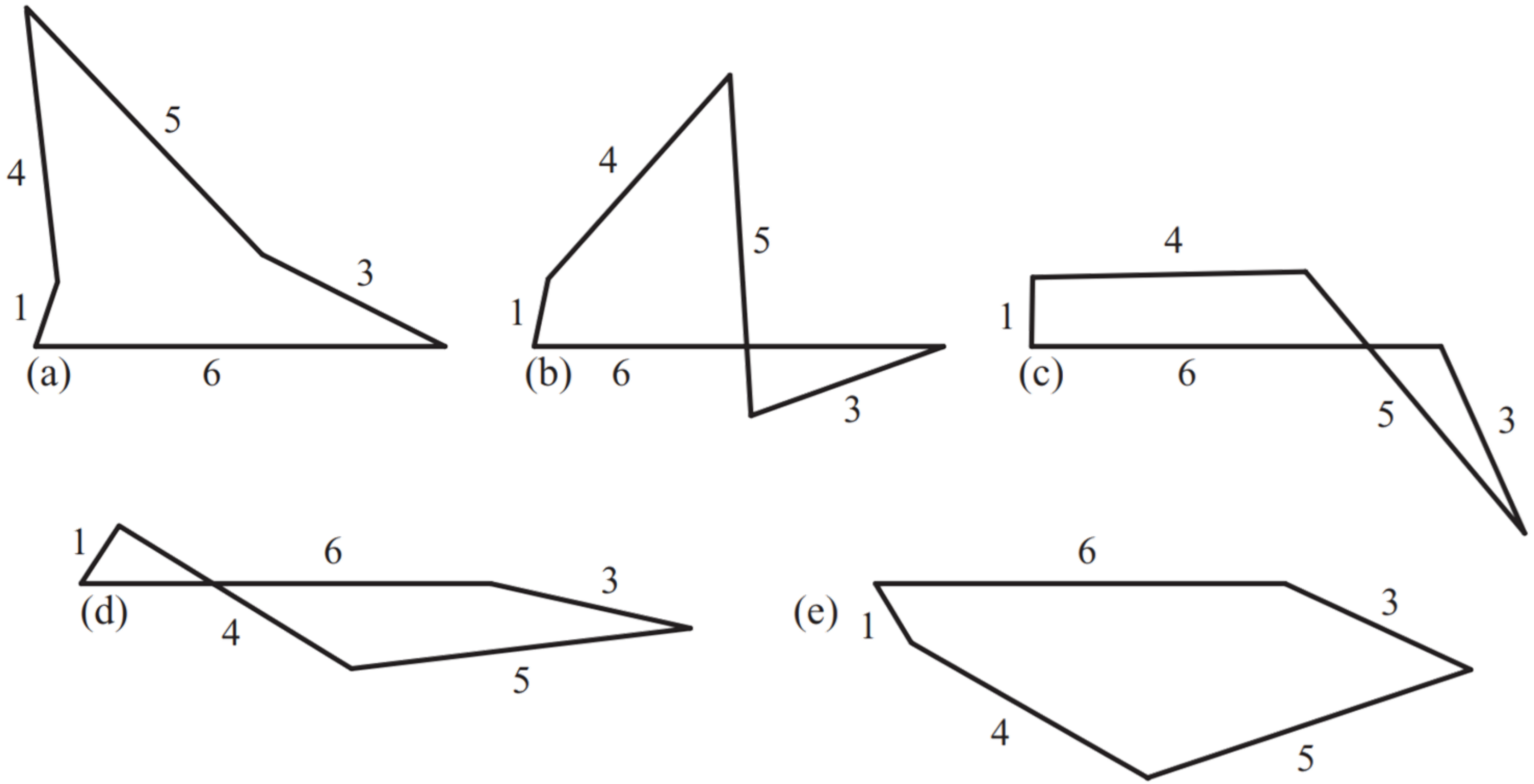




Polygon Inside-Out

- When can you turn a polygon inside-out?

(Self-intersections allowed)



