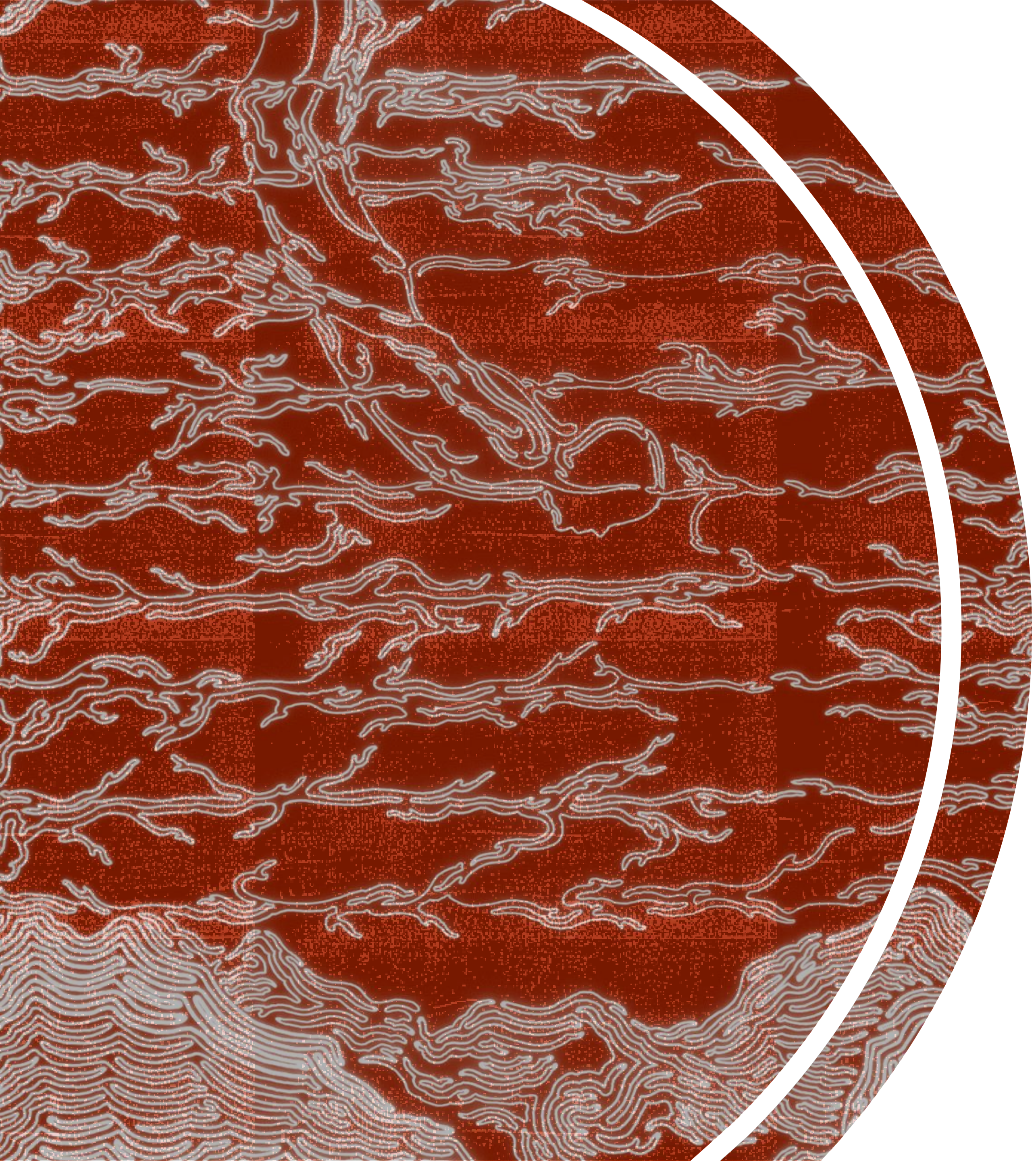


INTRODUCTION TO

COMPUTATIONAL TOPOLOGY

HSIEN-CHIH CHANG

LECTURE 3, SEPTEMBER 21, 2021

Administrivia

- Homework 1 is due 9/27 (next Monday)
  - Starting from Homework 1, group submission up to 2 people

Surfaces (2d Manifolds)

What is a Surface?

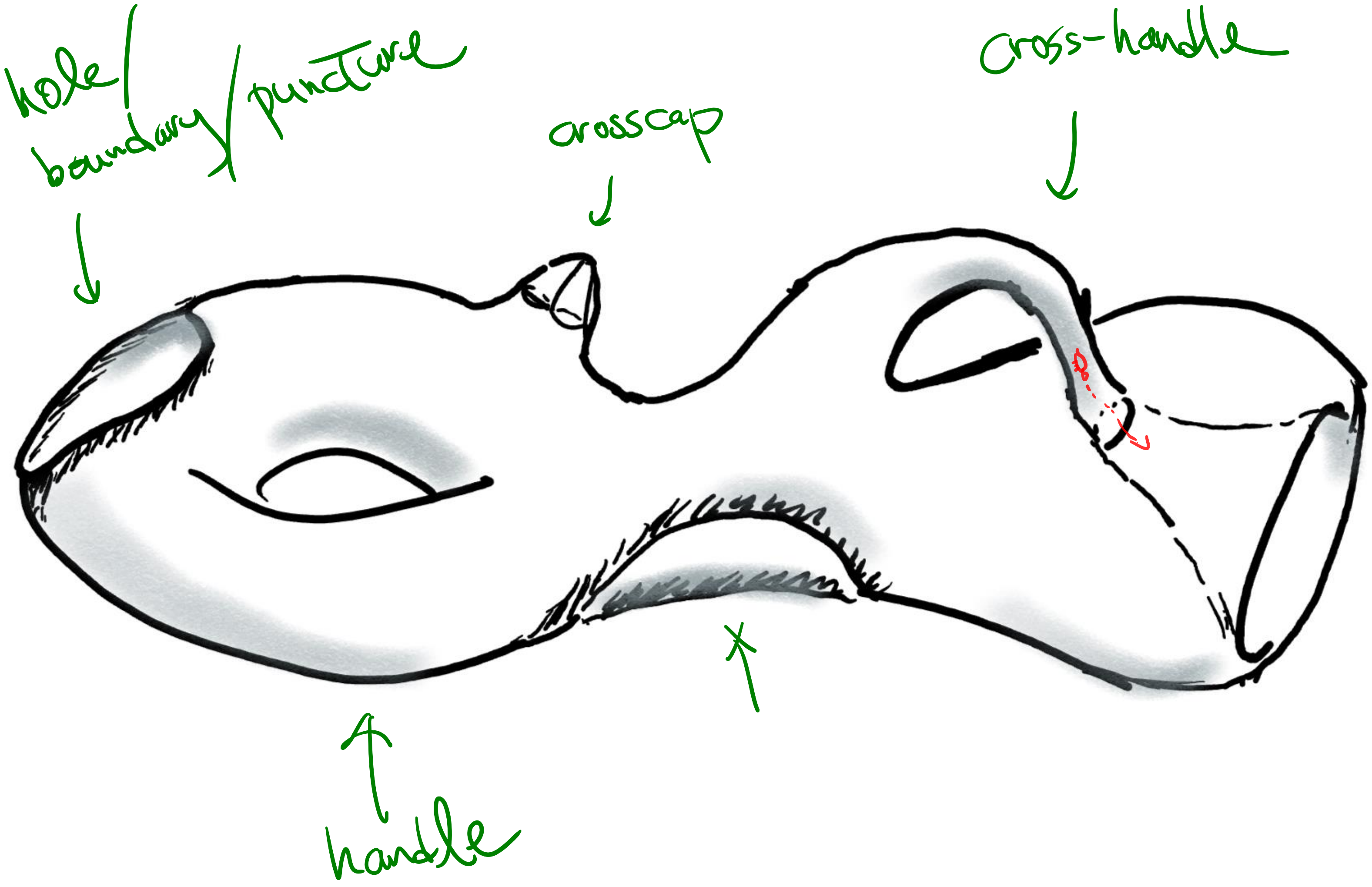
- Formally, a surface (without boundary) is

A Haussdorff 2nd-countable topological space,
that is locally homeomorphic to the plane.

What is a Surface?

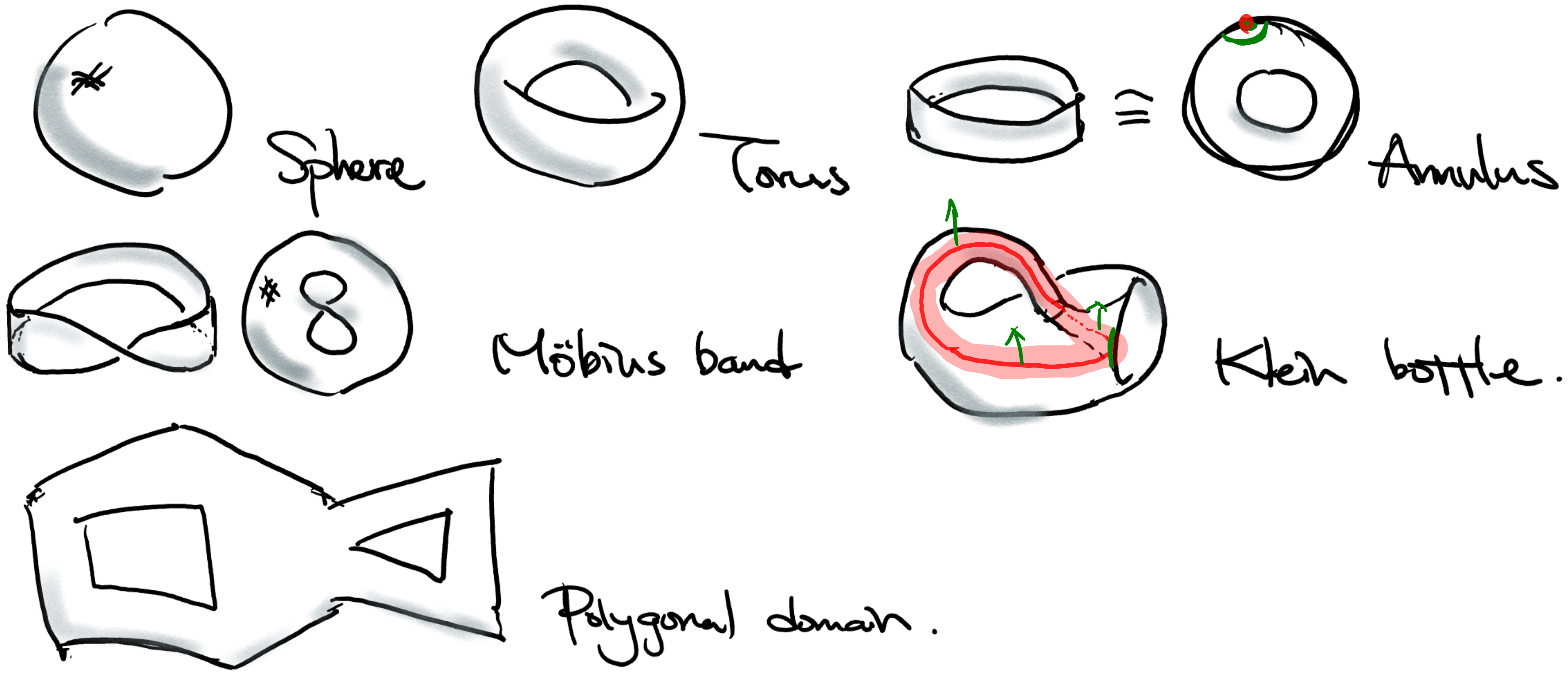
- Formally, a surface (with boundary) is

A Haussdorff 2nd-countable topological space,
that is locally homeomorphic to the plane or the half-plane.



