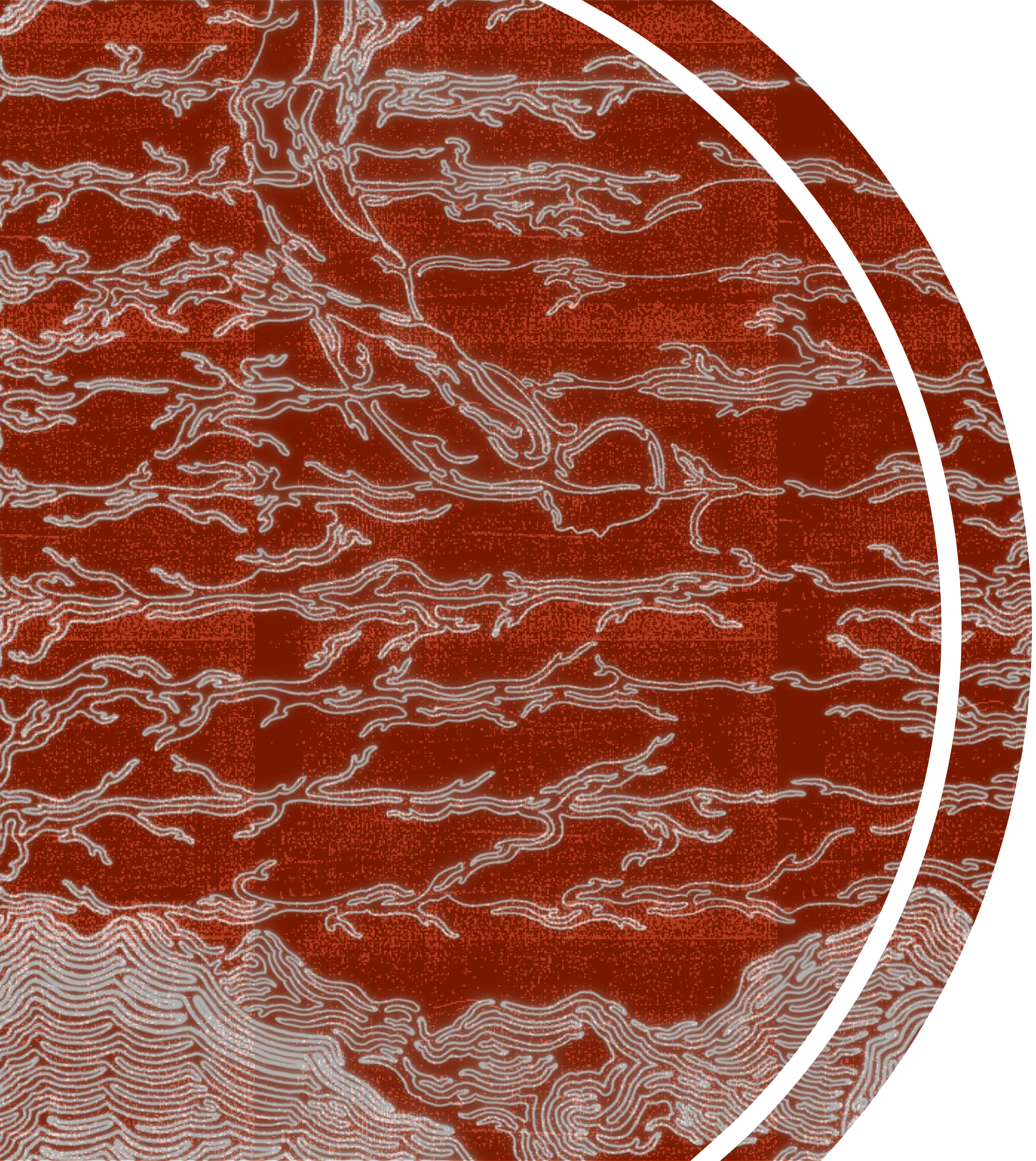


Introduction to Computational Topology

Hsien-Chih Chang

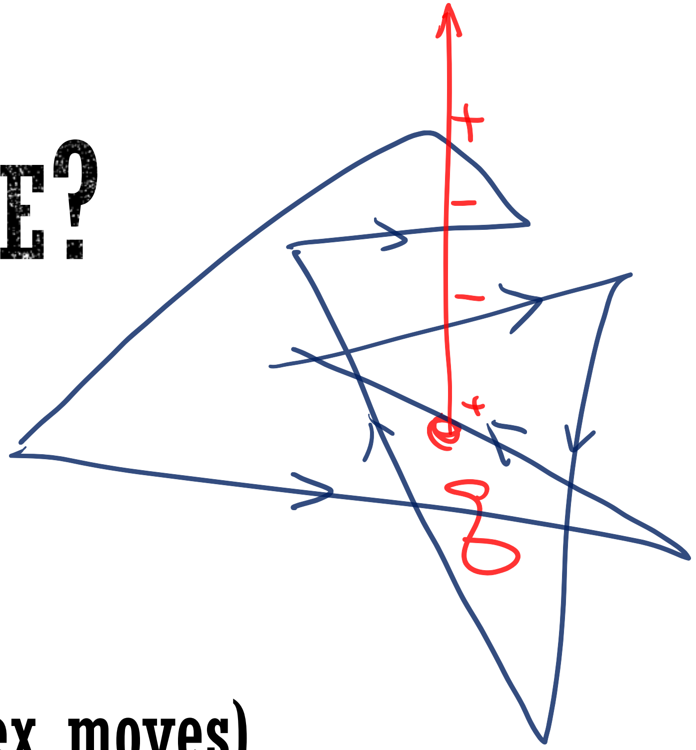
Lecture 2, September 16, 2021

Administrivia

- Homework 0 is due 9/20 (next Monday)
  - Starting from Homework 1, collaboration up to 2 people
  - Open-everything
- Come to the office hour tomorrow!

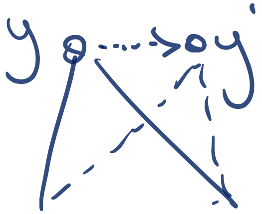
- Again, STOP me anything you have questions

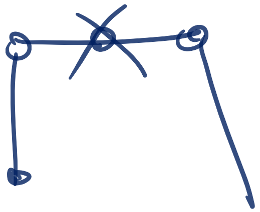
Where were we?