Polygon Inside-Out

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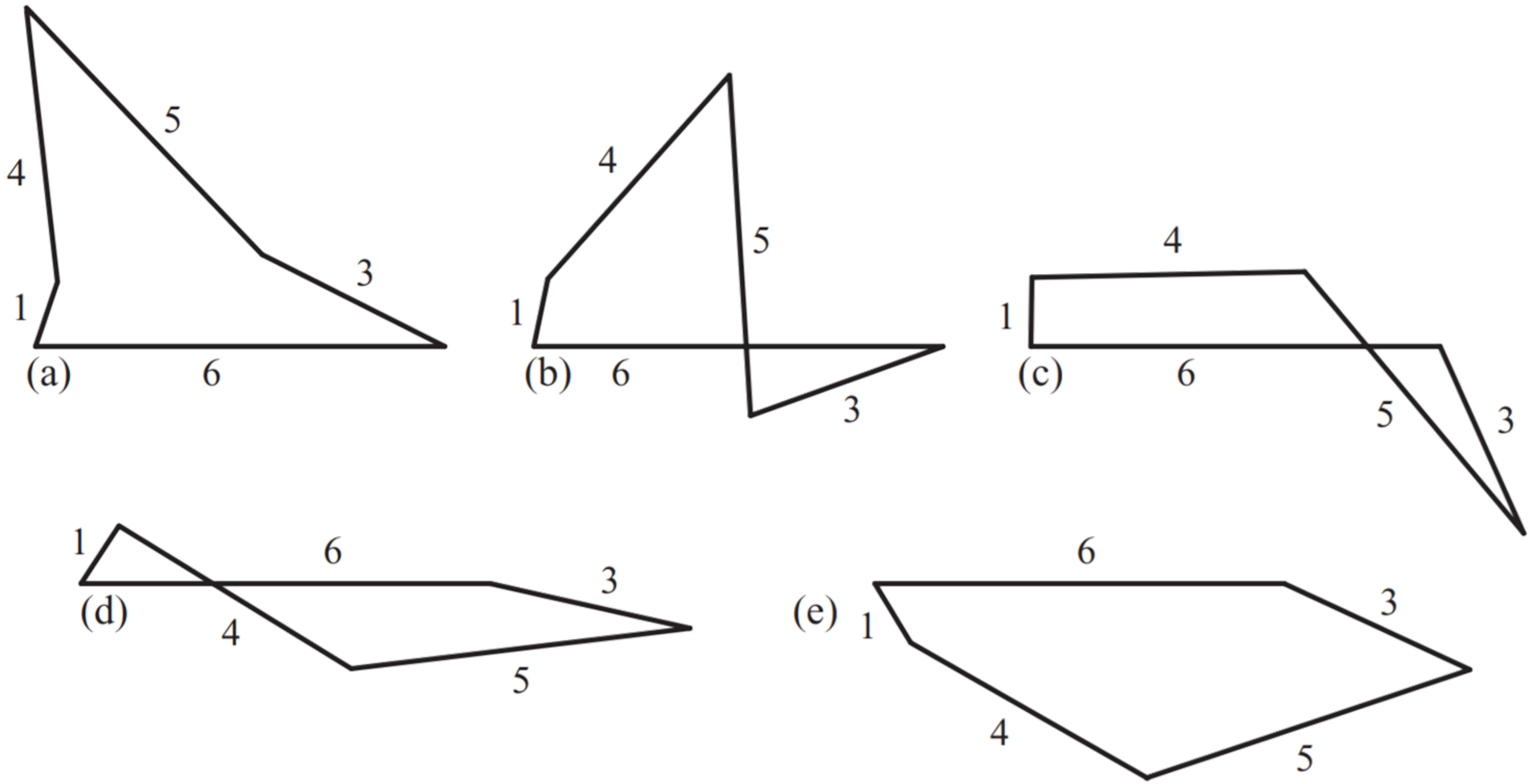


Polygon-Inverting Theorem

[Kapovich-Millson 1995] [Lenhart-Whitesides 1995]

A polygon is invertible if and only if

$$\ell_2+\ell_3 \leq \ell_1+\ell_4+\ldots+\ell_n$$

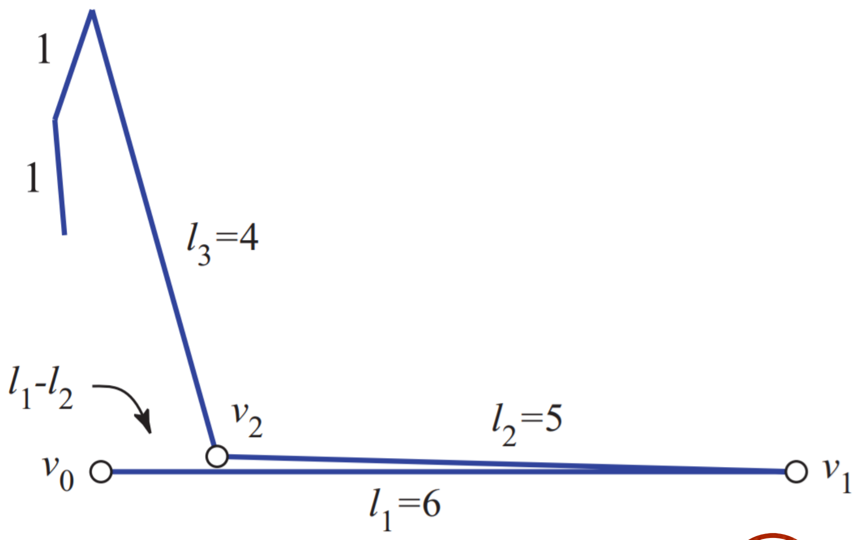
Necessity Proof of Polygon-Inverting Theorem

Rearrange bars so that second longest bar is e_2.

To flip e_2 below e_1, v_2 will lie on $\overline{v_0 v_1}$.

$(l_1 - l_2) + \sum_{i \geq 4} l_4 \geq l_3$ for the rest of the chain reaching v_0.