Mythic Creature Exhibition

Mythic Creature Exhibition

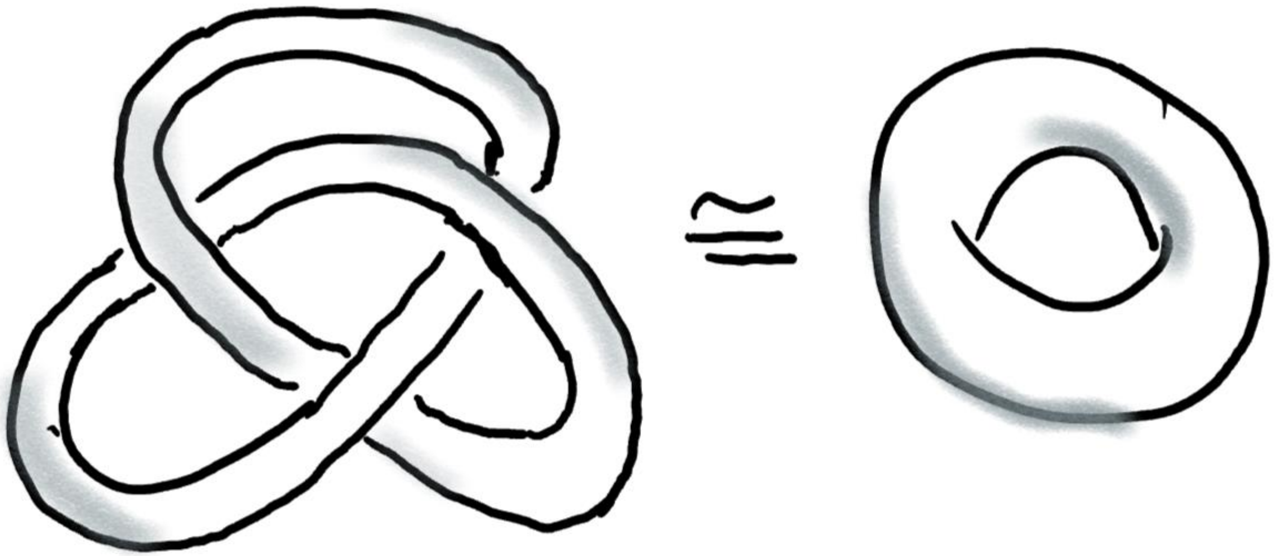


Tech-spec of the Fabric

- Bendable and stretchable

Tech-spec of the Fabric

- Bendable and stretchable
- Can phase-through itself



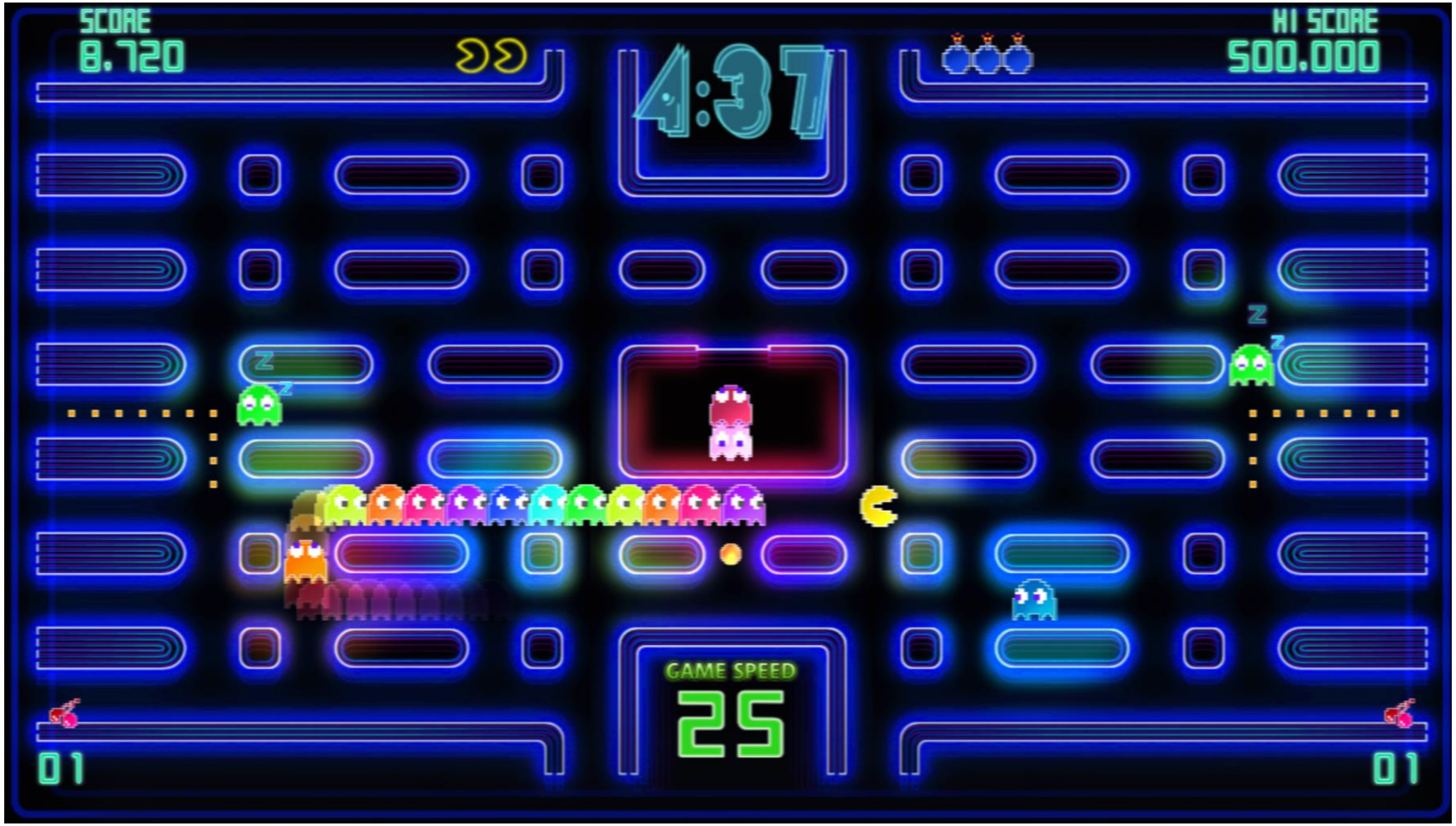
Tech-spec of the Fabric

- Bendable and stretchable
- Can phase-through itself
- NOT cuttable…

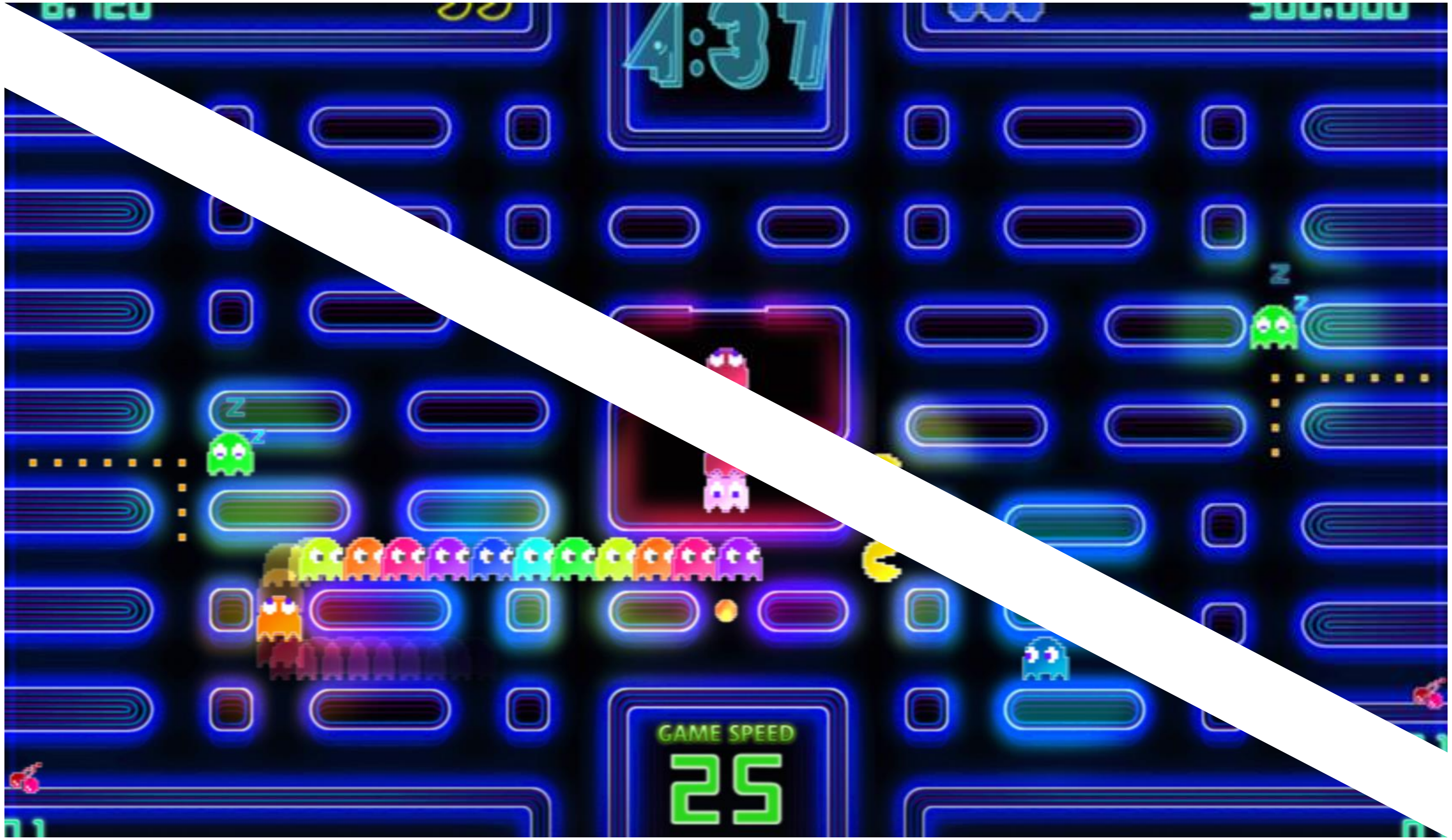
Tech-spec of the Fabric

- Bendable and stretchable
- Can phase-through itself
- NOT cuttable…
  …unless you glue it back