- $\text{Wind}_q(P)$

- Discrete homotopy (vertex moves)

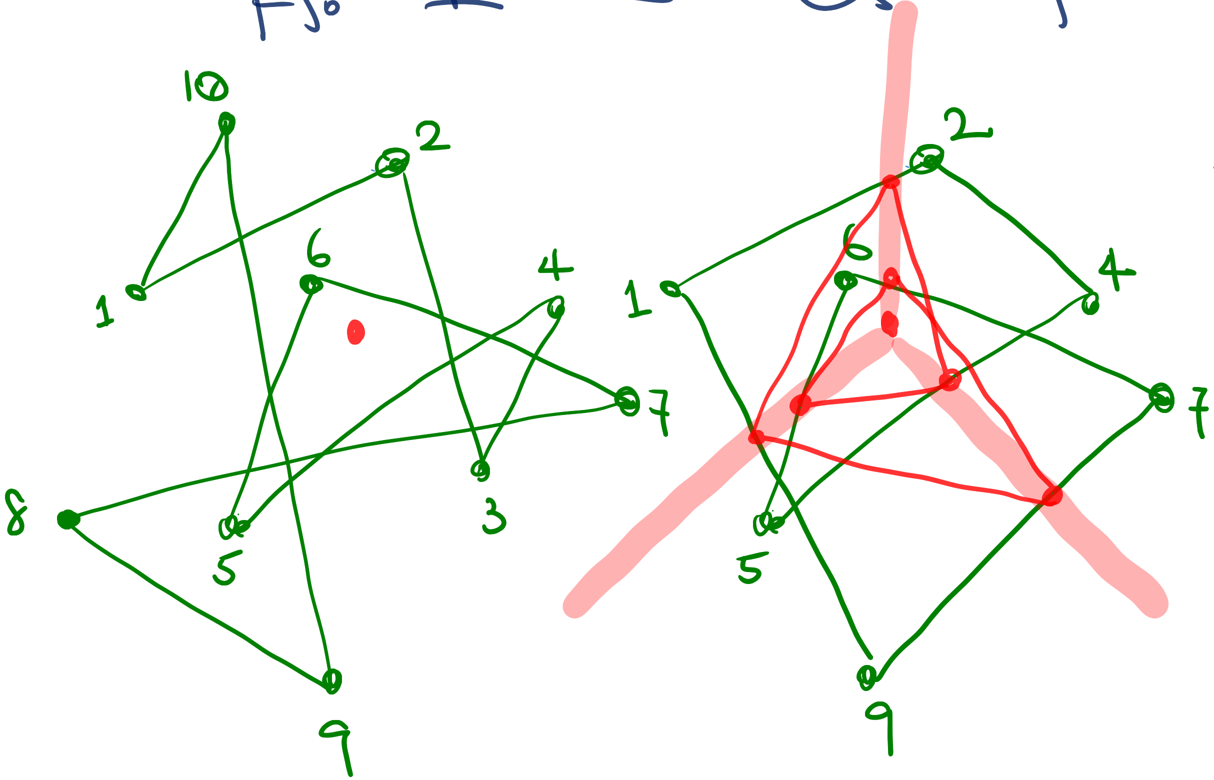




- **Lemma.** $\text{Wind}_q(P)$ is invariant under safe vertex moves.

Theorem. Two polygons P and Q are homotopic in $\mathbb{R}^2 \setminus q$ if and only if they have the same Wind_q. [Hopf 1935]

Pf. $P \rightsquigarrow \Delta^k \rightsquigarrow Q$ if $\text{wind}(P) = \text{wind}(Q) = k$.



3-coin(P):

```
(p, q, r) ← (1, 2, 3)
do:
    if Δ(p, q, r) is safe:
        (p, q, r) ← (p, r, succ(r))
    else
        (p, q, r) ← (q, r, succ(r))
while r ≠ 3
```

(I made mistake in class)

Wind_q is a Complete Homotopic Invariant!

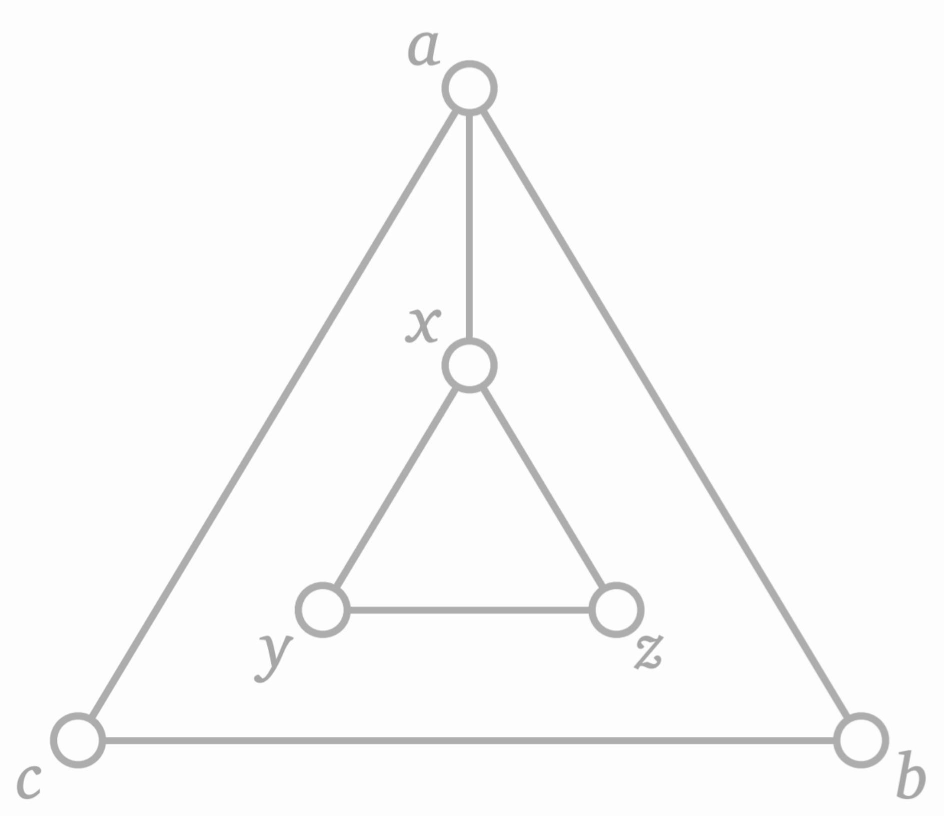
Takeaway.
Planar curve in punctured plane is described by #times it goes around the puncture.

Remark. The punctured plane $R^2\backslash q$ is different from the plane as spaces.