$\Leftrightarrow \quad l_1 + \sum_{i \geq 4} l_i \geq l_2 + l_3$



Sufficiency Proof of Polygon-Inverting Theorem

$B_1 : B(x, \sum_{k=1}^{j-1} l_k)$

$B_2 : B(z, l_j)$

Δ-form exists, because

1. $l_1 + \cdots + l_{j-1} \le l_j + \cdots + l_n$: B_1 not containing B_2
2. $l_j \le l_1 + \cdots + l_{j-1} + l_{j+1} + \cdots + l_n$: B_2 not containing B_1.

(o.w. no Δ-form)

3. $(l_1 + \cdots + l_{j-1}) + l_j \overset{(2)}{>} L/2 \overset{(2)}{>} l_{j+1} + \cdots + l_n$

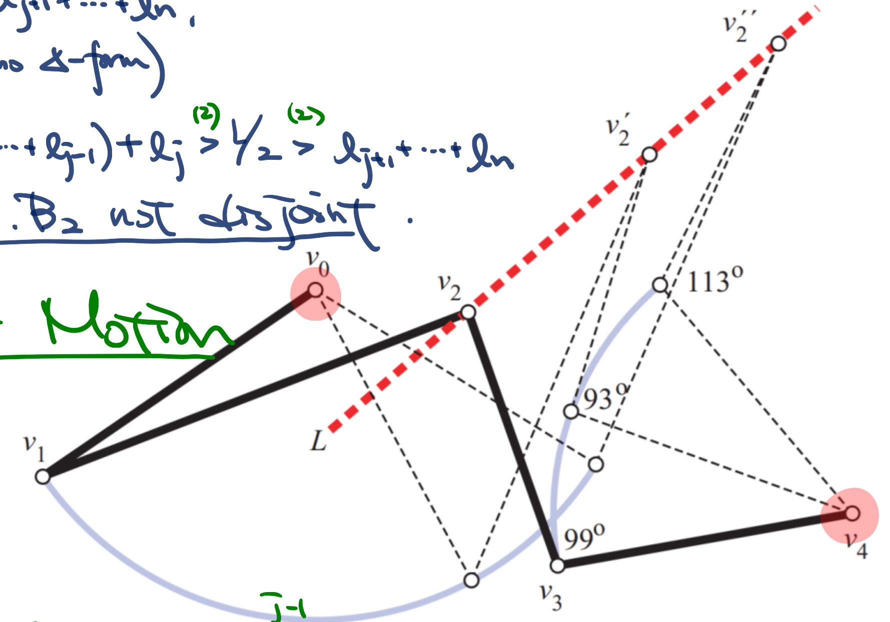
B_1, B_2 not disjoint.

Δ-Form(P):

- remove all straight joints
- while ∃ ≥ 4 joints left:
  - LinearMotion(v_0, v_1, v_2, v_3, v_4) (until either v_1 or v_3 is straight)
  - remove all straight joints
- // call resulting $\Delta = (u, v, w)$
- adjust(x) = LinearMotion(w, u, x, v, w) if x is on $\overline{uv}$.
- adjust(y)
- adjust(z)

all non-straight. v_4 might be v_0.

Linear Motion



Δ-form: (1) $\sum_{k=1}^{j-1} l_k \le L/2$.

(2) $\sum_{k=1}^{j} l_k > L/2$

$L := \sum_{k=1}^{n} l_k$

x, y, z, l_1, l_{j-1}, l_j

Intermission

Food for thought.