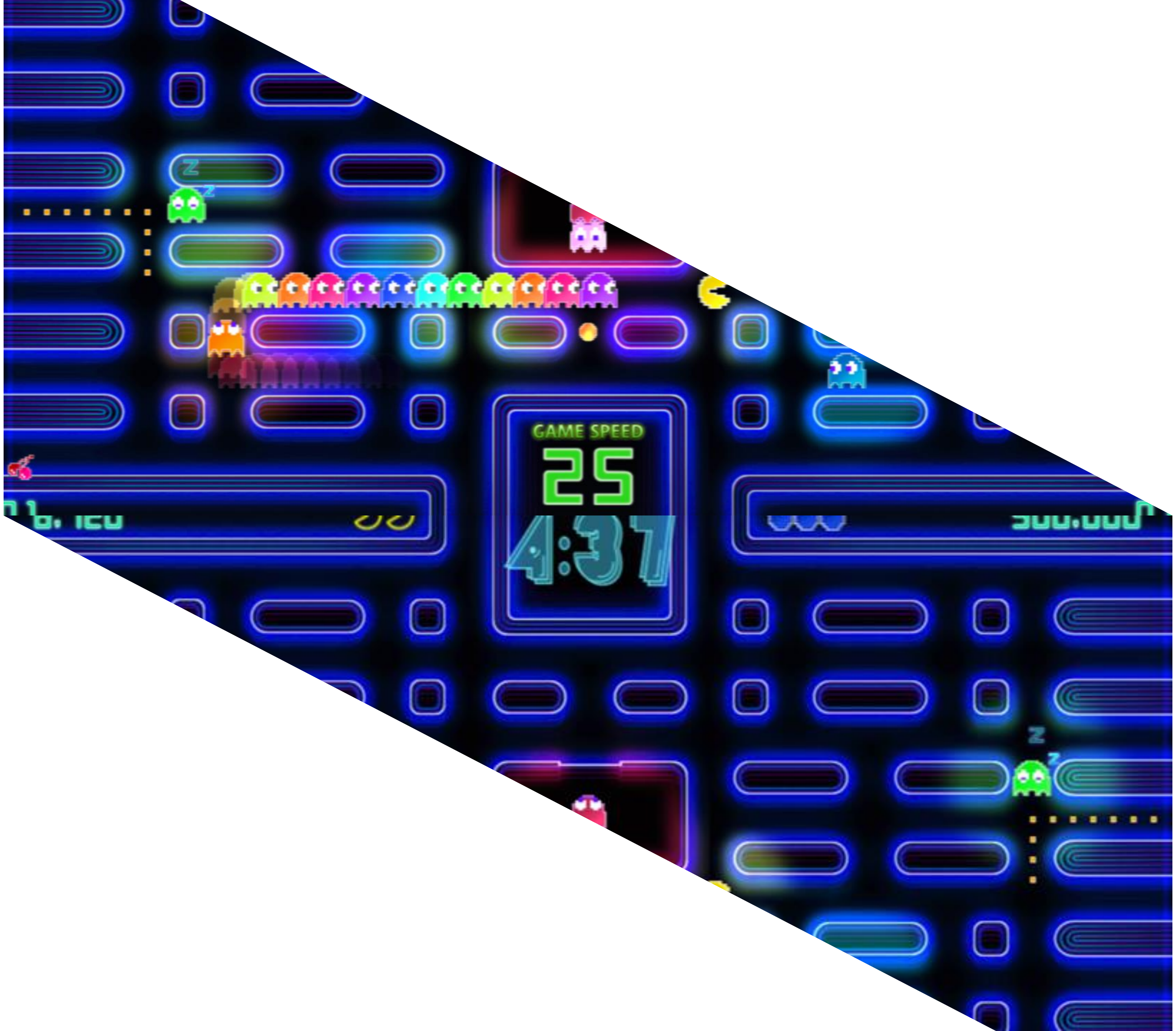
Example: Pacman Space



Example: Pacman Space

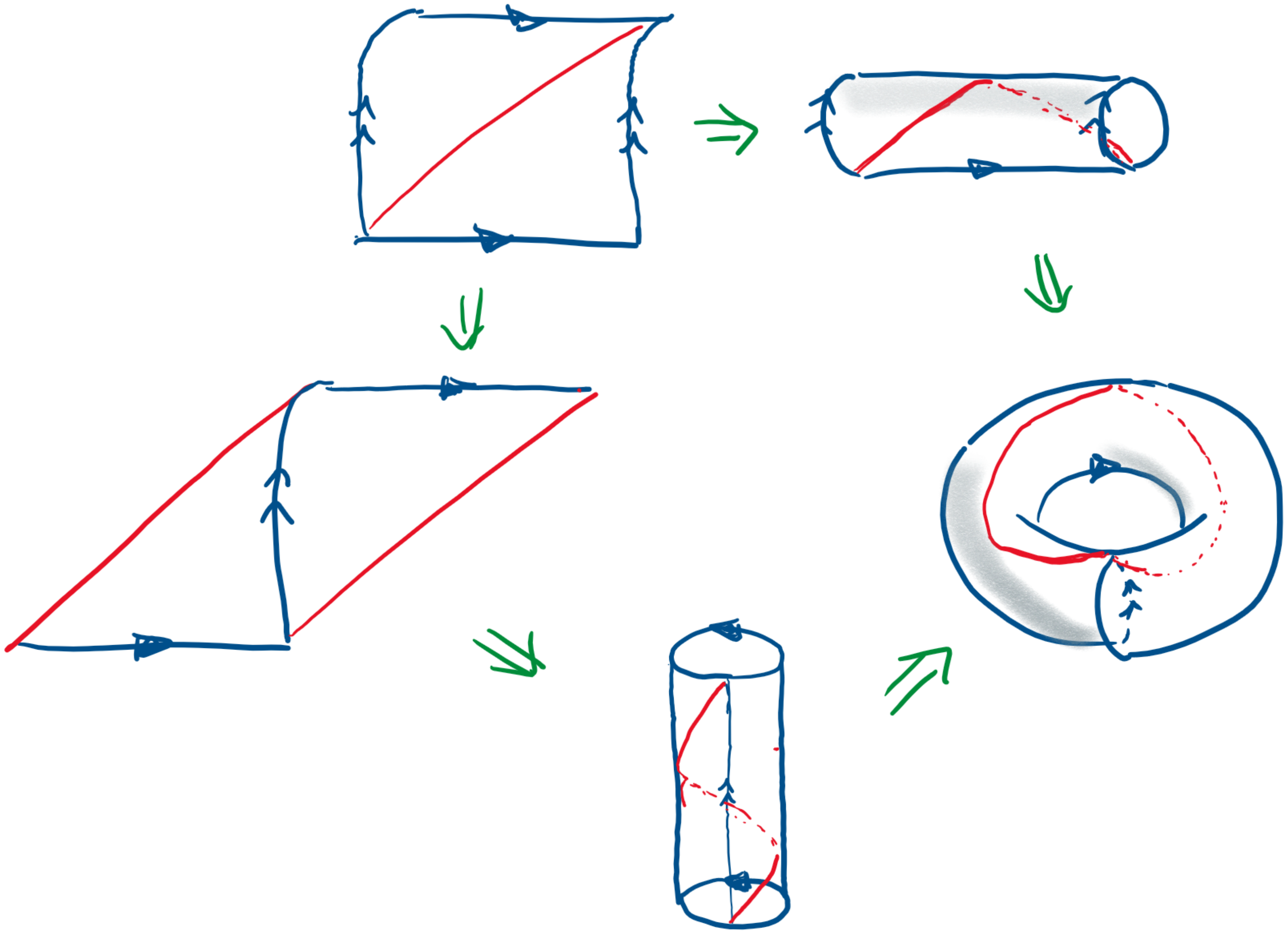


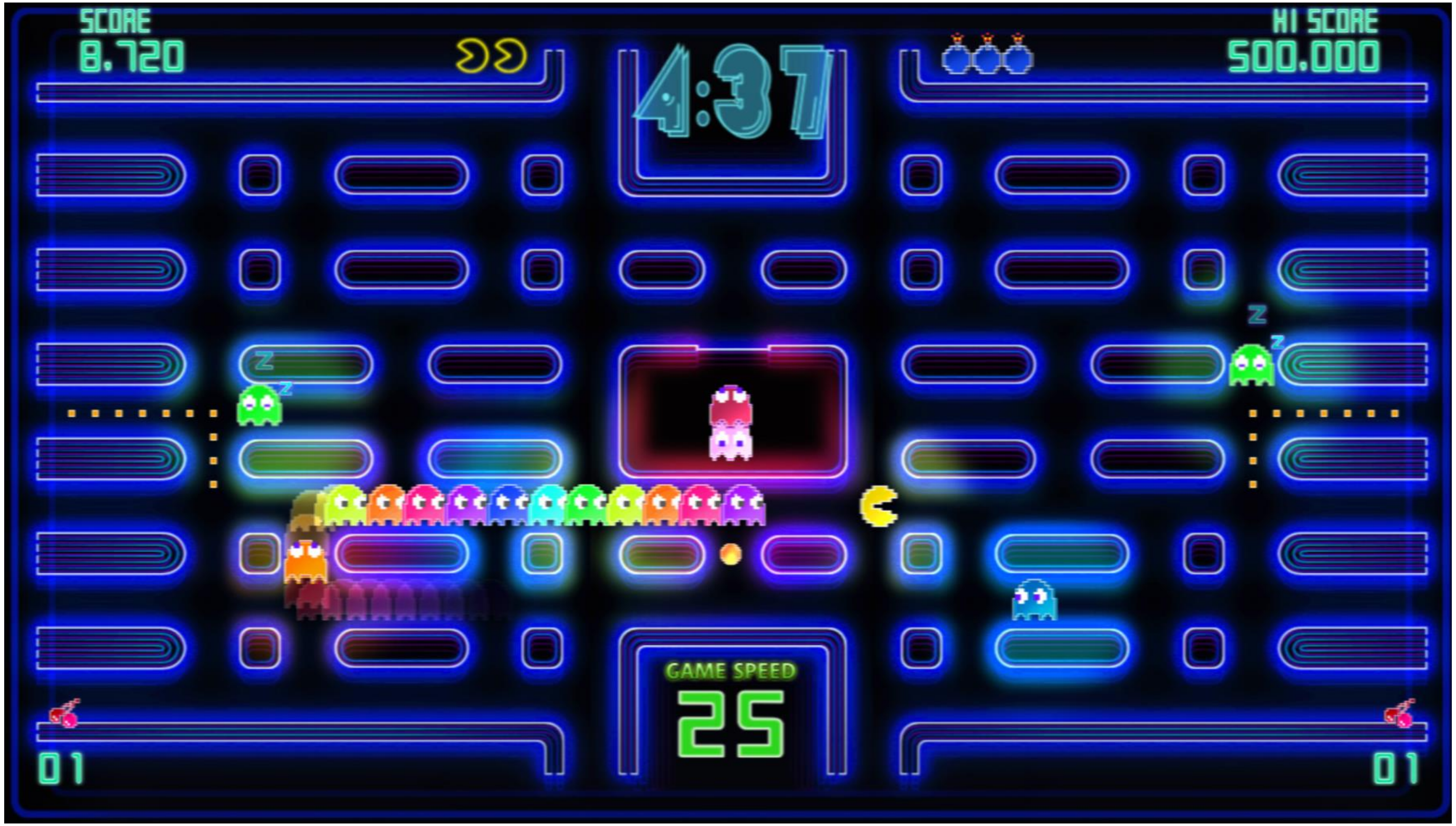
Example: Pacman Space



Example: Pacman Space

Example: Pacman Space





Upside-down Pacman?

Upside-down Pacman?

Exercise: What is this Surface?

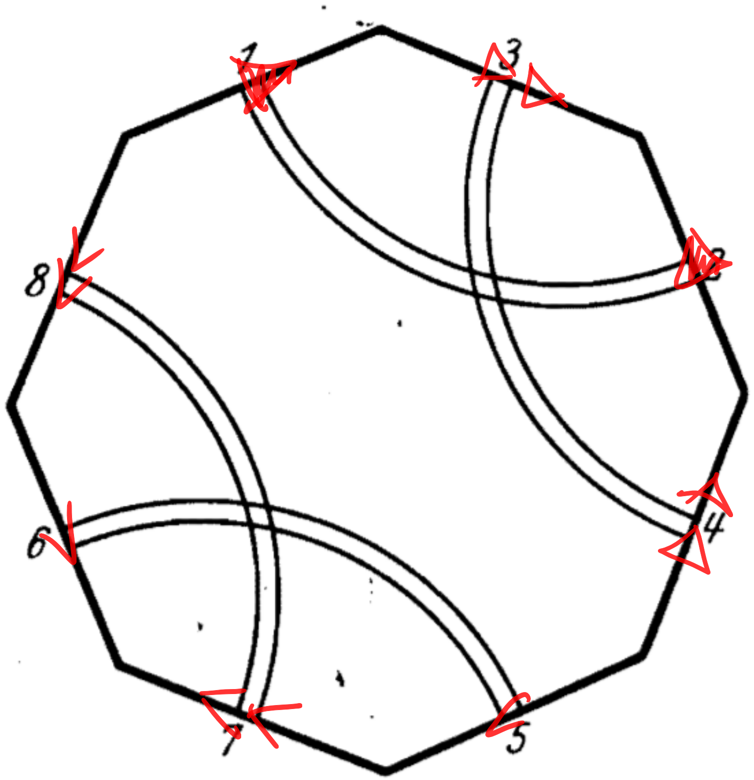


Fig. 286a

Exercise: What is this Surface?

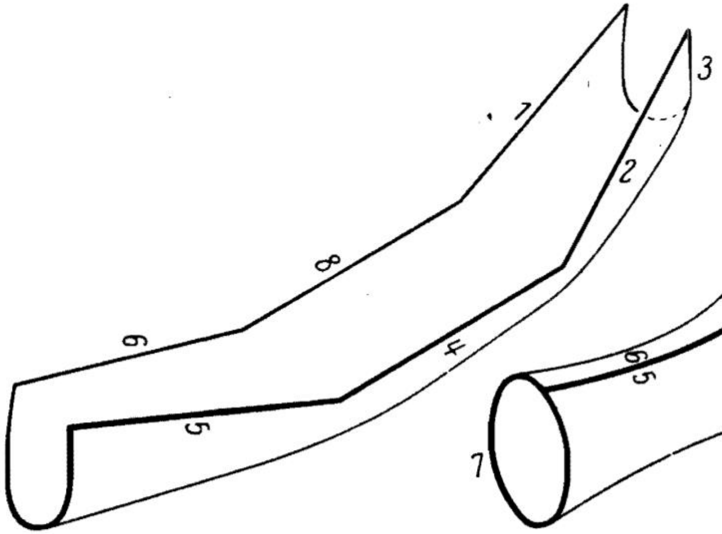


Fig. 286b

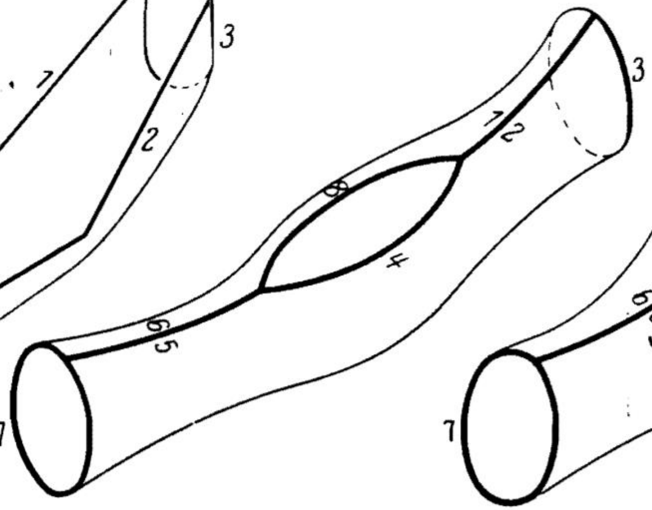


Fig. 286c

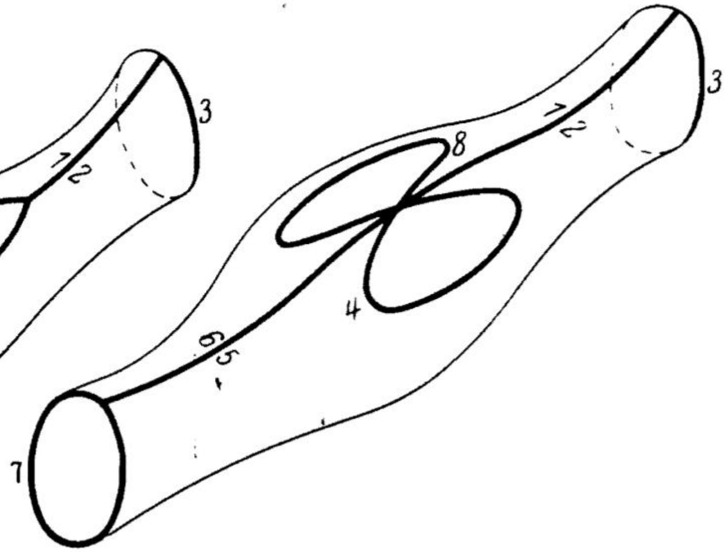


Fig. 286d

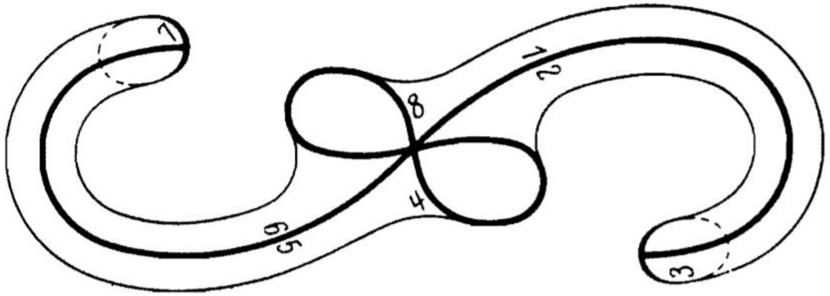


Fig. 286e

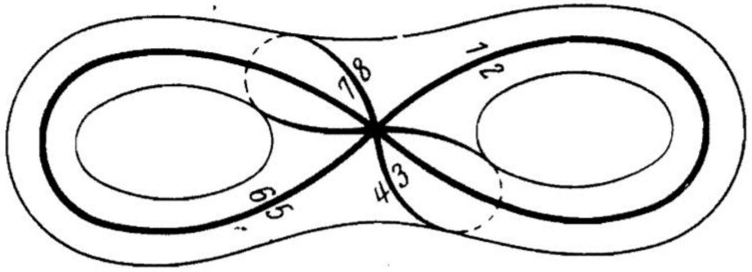
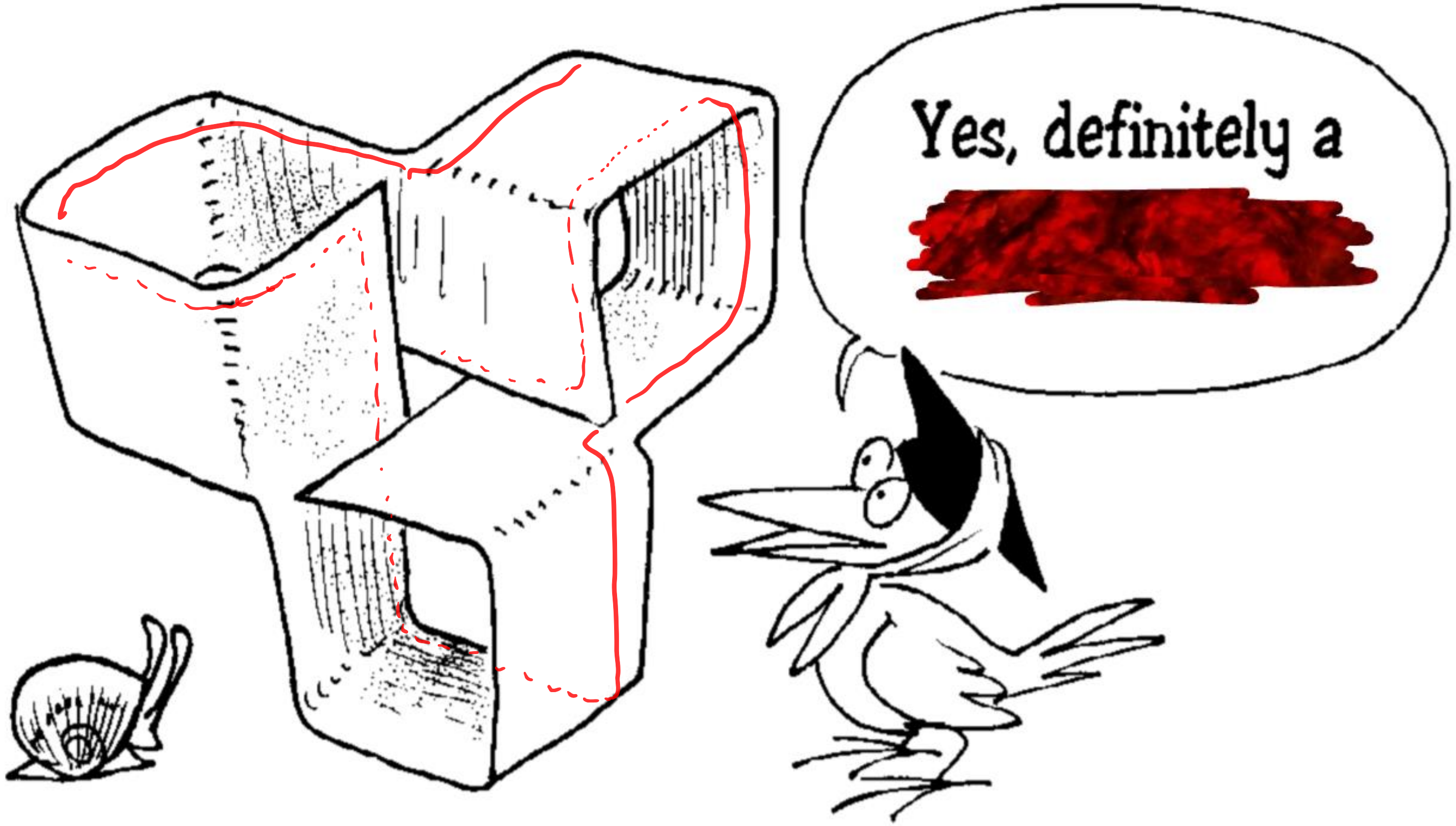
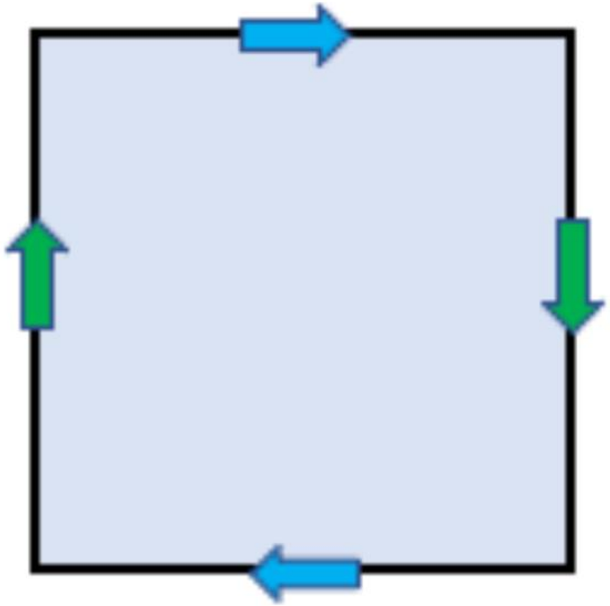