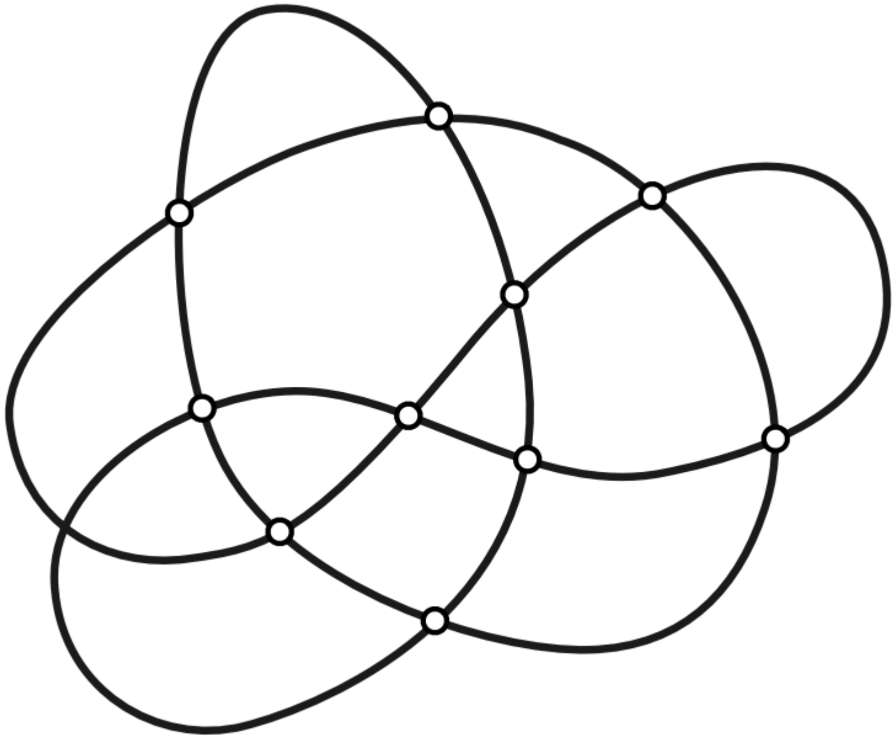
Regular Homotopy and Rotation Number

Switching Views

- Polygonal



- Generic

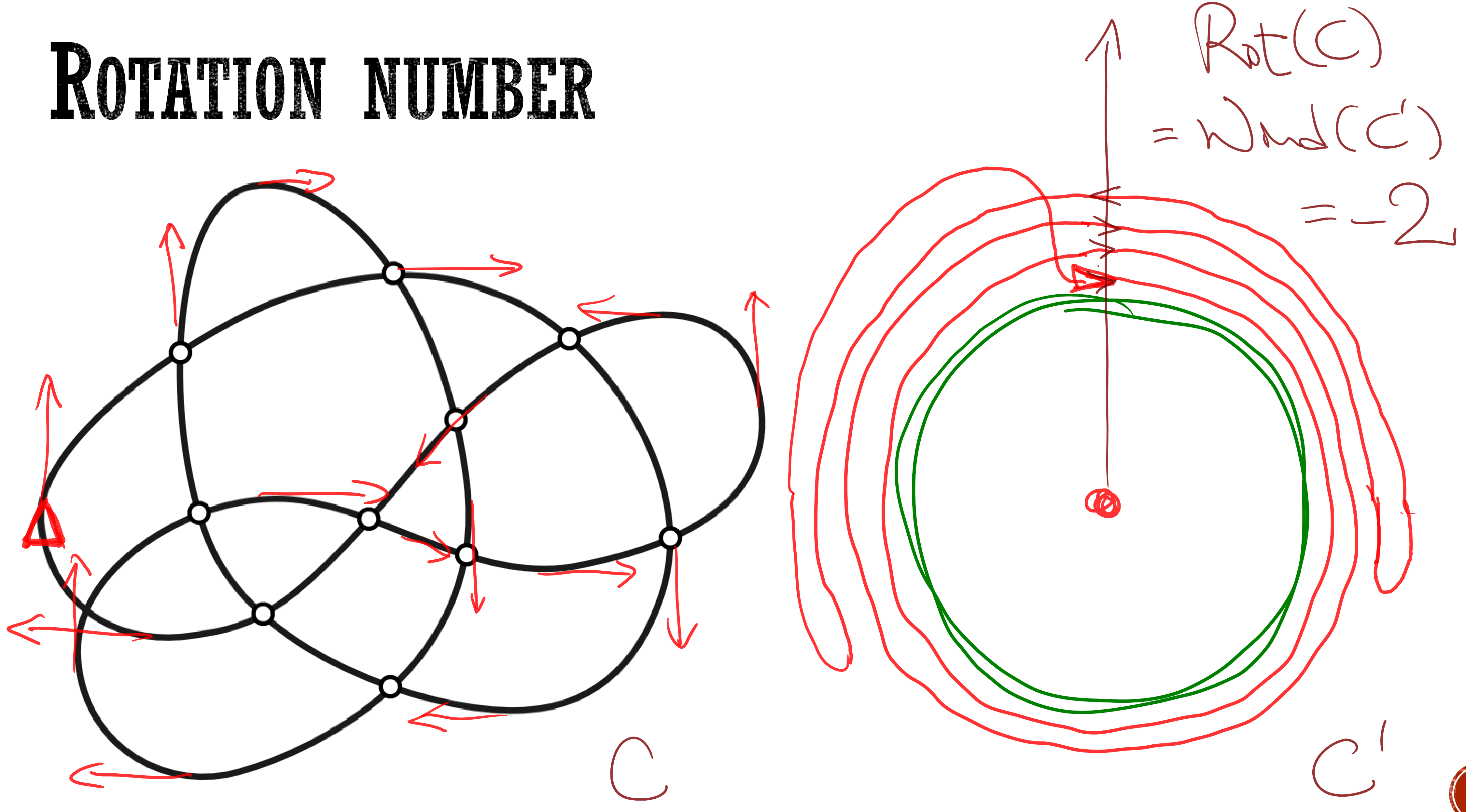


Untangling Gardening Hose

Can you untangle the hose without lifting or twisting?

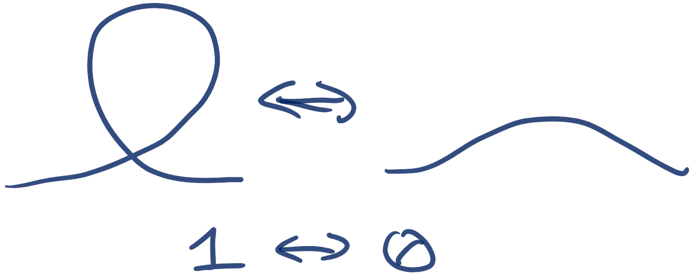
(Cables magically pass through each other)