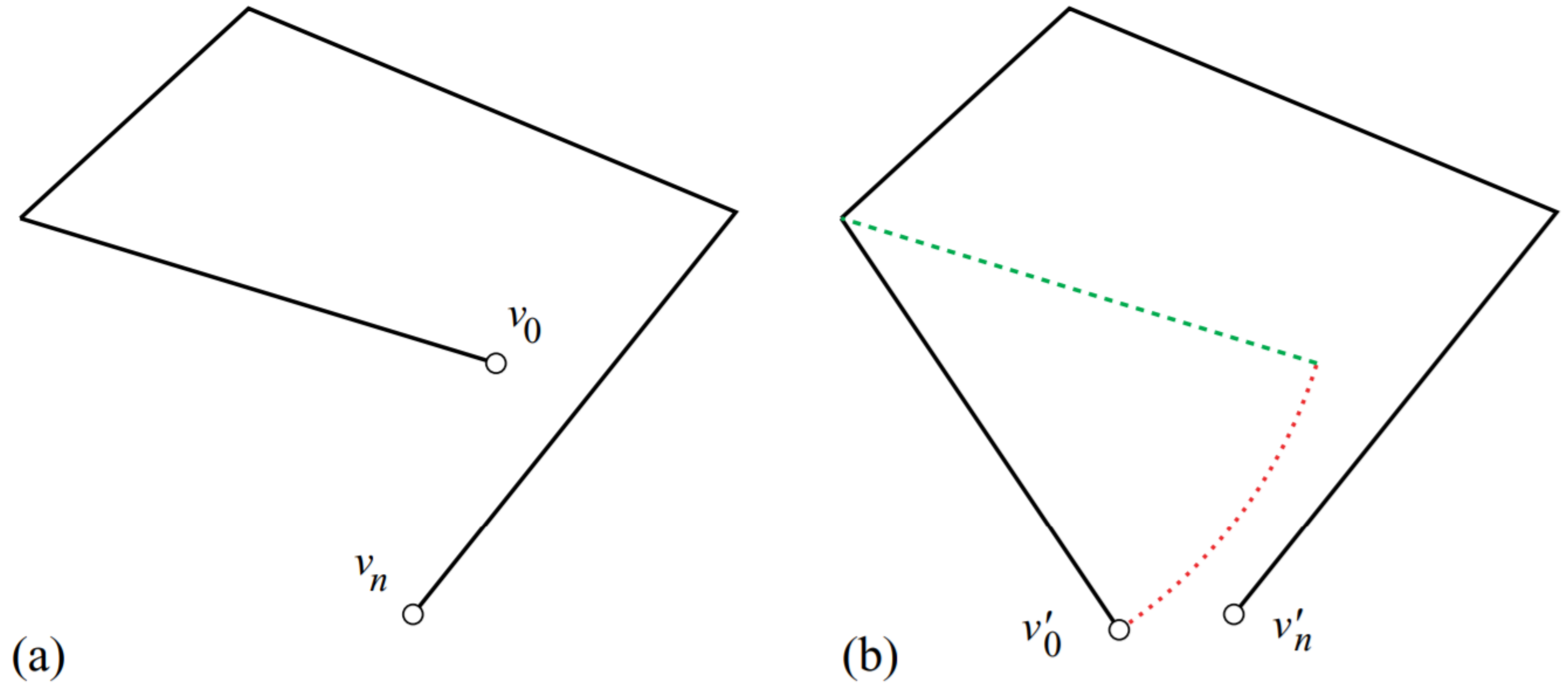
Linear-time implementation?

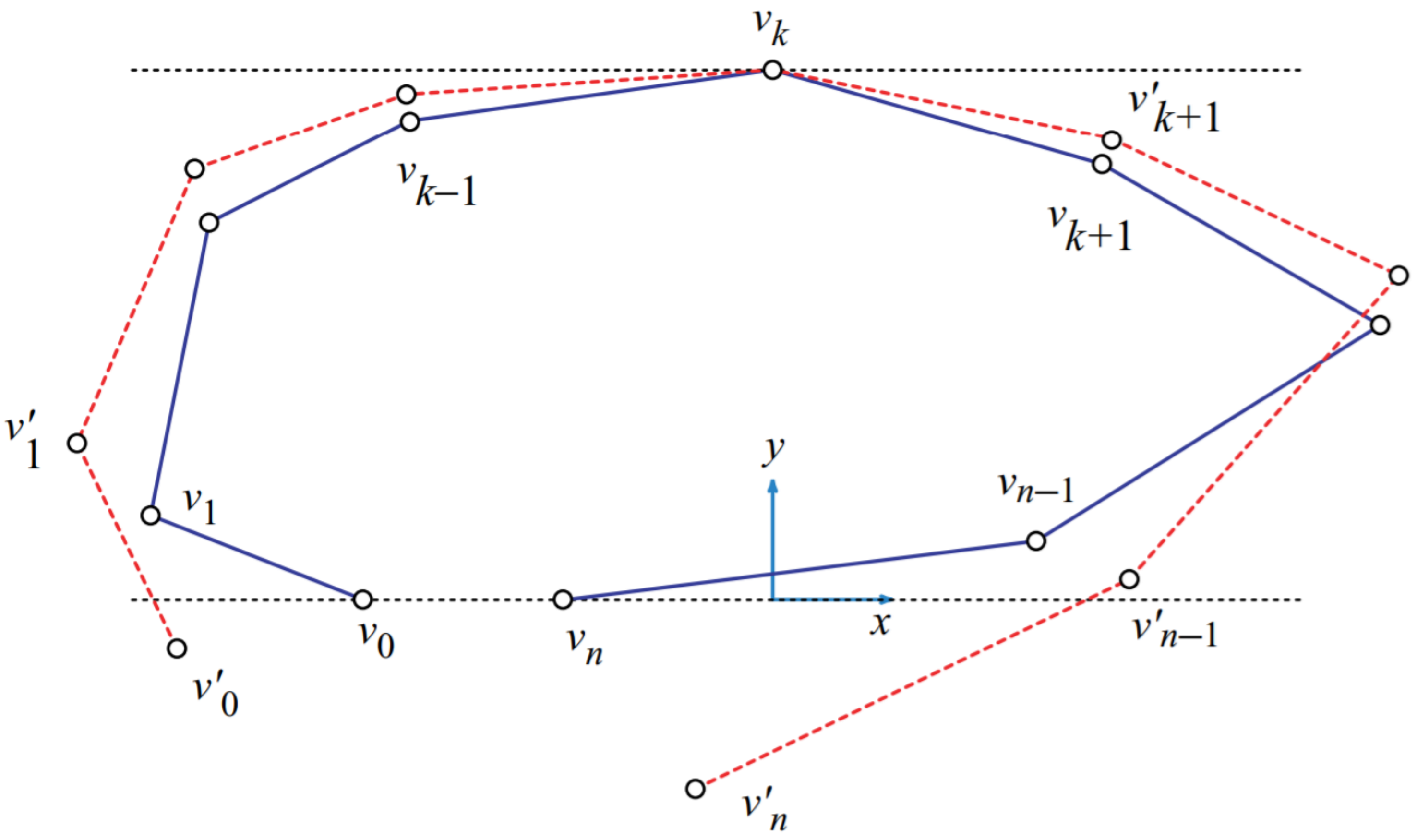
What if Self-Intersections are Not Allowed?