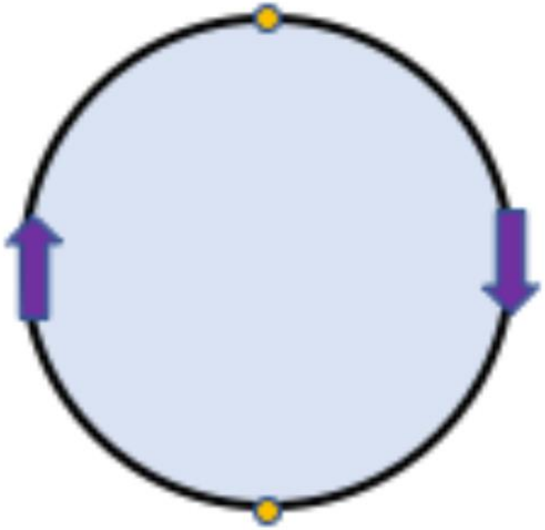
Fig. 286f

Exercise: What is this Surface?

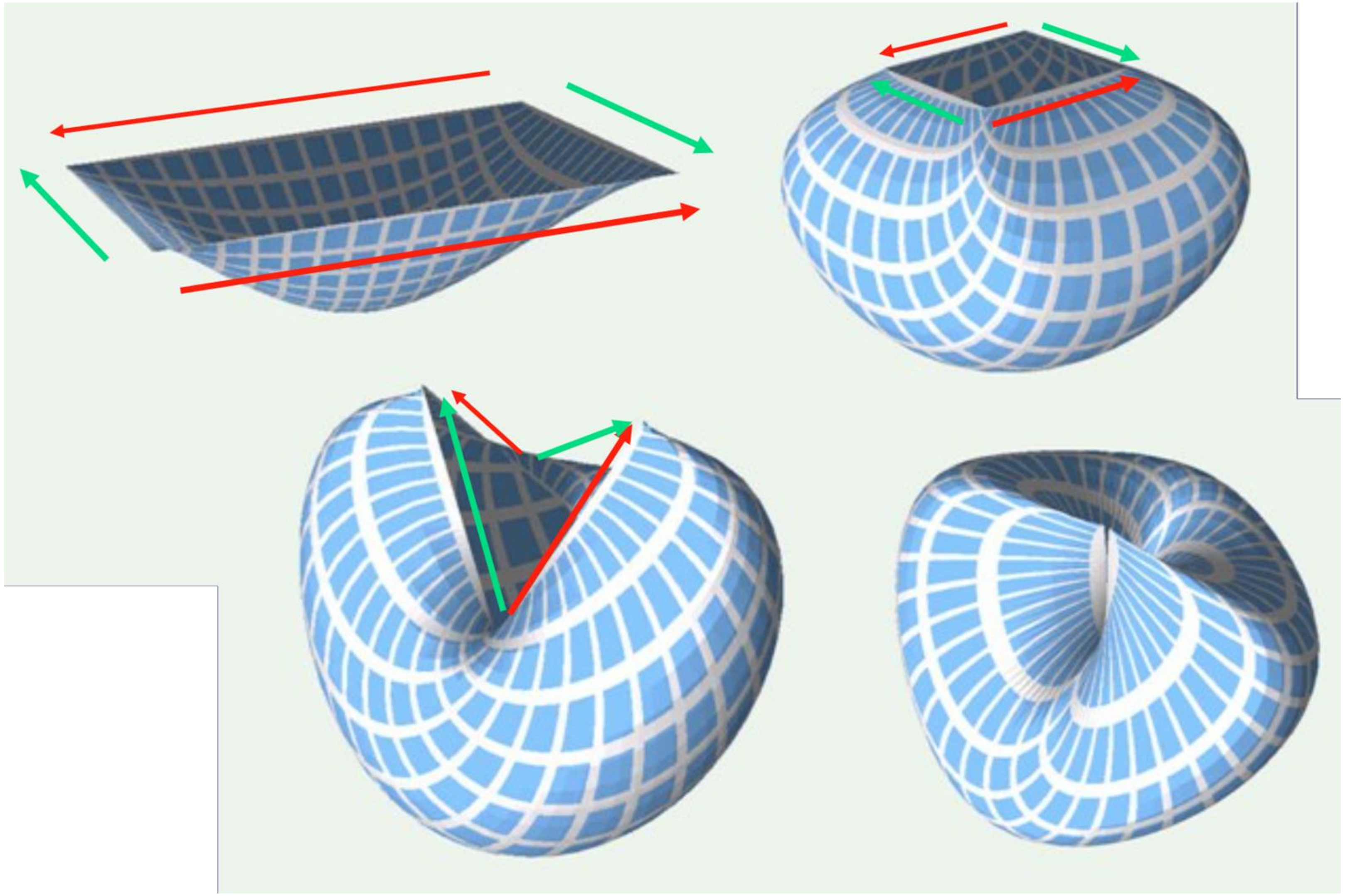


Exercise: What is this Surface?

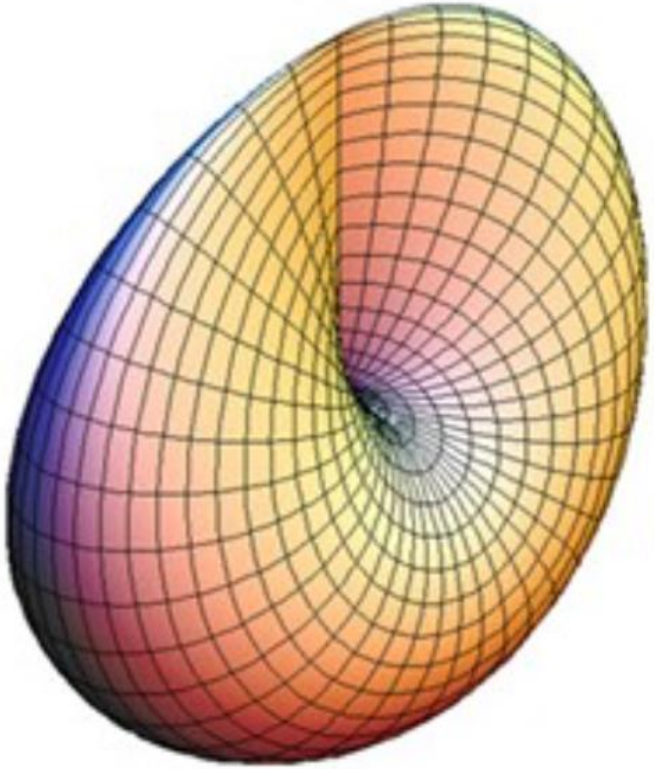
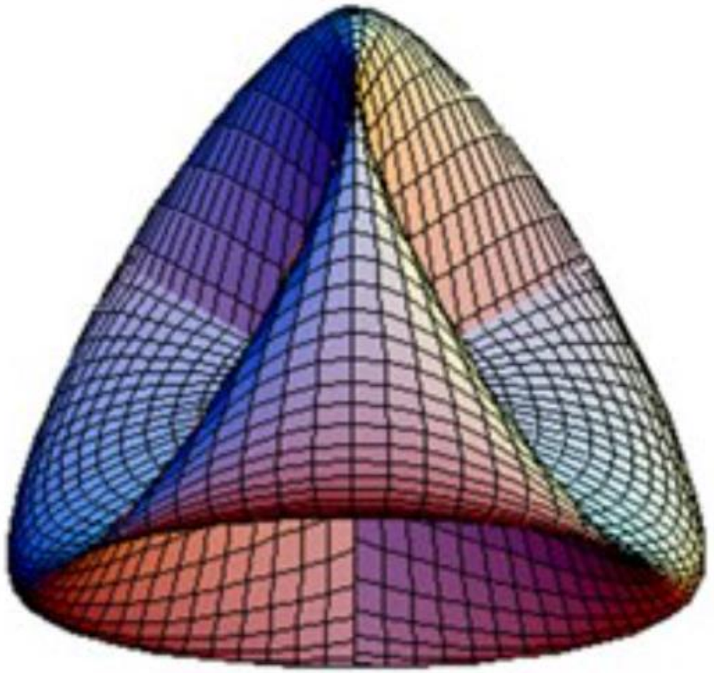
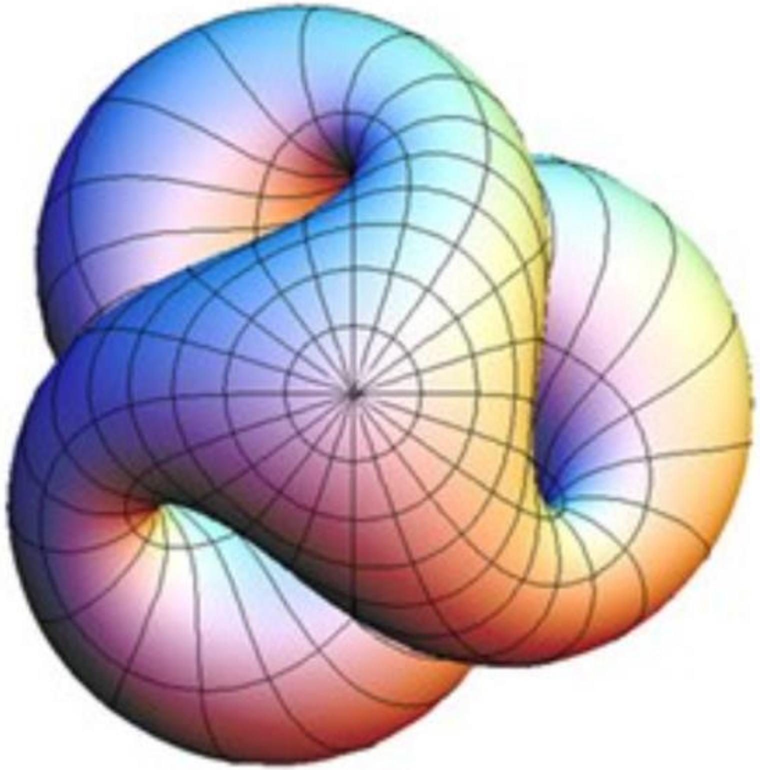




Exercise: What is this Surface?



Exercise: What is this Surface?

Can we get all surfaces through cut-and-paste?

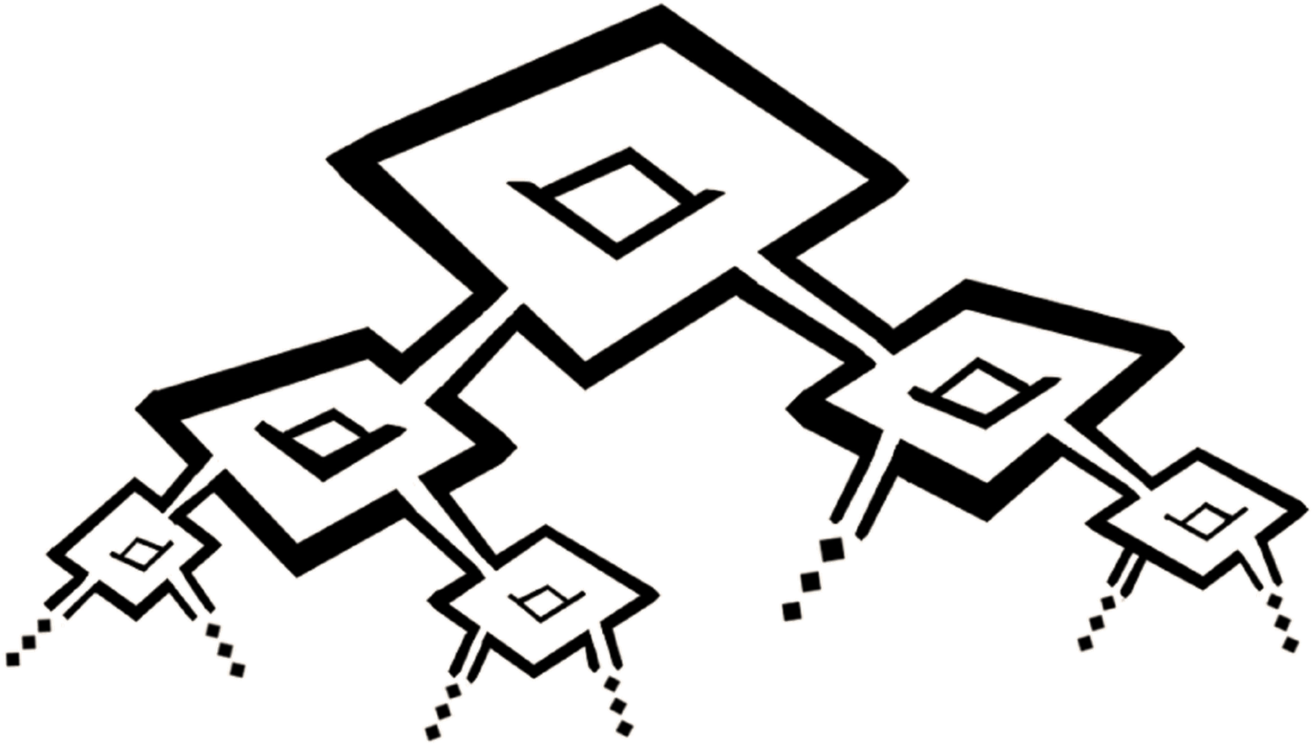
Equivalence

- Homeomorphism

$f: X \to Y$ cont. bijection

& f^{-1} is also cont.