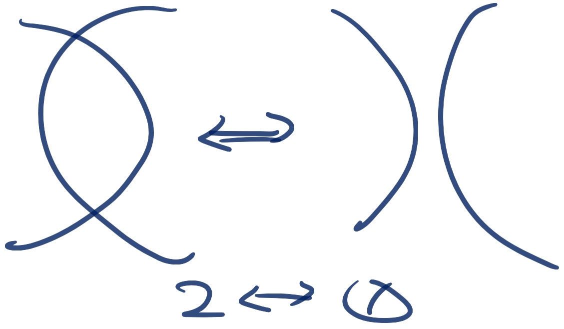
Rotation number

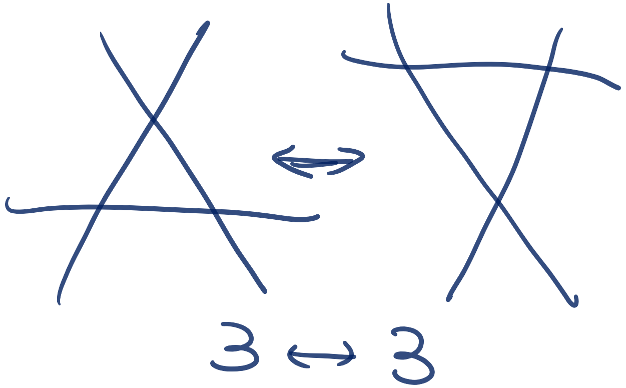


Definitions

- Homotopy moves

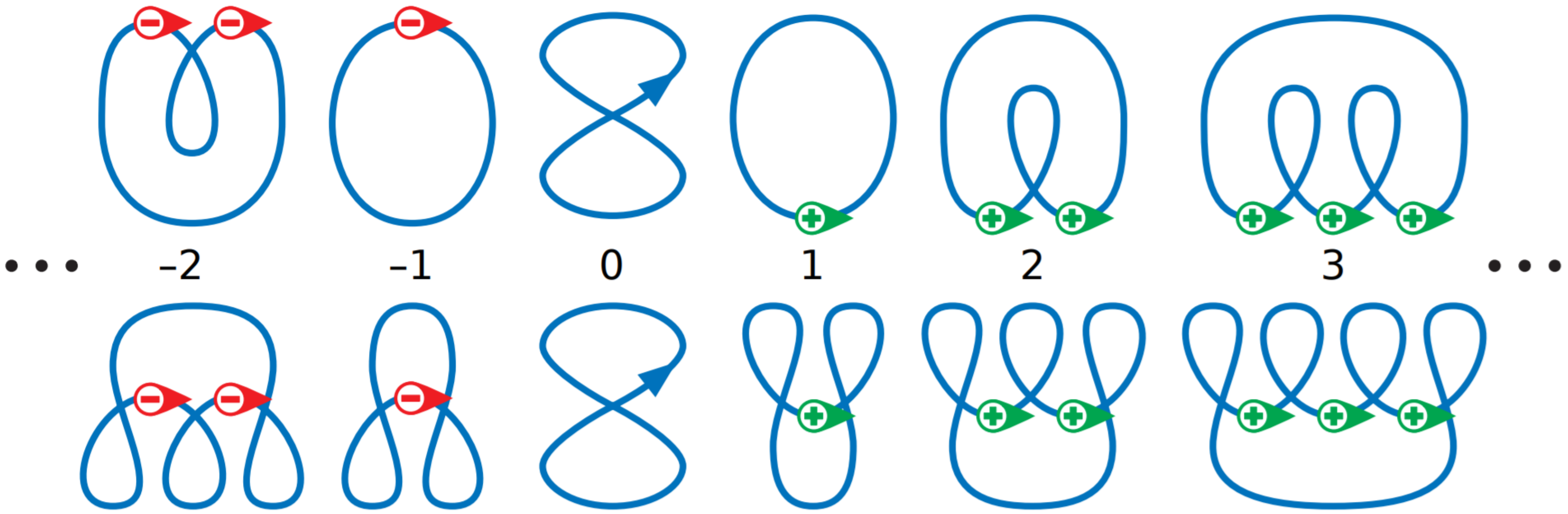






- Regular homotopy

only 2 → ∅ moves
∅ → 2
3 → 3



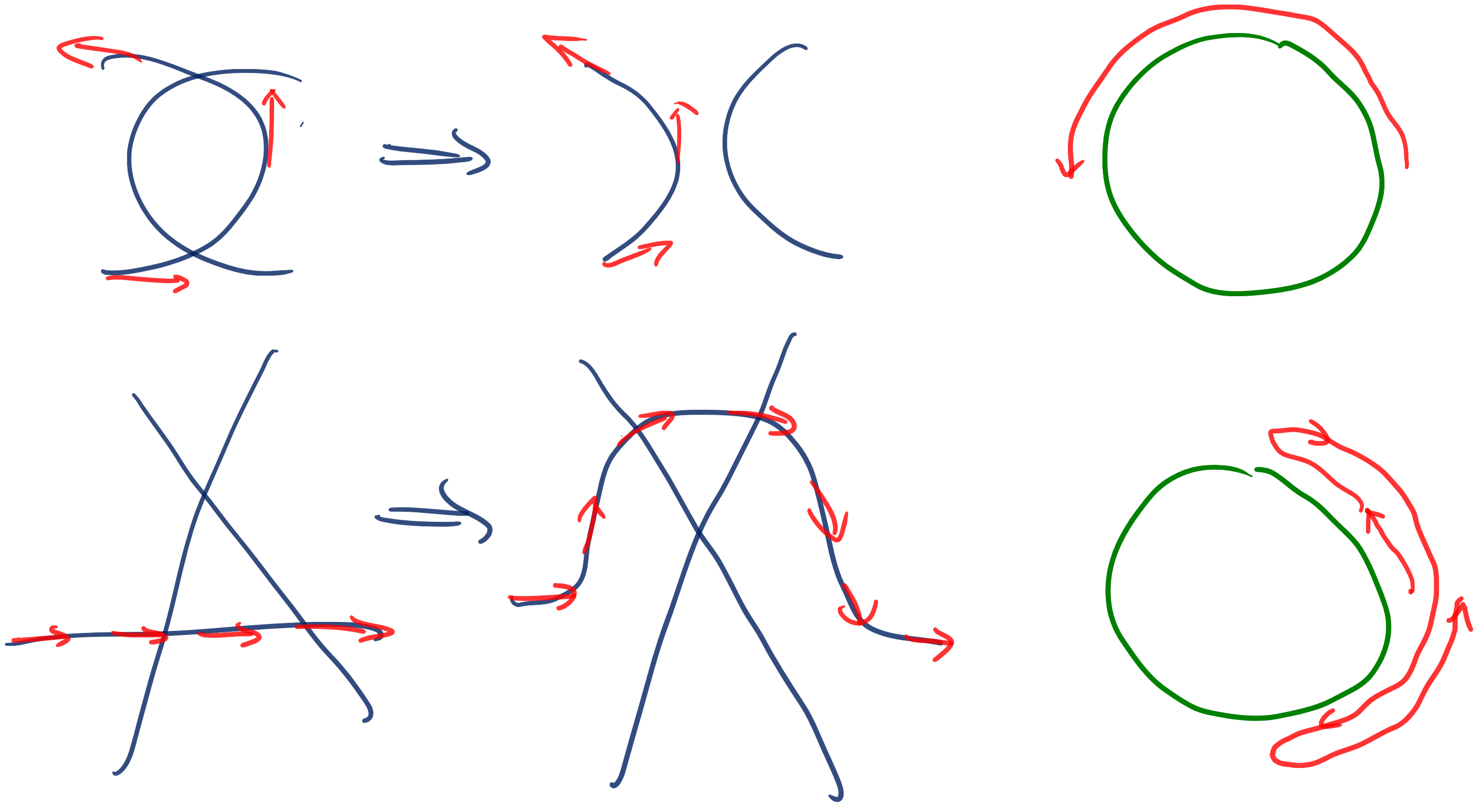
Whitney-Graustein Theorem

[Whitney 1937] [Boy 1901] [Meister 1770]