Convex Open Chain

- Reconfiguration?

(Self-intersections disallowed)

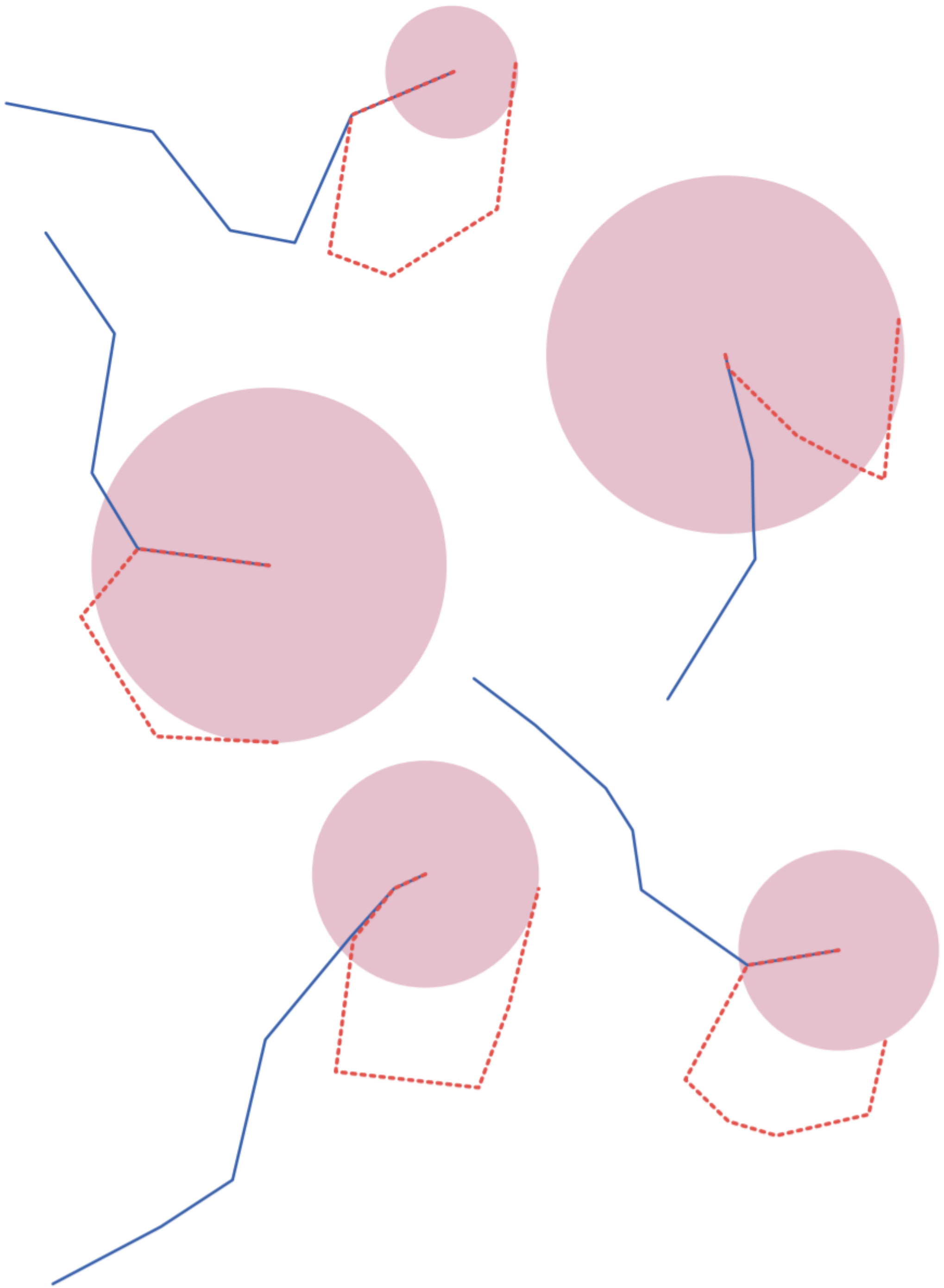




Arm Lemma

[Cauchy 1813] [Steinitz-Rademacher 1934] [Zaremba 1967]