- Homotopy equivalence



Surface Classification

[Möbius 1861] [Dehn-Heegaard 1907] [Radó 1925]

Every connected surface is homeomorphic to the following:

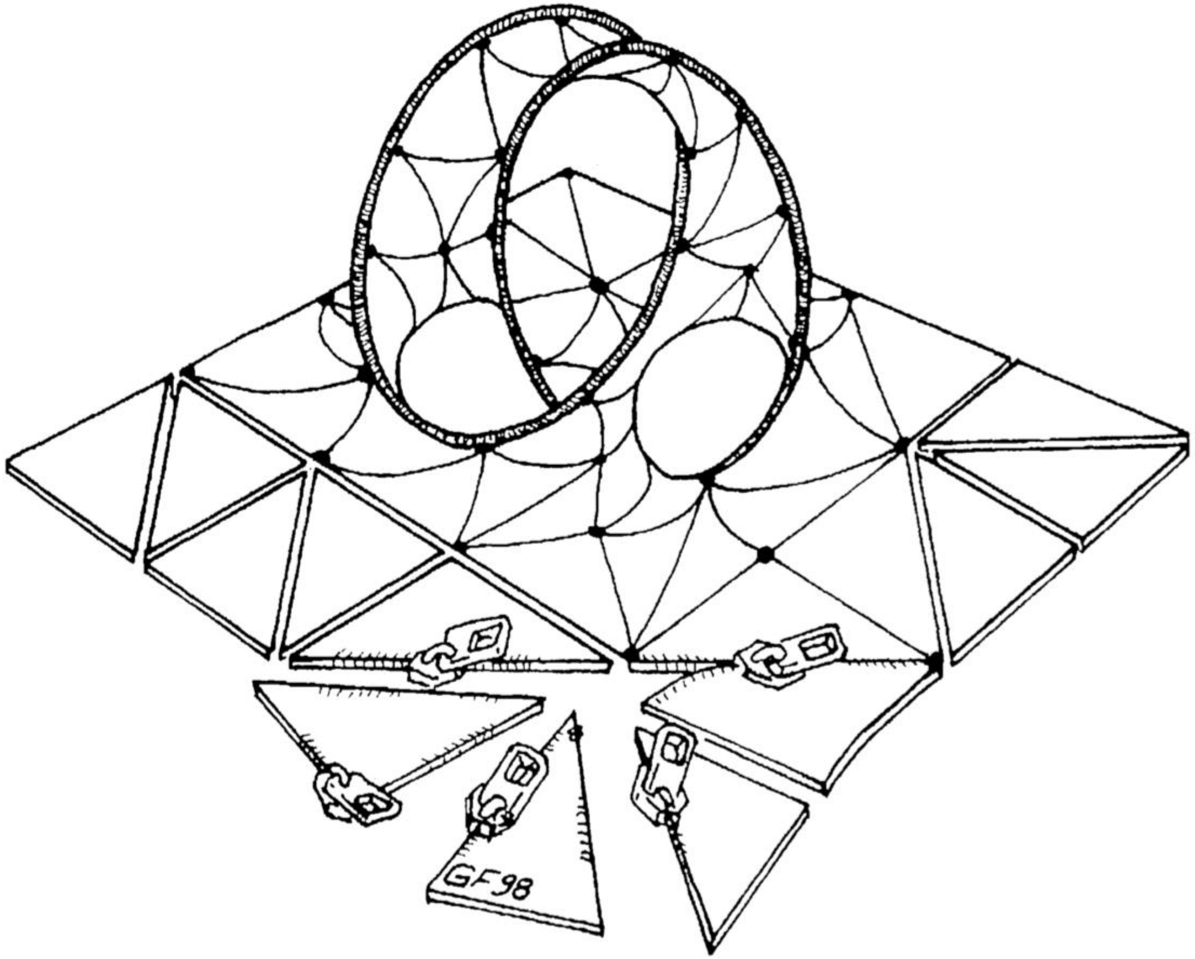
- Sphere with g handles $\Sigma(g, 0)$
- Sphere with r cross-caps $\Sigma(0, r)$

(plus boundaries)

Theorems we secretly assumed

- Triangulation Theorem [Kerékjártó-Radó 1925]
  - Any surface can be cut into triangles
- Refinement Theorem [Moise 1977]
  - Any two triangulations have a common refinement