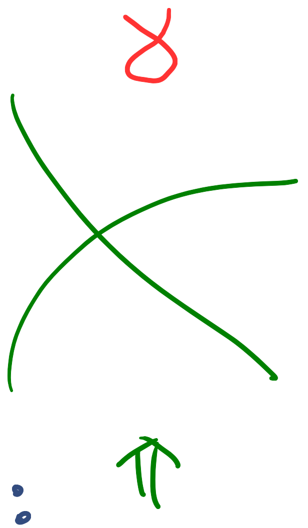
Any two regular curves are regular homotopic if their rotation numbers are the same.

Proof of Whitney-Graustein Theorem

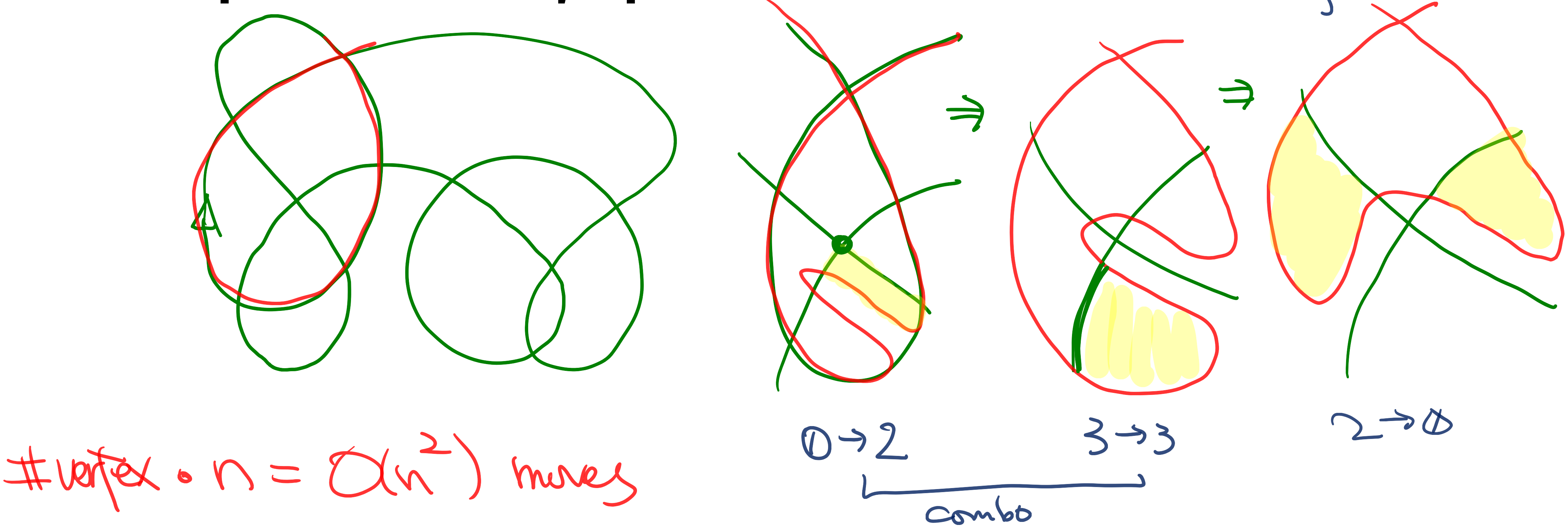
- Rotation number is invariant under regular homotopy:



Proof of Whitney-Graustein Theorem

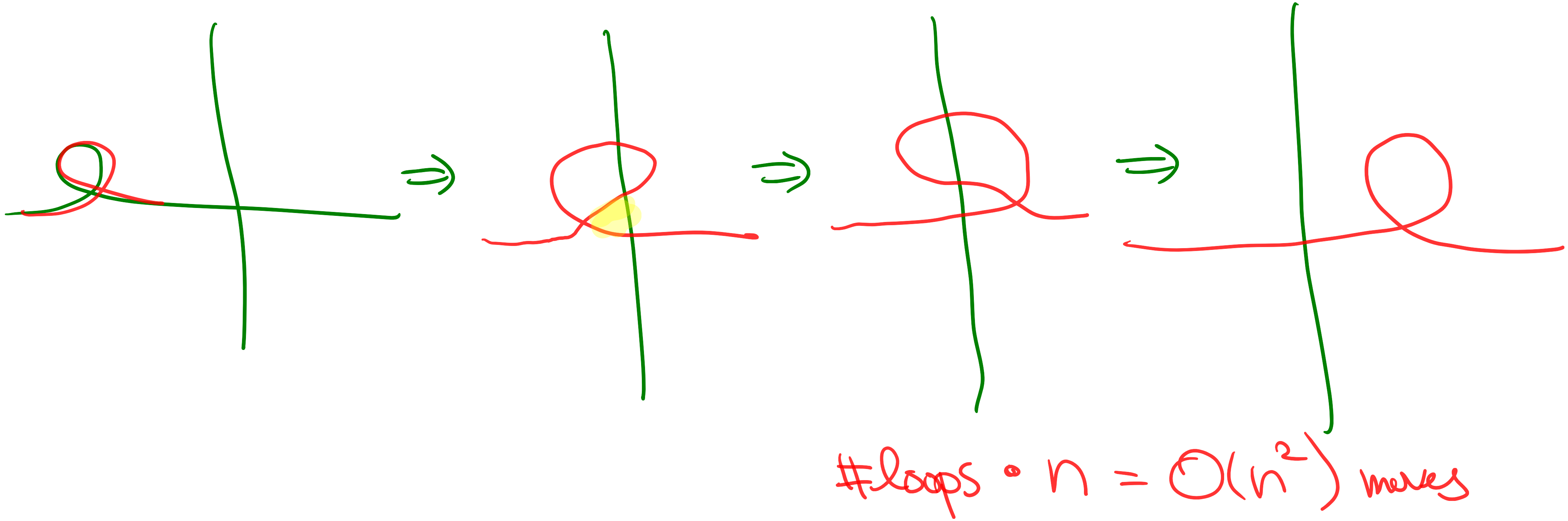


- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 1. Shrink an arbitrary loop