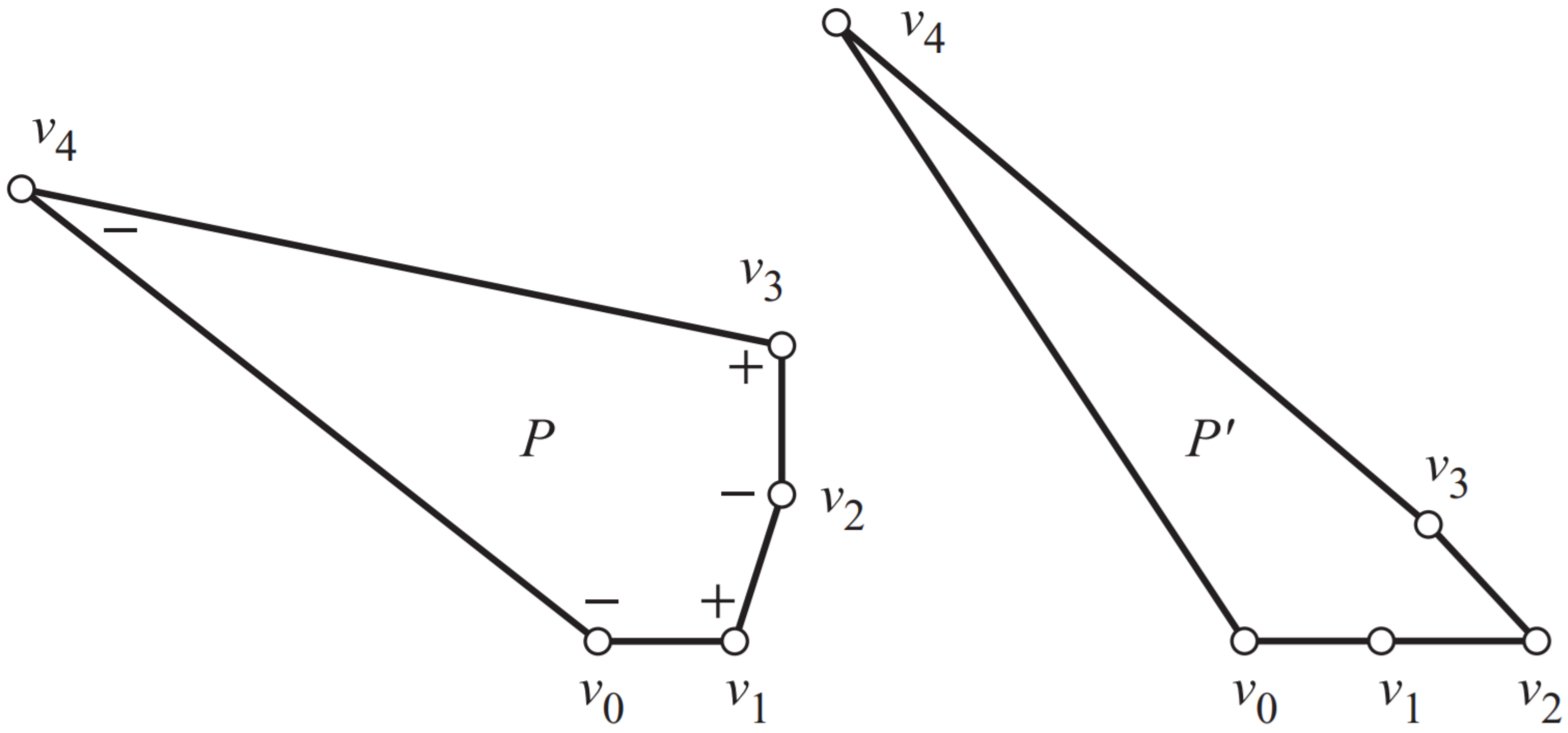
Opening a convex chain by increasing its internal angles increases the distance between v_0 and v_n.



Generalization

- $\theta_i' \in [-\theta_i, \theta_i]$ for all i

  [O'Rourke 2001]