Conway's ZIP Proof



Conway's ZIP Proof

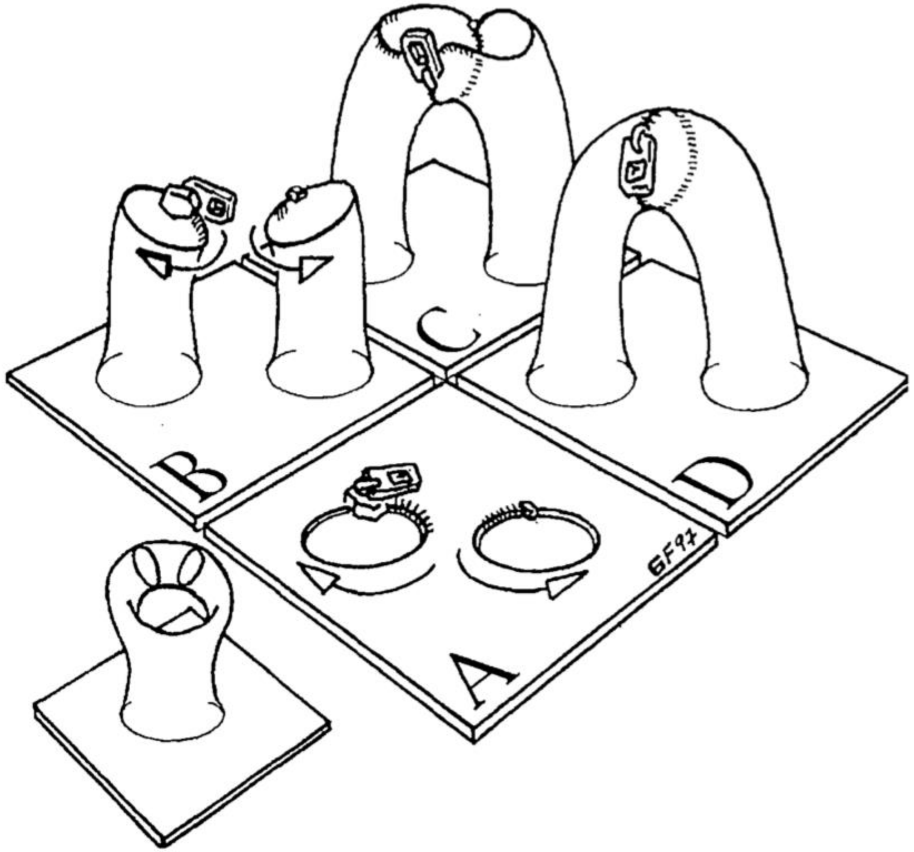


Figure 1. Handle

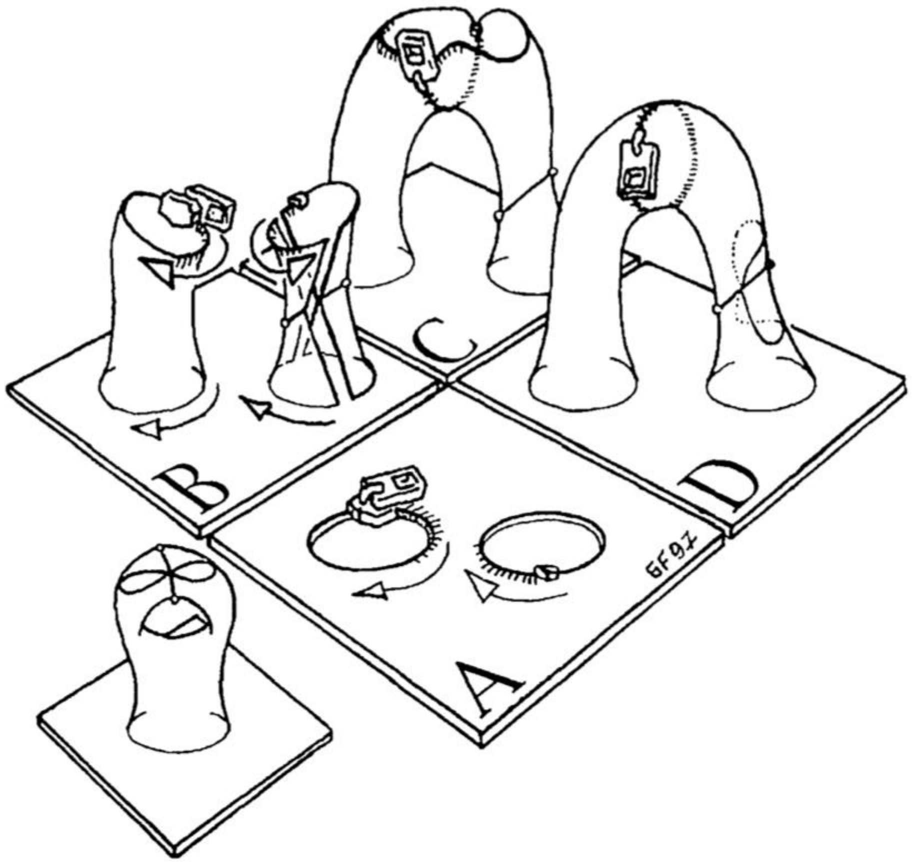


Figure 2. Crosshandle

Conway's ZIP Proof

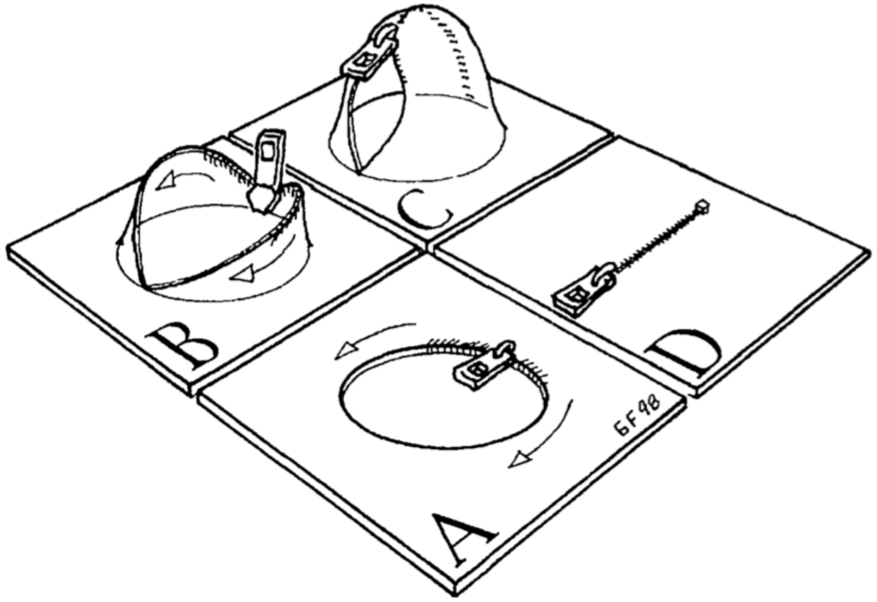


Figure 3. Cap

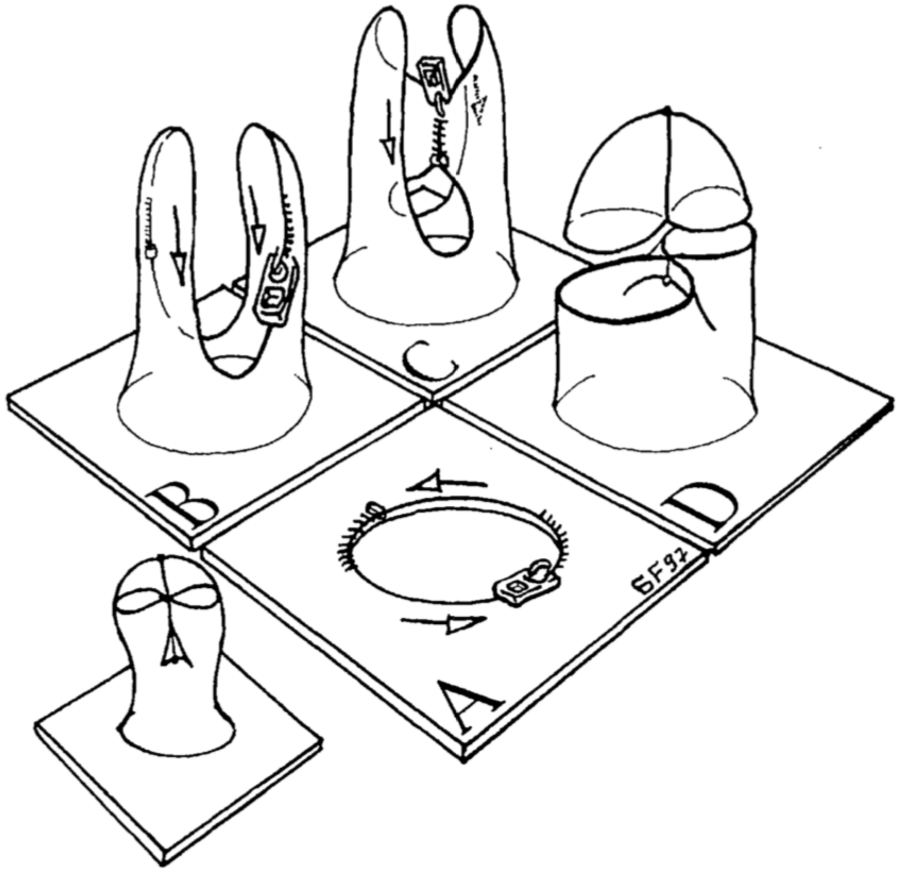
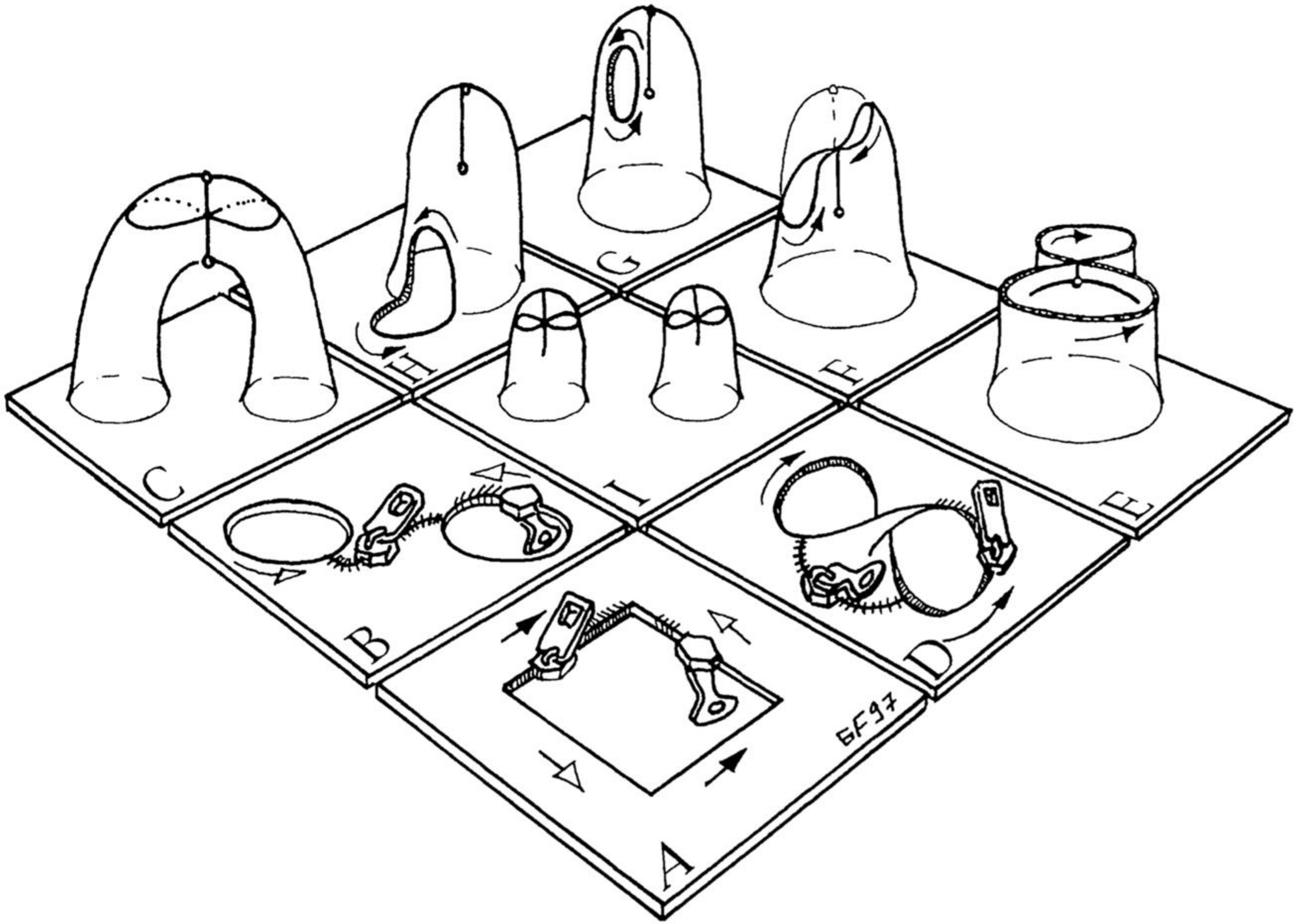
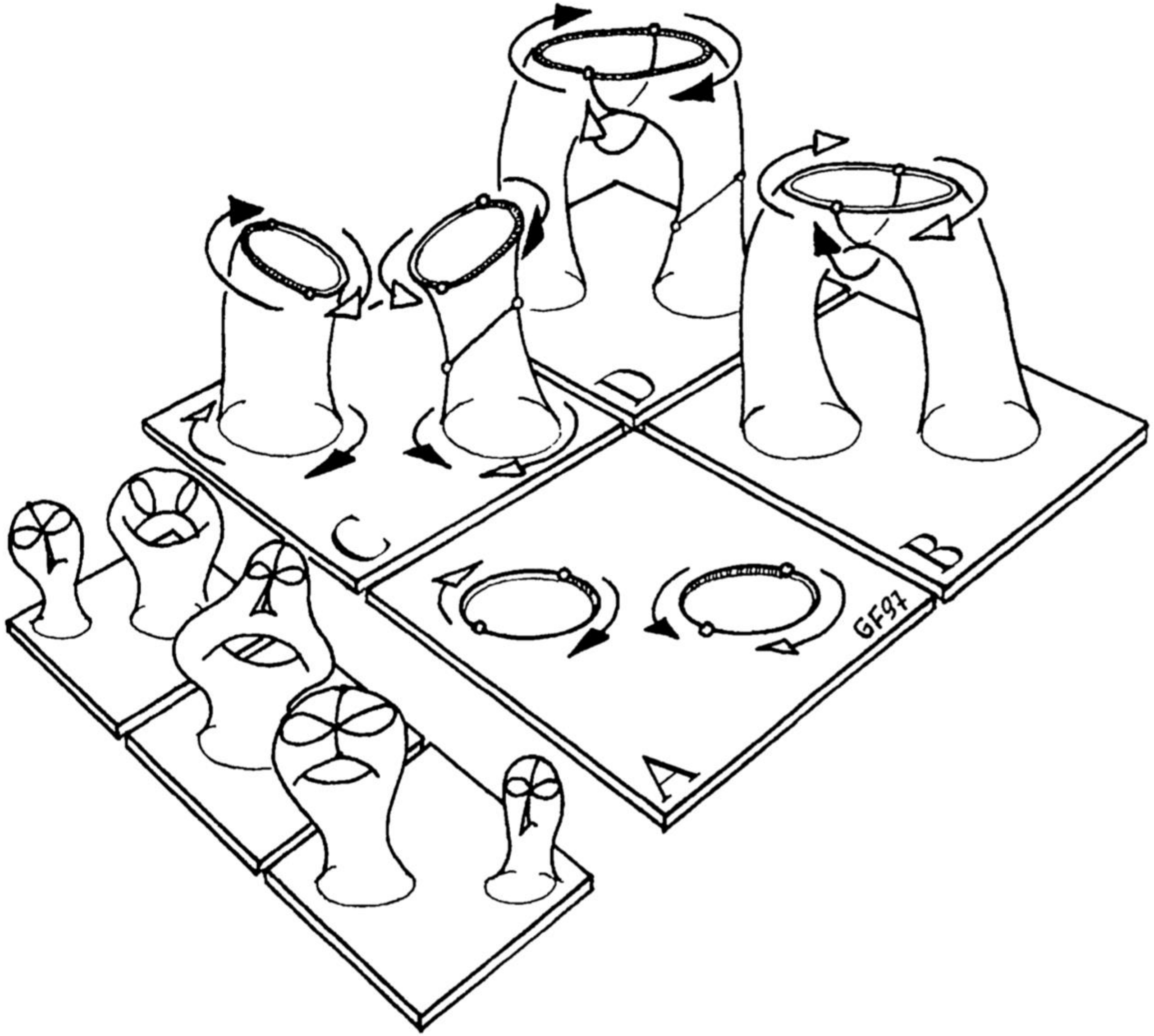


Figure 4. Crosscap



K=P#P

Exchanging two cross-caps for a cross-handle



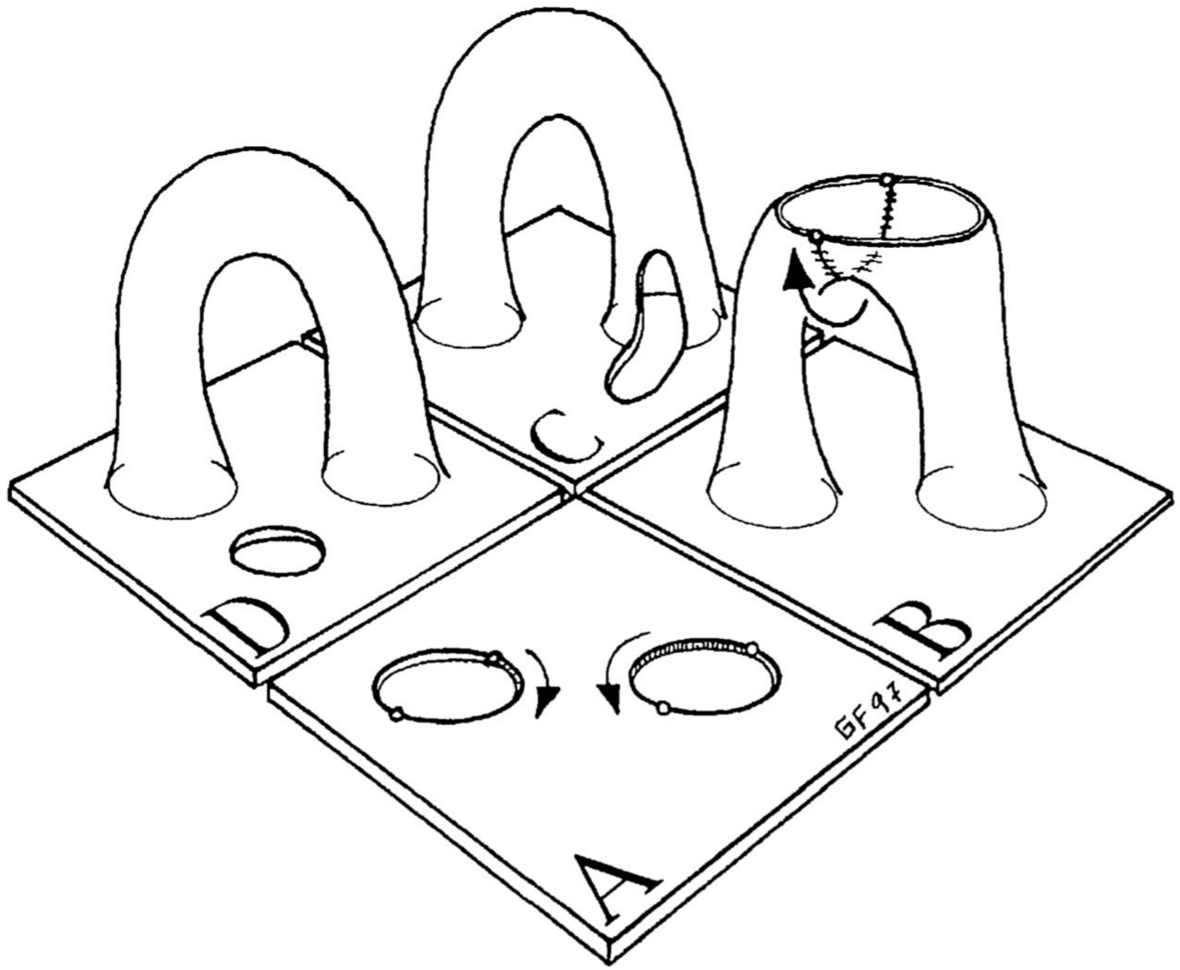
T#P=K#P

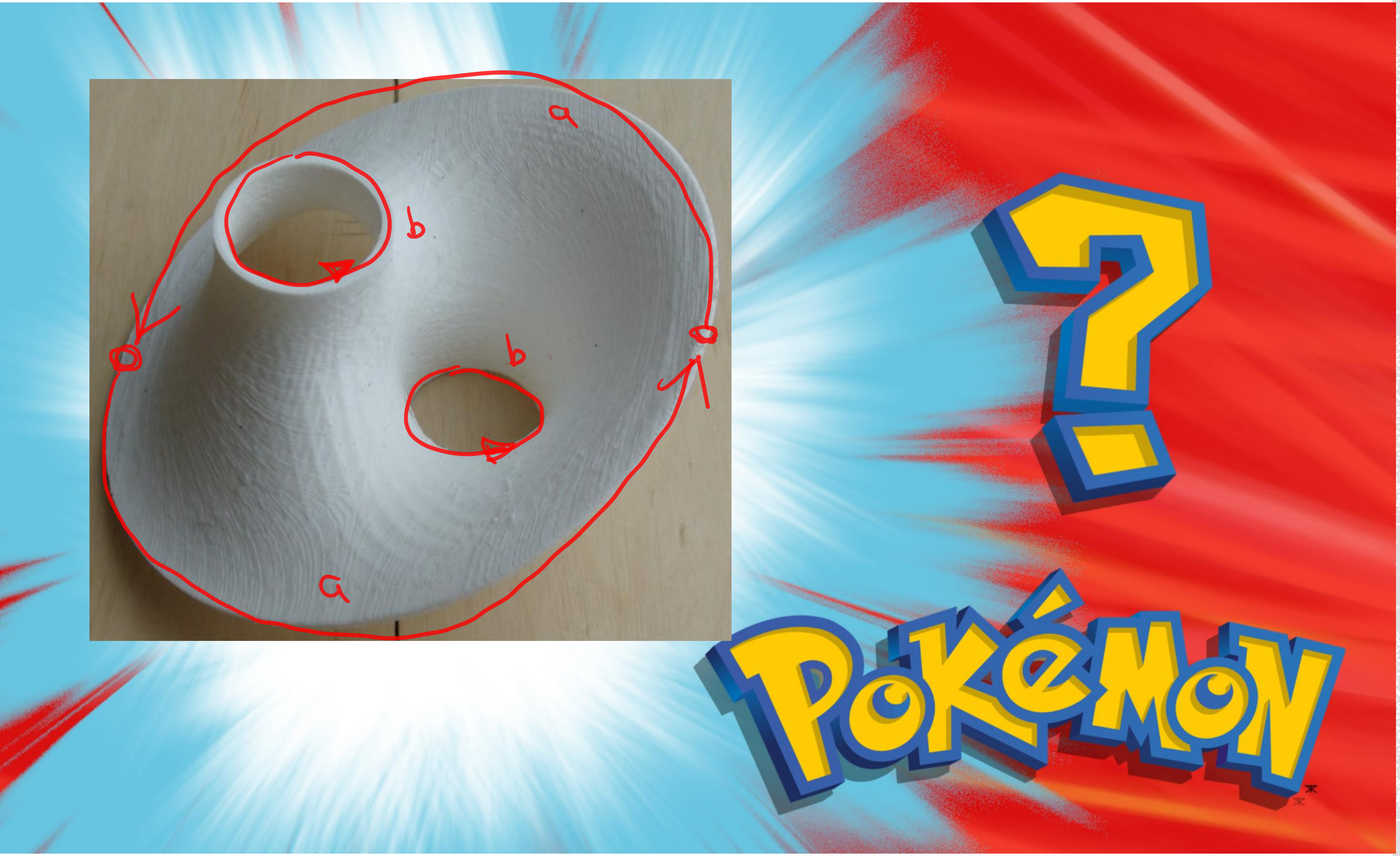
Handles and cross-handles are the same when cross-caps are presented