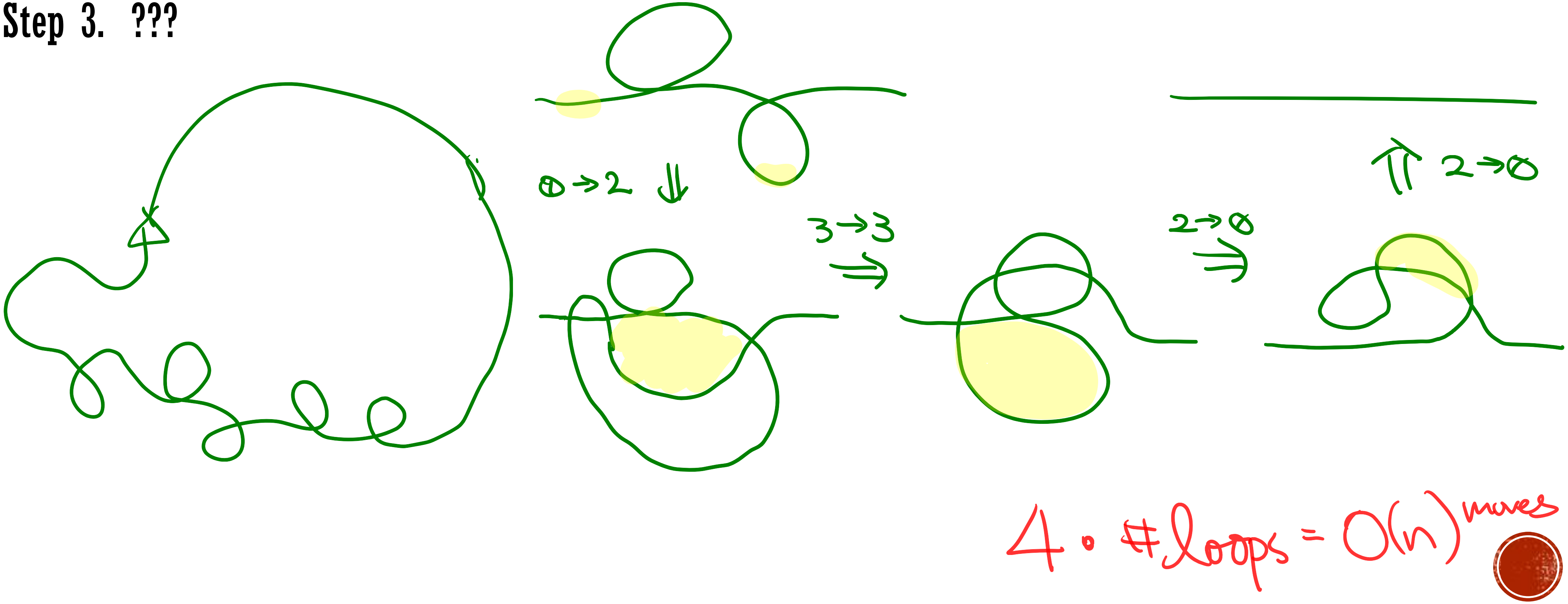
Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 2. Move empty loop to outside



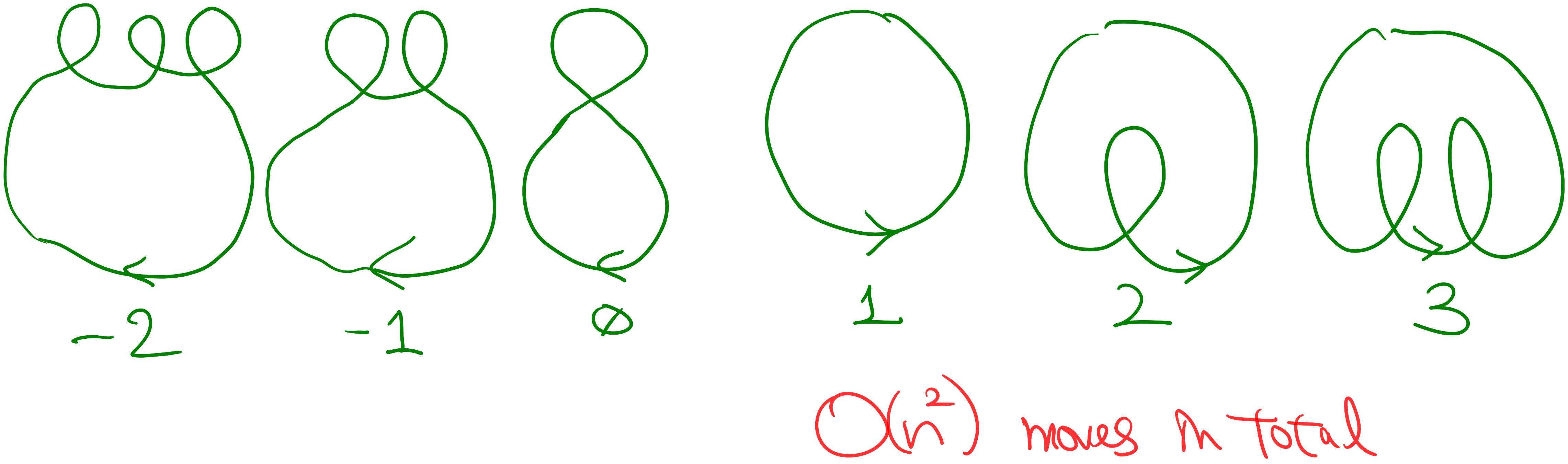
Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 3. ???



Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $O^{Rot(C)}$:
  - Step 4. PROFIT



Rot is a Complete Regular-Homotopic Invariant!

Takeaway.
Planar curve can be described by how many times its derivative winds around the origin.

Intermission

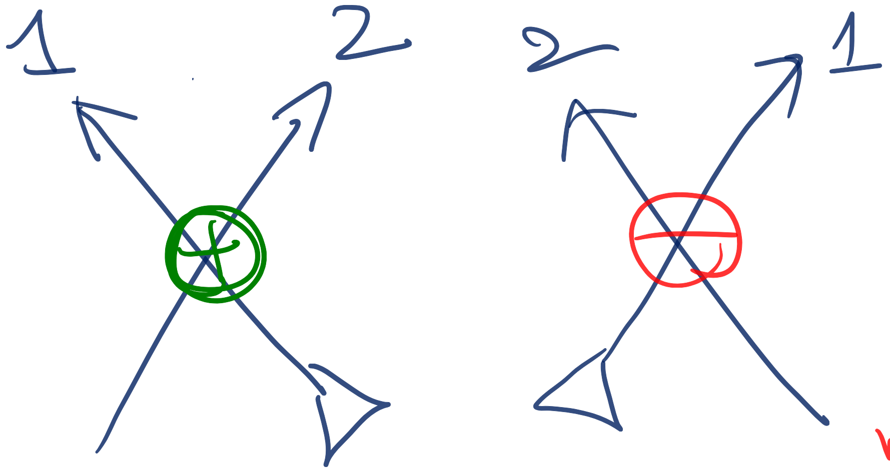
Food for thought.