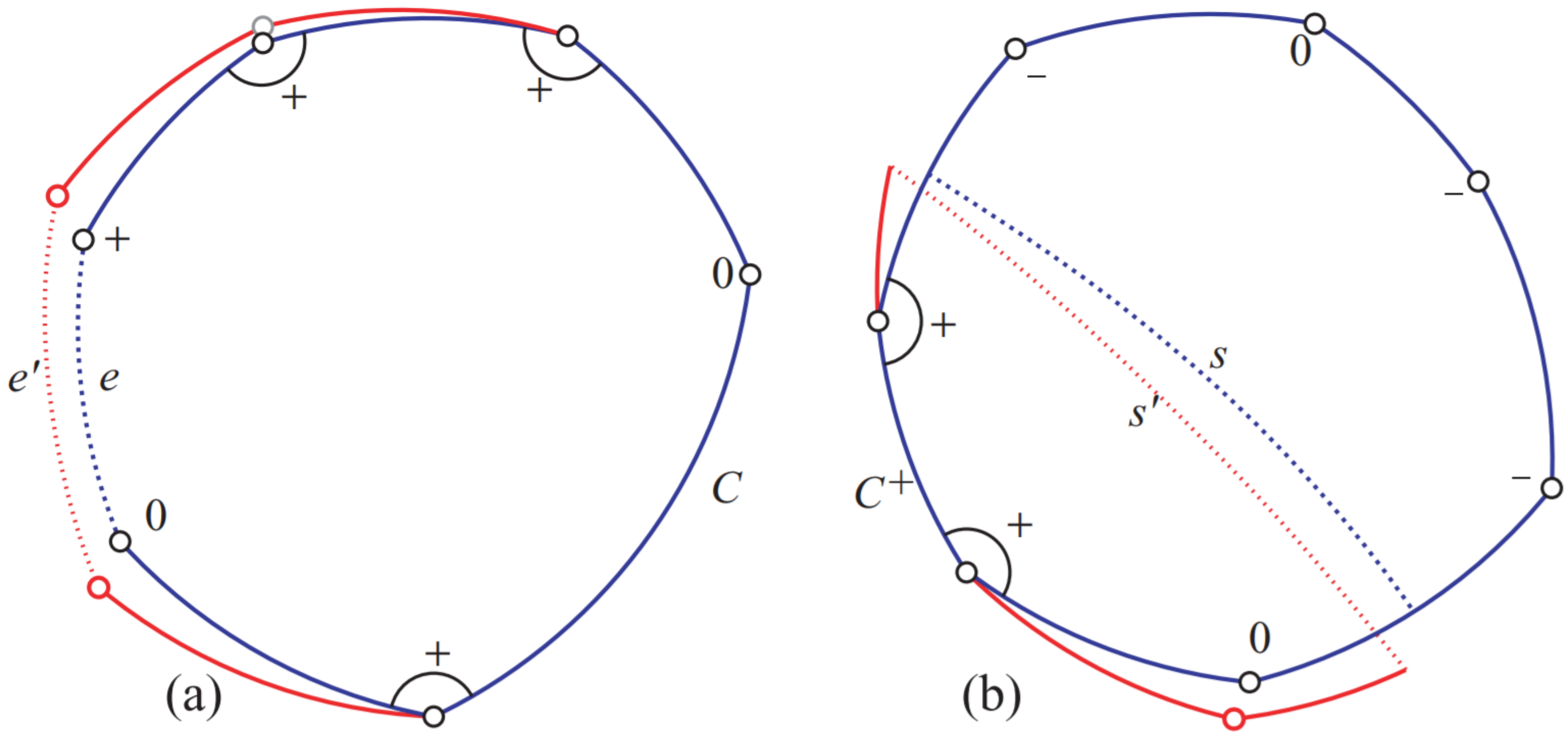
Convex Polygon

- Reconfiguration?

  (Self-intersections disallowed)

- **Observation.**
  Rotation number has to preserve; no flipping

Cauchy-Steinitz Lemma. Any ±-labeling on the corners of convex polygon must have 4 alternations.



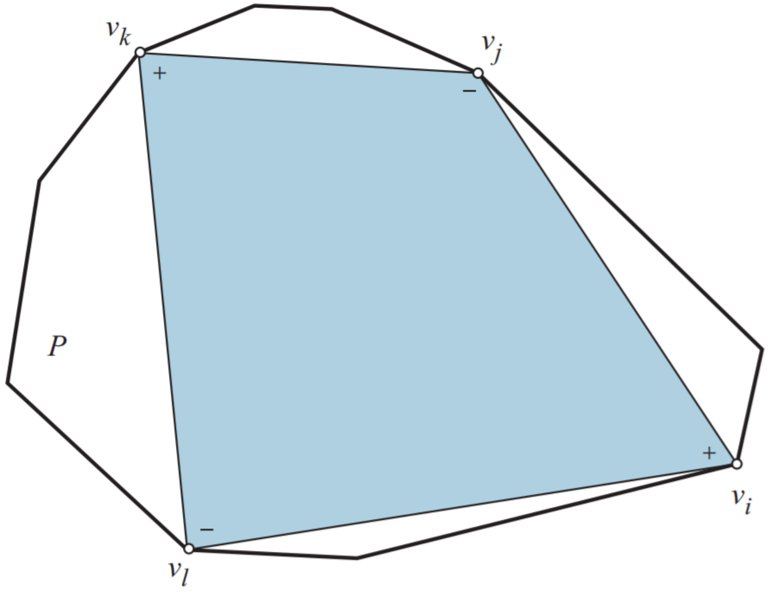
Reconfig Convex Polygons

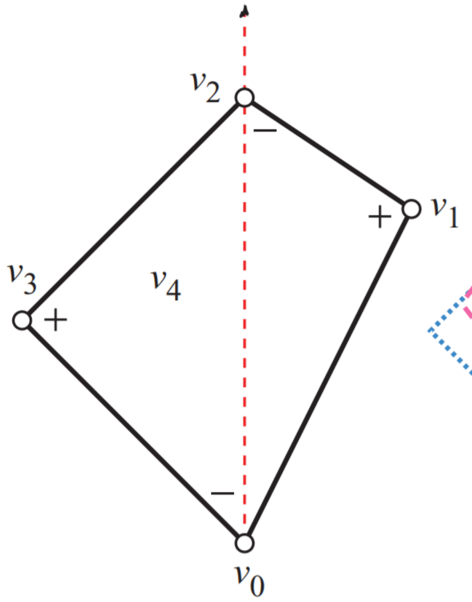
Reconfig(P, Q):