Trading

- When cross-handles or cross-caps are presented
  - Turn all handles and cross-handles into cross-caps
- Otherwise, only handles exist

Dealing with Boundaries





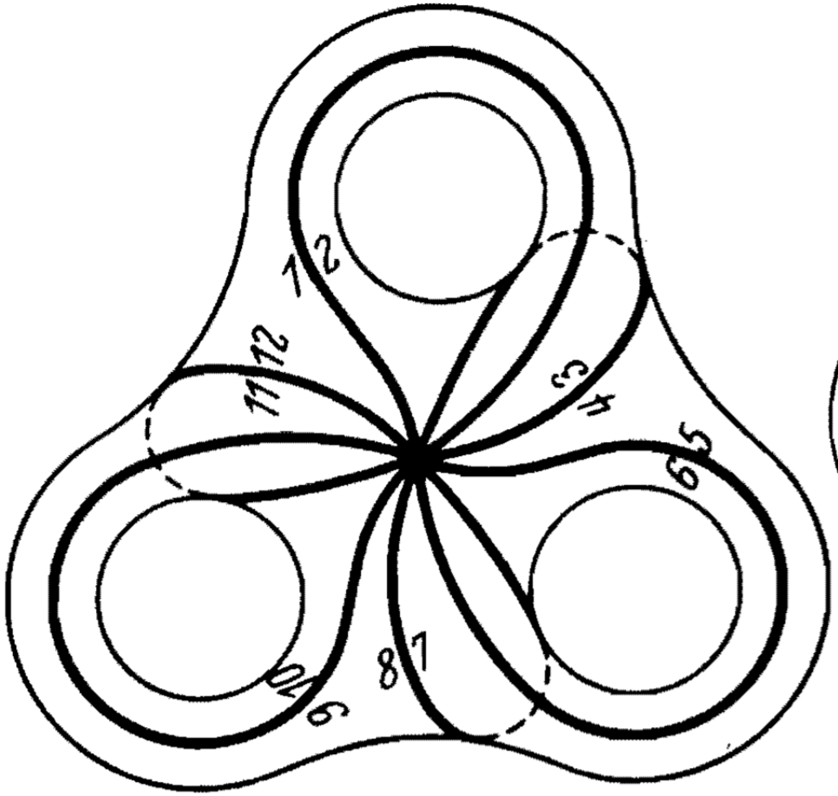
Intermission

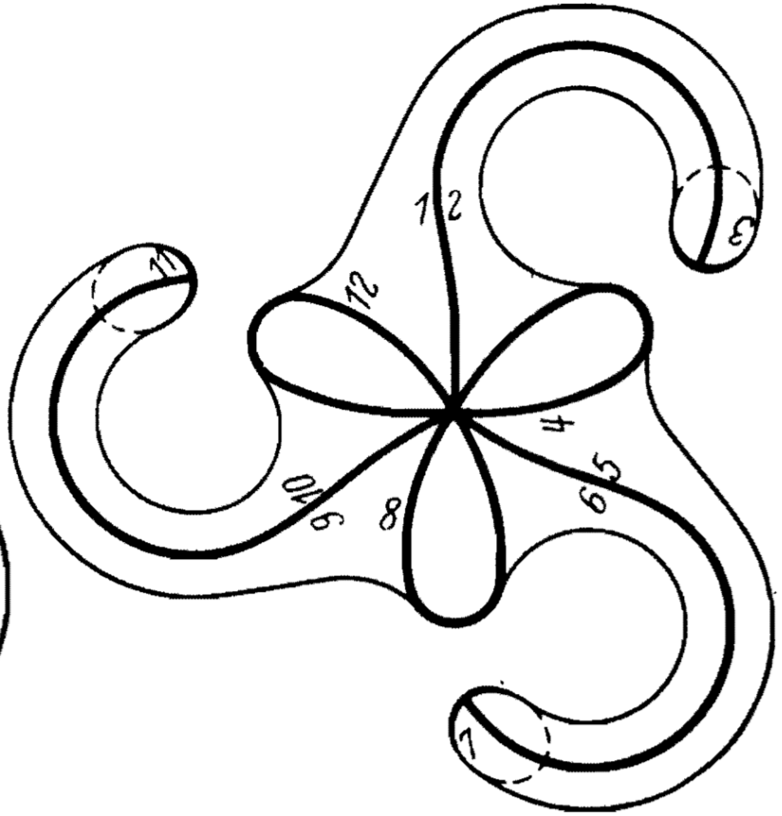
Exercise:

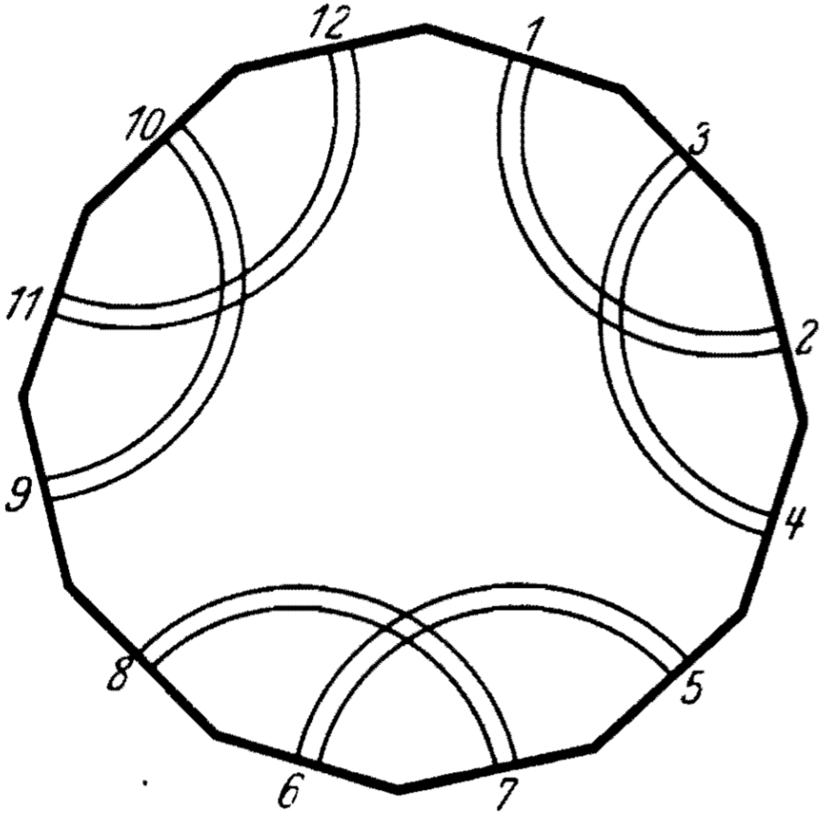
What is this surface?

Surface Graph and Rotation System

Treat Cutting-lines as Graphs







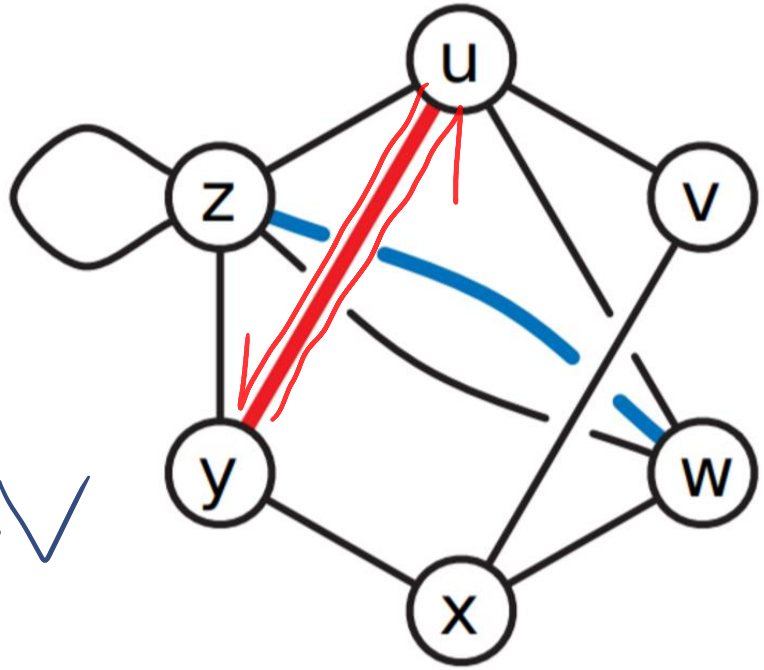
Treat Cutting-lines as Graphs

Abstract Graph

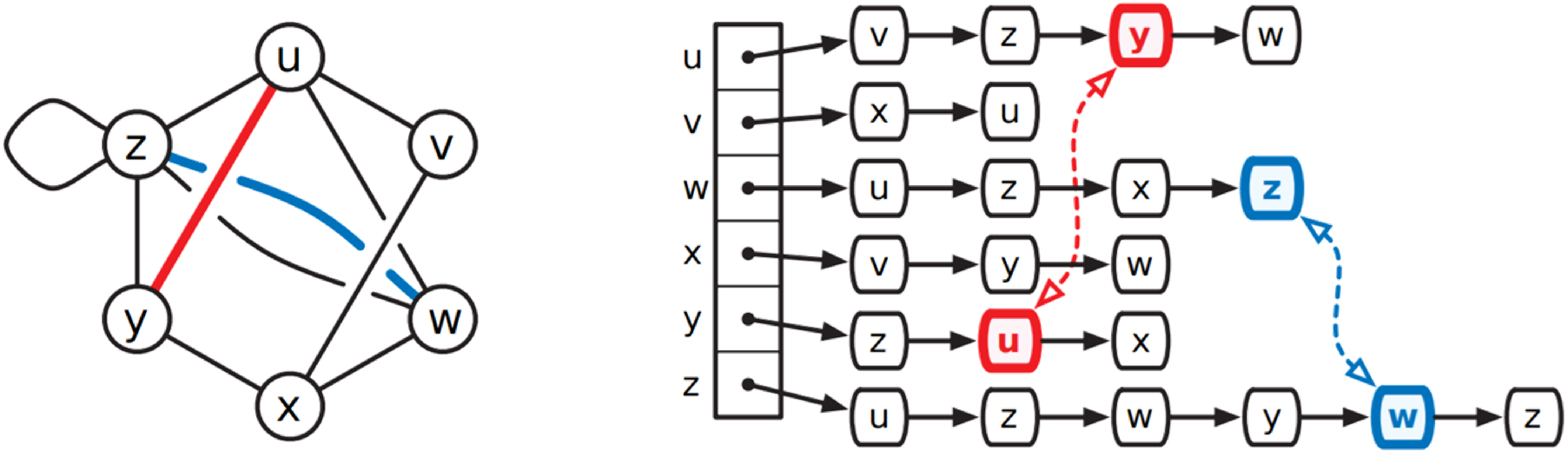
- V : vertices
- D : darts
- rev : reversal map D → D

  $\mathrm{rev} \circ \mathrm{rev}(d) = d$

- head : head vertex of a dart, D → V
- tail : head ∘ rev
- edge : $\{d, \mathrm{rev}(d)\}$



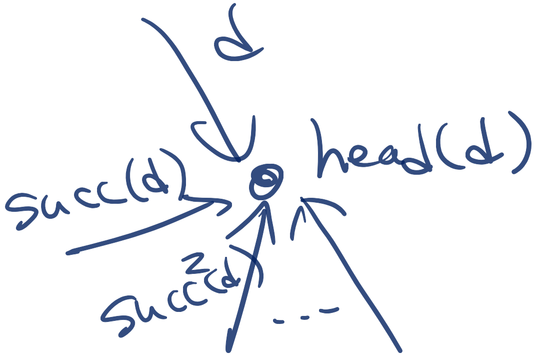
Graph Data Structure

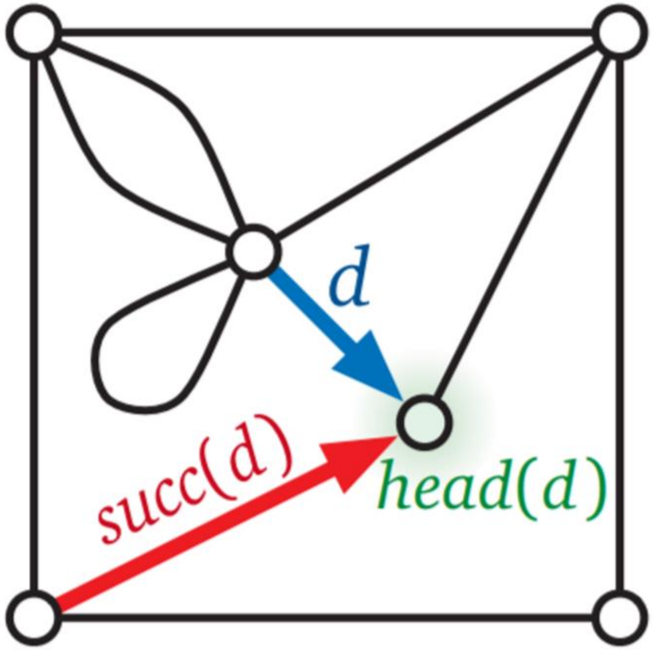


An incidence list representation of a graph, with the dart records for two edges emphasized. For clarity, most reversal pointers are omitted.