WIND_q and Rot are really the same. Why?

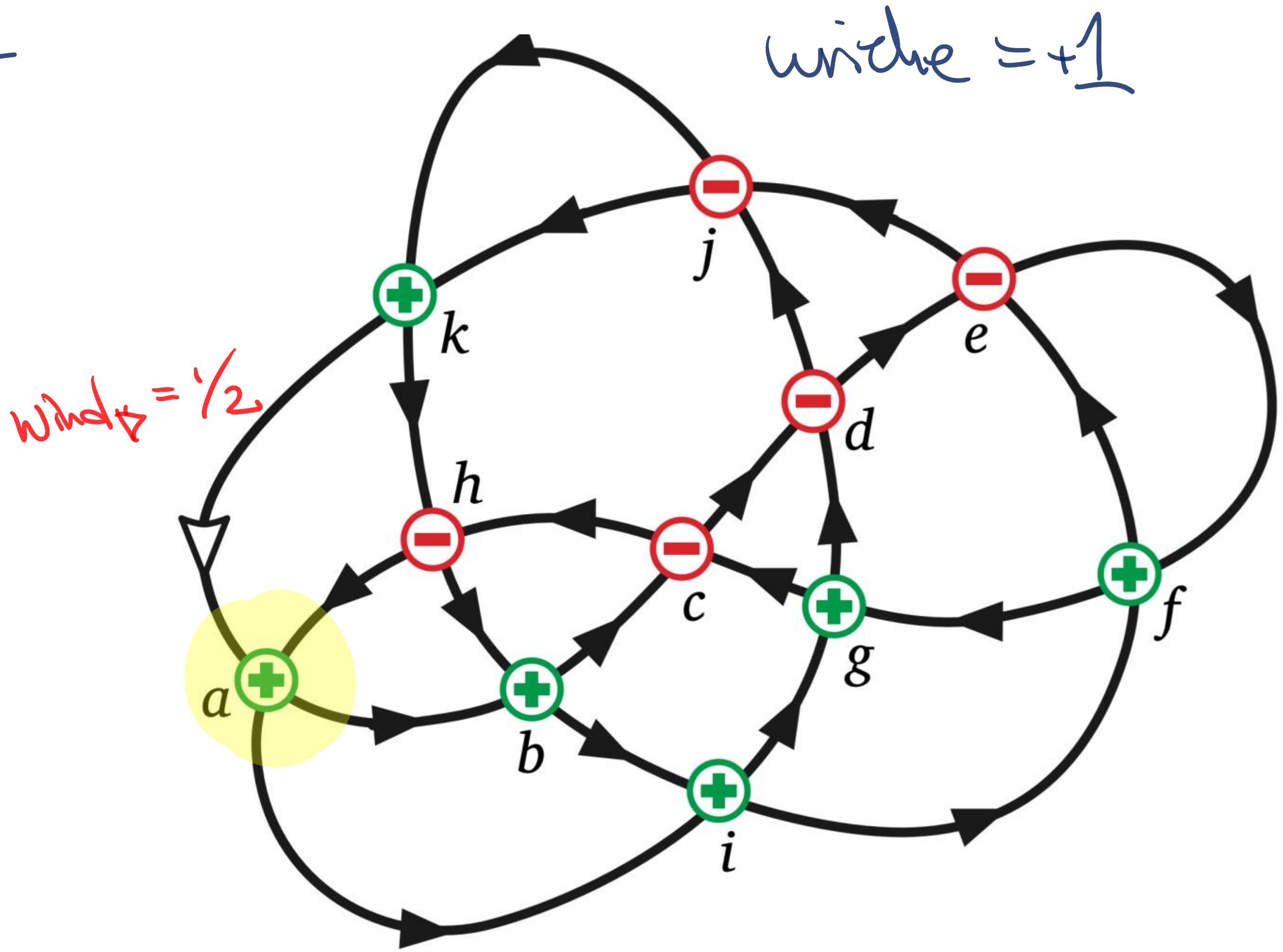
Combinatorics of Curves

Q. Is the $O(n^2)$ Bound Tight?

Gauss Signing and Writhe



$$\mathrm{Writhe}(C) := \sum_{x} \mathrm{sgn}(x)$$



PROPOSITION. $\mathrm{Rot}(C) = 2\mathrm{Wind}_{\nabla}(C) + \mathrm{Writhe}(C)$.

[Titus 1960]
[Gauss ~1823]

Pf sketch. Prove that equation holds under regular homotopy.

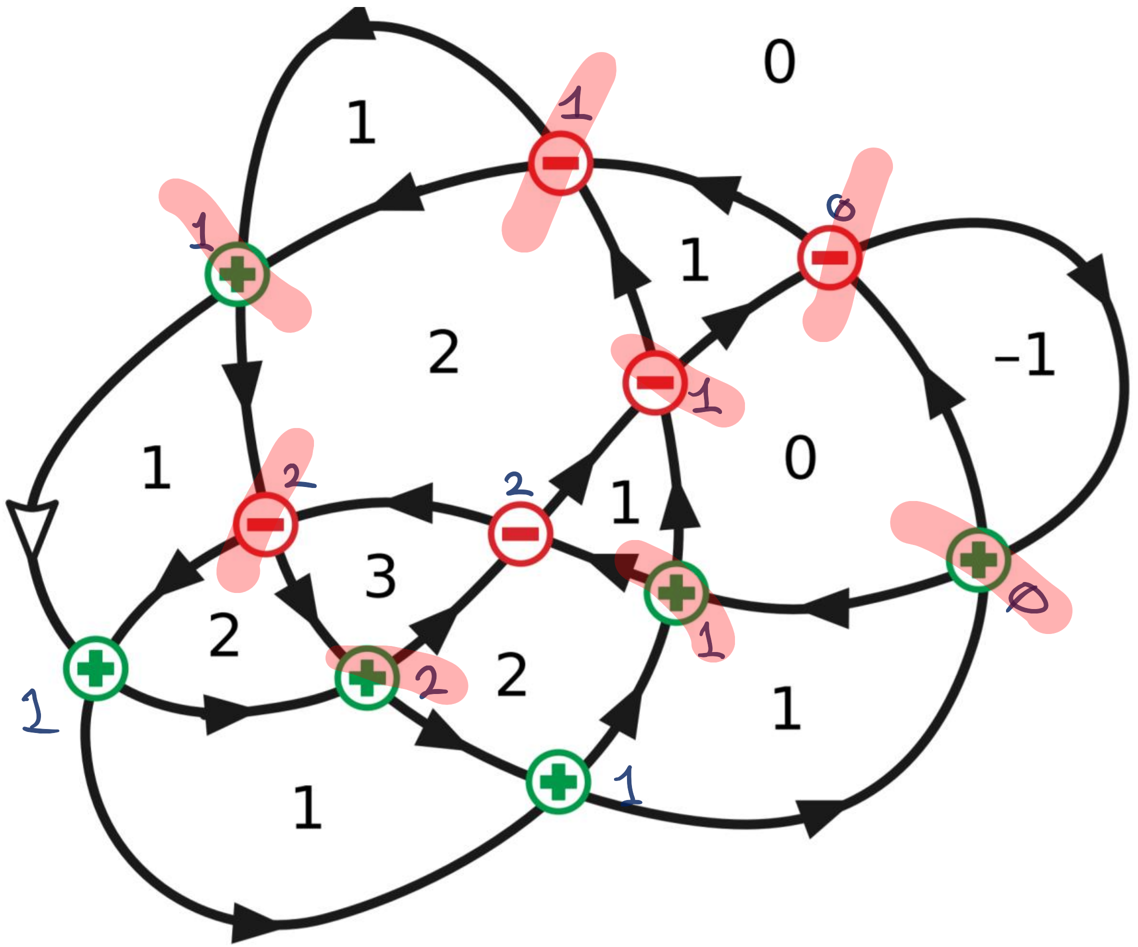
So just check the canonical curves $\circ^{\mathrm{Rot}(C)}$.