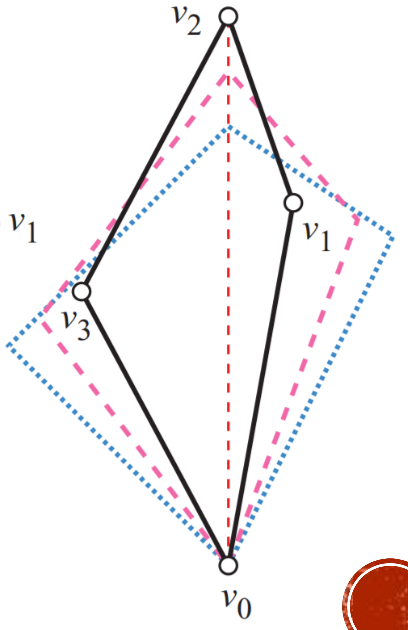
do:

- compute ±-labeling for all corners
- find 4 corners with alternating signs.
- LinearMotion(v_0, v_1, v_2, v_3, v_0) until one label becomes 0.

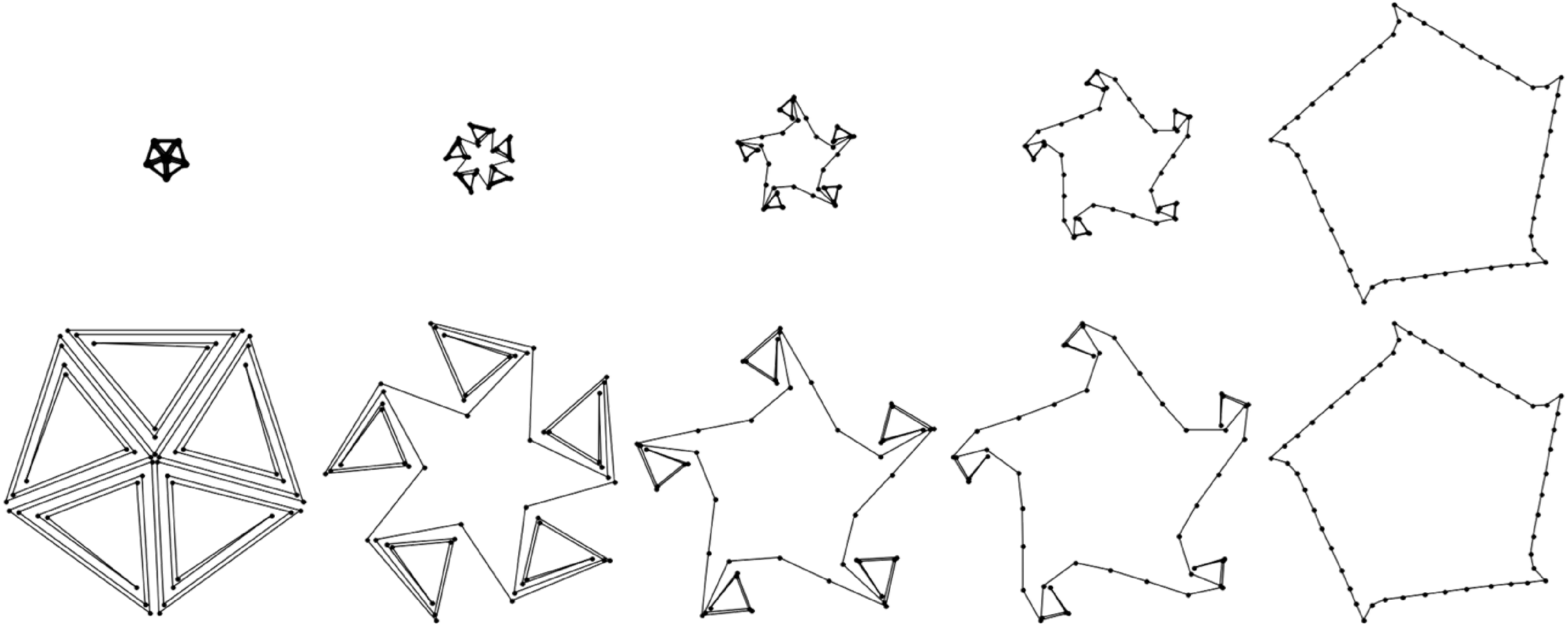
repeat







What about Actual Chains?



Carpenter's Rule Theorem [Connelly-Demaine-Rote 2003]

Any two equivalent 2D chains can be reconfigured into each other in the plane.

Closing Q. Reconfiguration using Regular Homotopy?

Next time.

Can you flat-fold this crease pattern?

When is a polyhedron unique?