Surface Graph

- succ : D → D

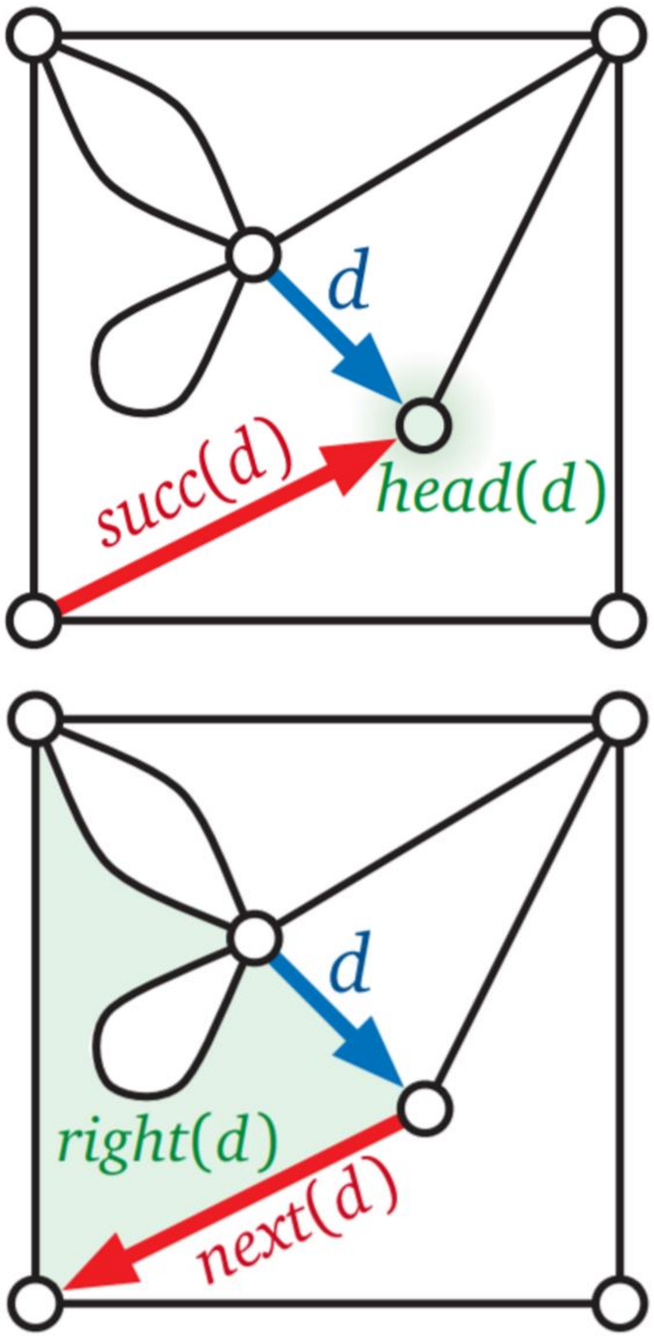




Surface Graph

- succ : next dart ccw around head(d)
- next : rev ∘ succ

  next dart cw. around "right" face incident to d.
- face : orbits of <u>next(·)</u>



Dual Surface Graph

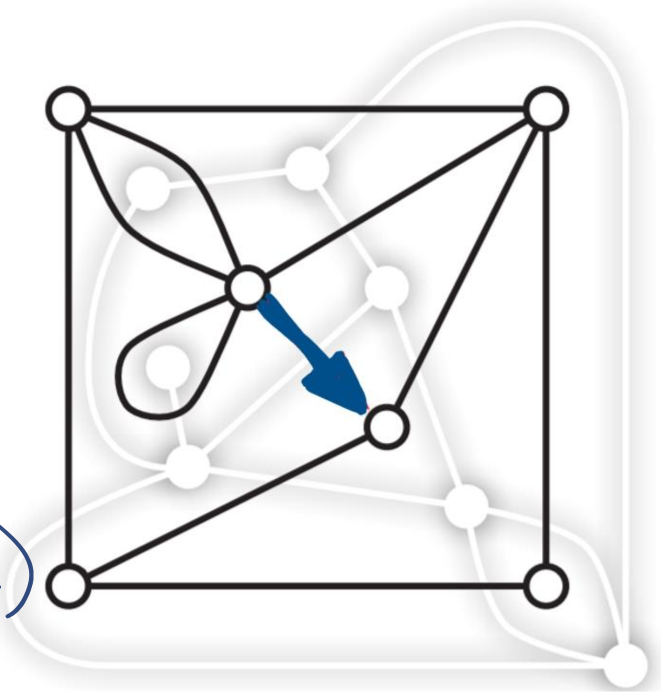
$(\text{succ}, \text{rev}) \Longleftrightarrow (\text{next}, \text{rev})$

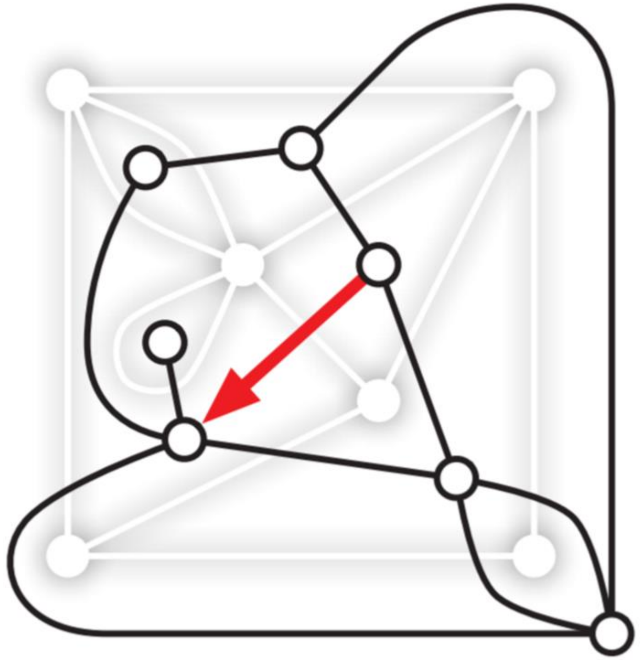
dual graph $G^* = (V^*, D^*, \text{rev}, \text{rev} \circ \text{succ})$, where $\text{rev} \circ \text{succ} = \text{next}$

V^*: orbits of $\text{next}(\cdot)$

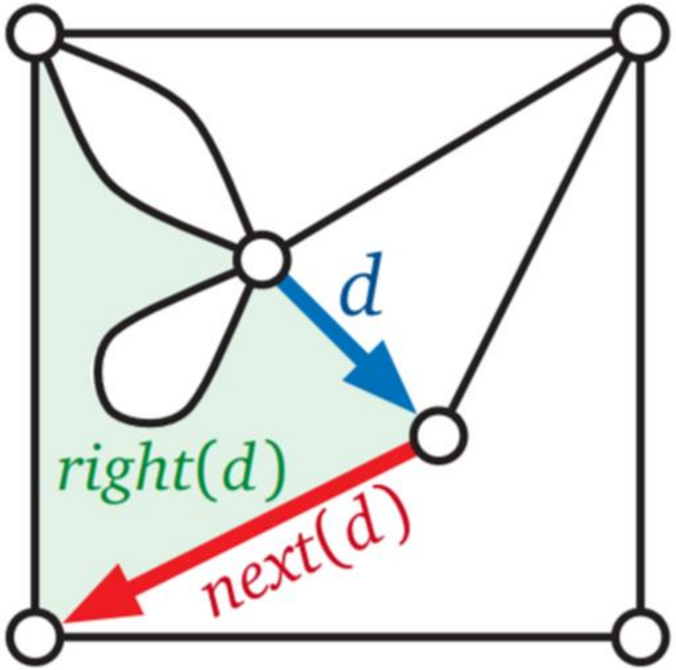
D^*: same as D

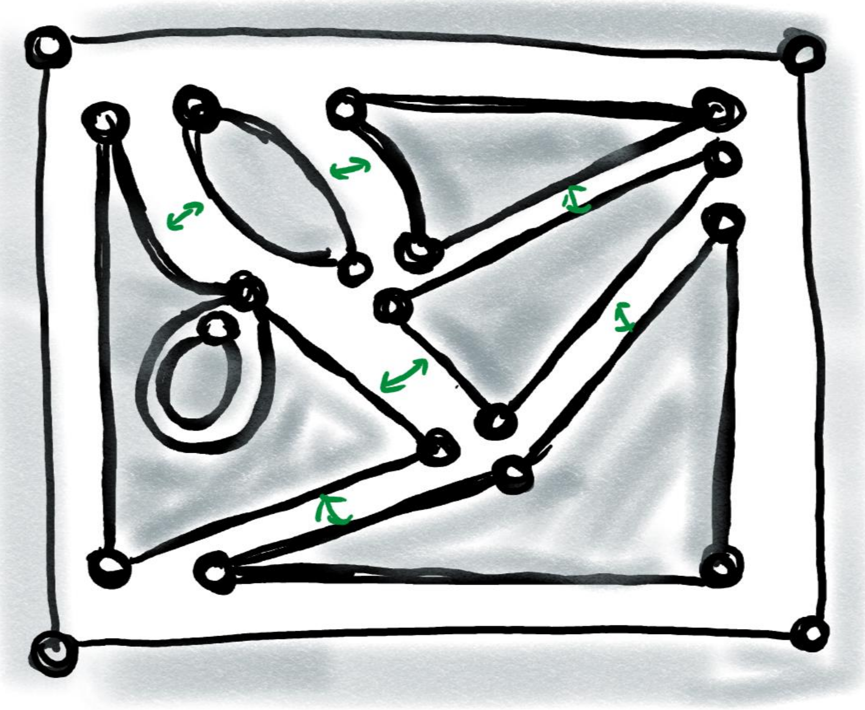
F^*: orbits of $\text{succ}(\cdot)$

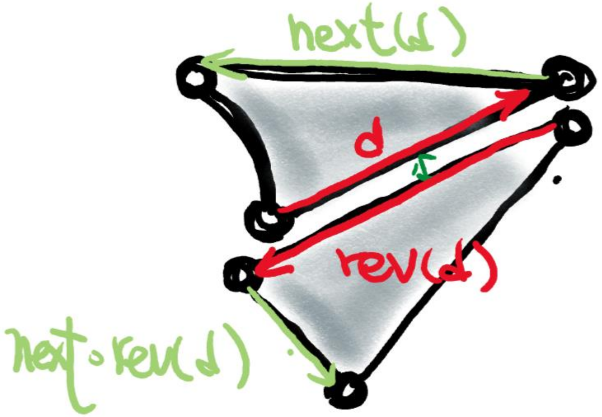




Polygonal Schema IS Rotation System







Theorems we secretly assumed

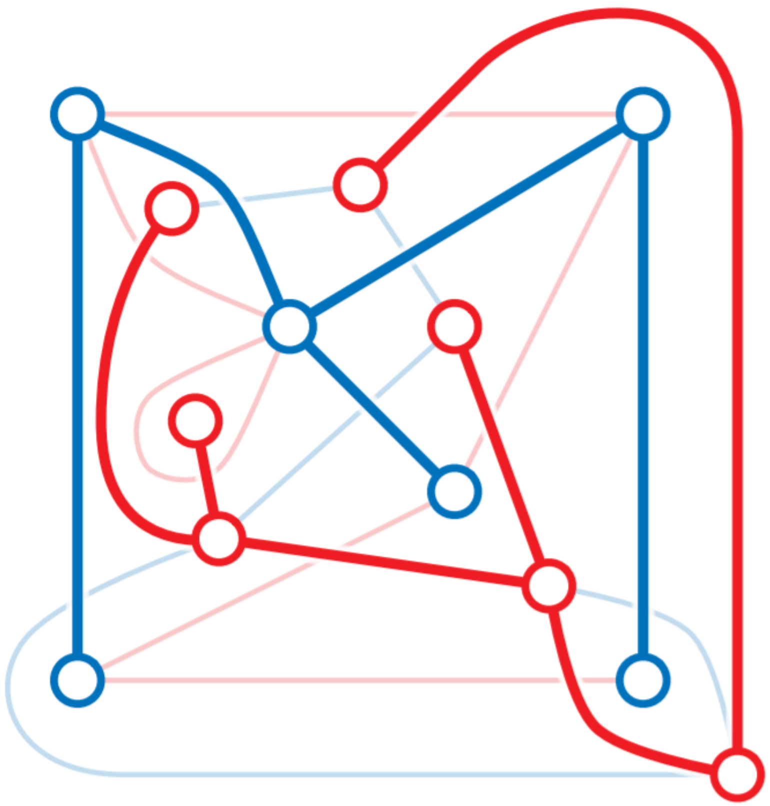
- Triangulation Theorem [Kerékjártó-Radó 1925]
  - Any surface can be cut into triangles
- Refinement Theorem [Moise 1977]
  - Any two triangulations have a common refinement
- Existence of rotation system
  - Every surface-embedded graph has a rotation system

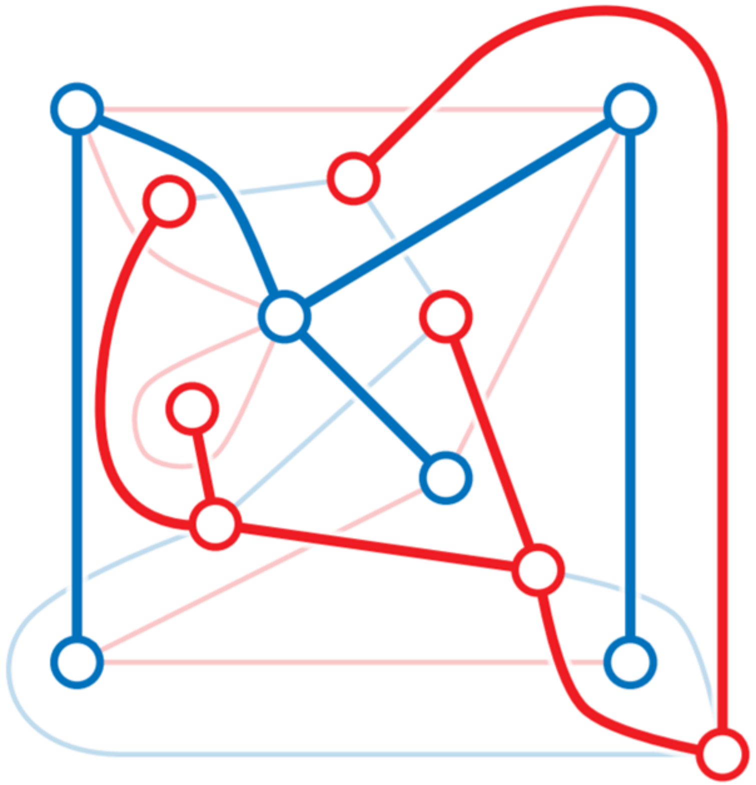
Let's Focus on Plane Graphs:

Tree-Cotree Decomposition

primal G	dual G^*	primal G	dual G^*
vertex v	face v^*	empty loop	spur
dart d	dart d^*	loop	bridge
edge e	edge e^*	cycle	bond
face f	vertex f^*	even subgraph	edge cut
$tail(d)$	$left(d^*)$	spanning tree	complement of spanning tree
$head(d)$	$right(d^*)$	$G \setminus e$	G^* / e^*
$succ$	$rev \circ succ$	G / e	$G^* \setminus e^*$
clockwise	counterclockwise	minor $G \setminus X / Y$	minor $G^* \setminus Y^* / X^*$

Correspondences between features of primal and dual planar maps





Euler's Formula [Euler 1750] [Legendre 1794] [Cayley-Listing 1861]

For any plane graph G,

$$V_G - E_G + F_G = 2$$

Q. Does Euler's Formula Hold for Surface Graphs?

Next time.

Surface hard to visualize?

The space is weird?