STRANGENESS

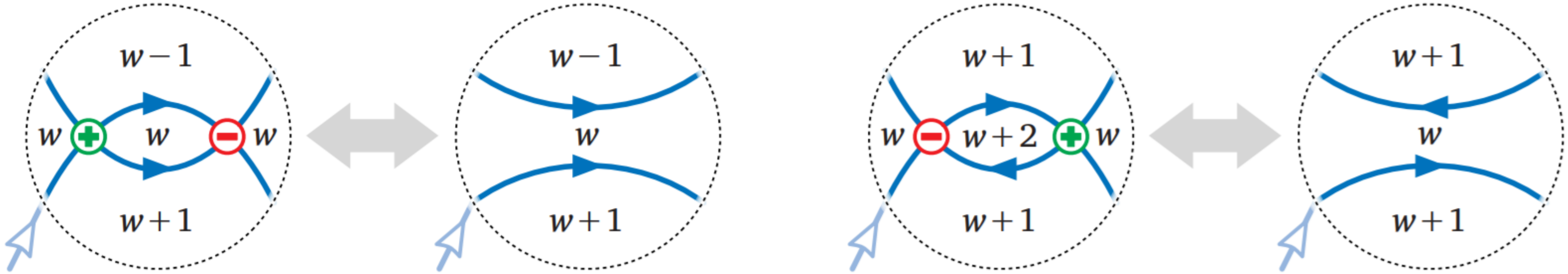
Arnold
[~~Arnaud~~ 1994]

- $\mathrm{St}(C) = \sum_{\text{vertex } x} \mathrm{sgn}(x) \cdot \mathrm{Wind}_x(C)$

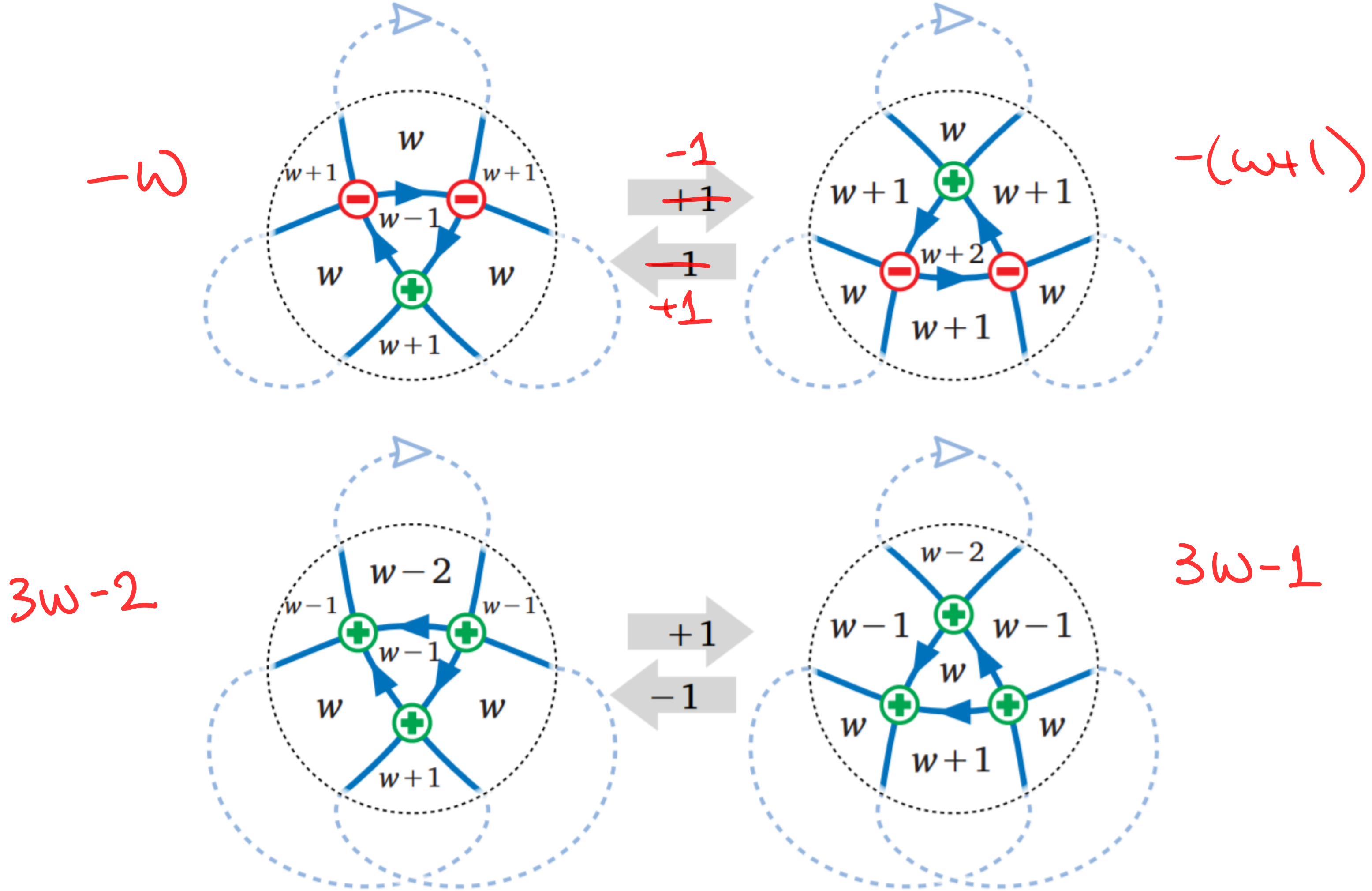
$St(C) = -2 + 1 + 1 = 0$



Theorem. Some generic curves require $\Omega(n^2)$ regular homotopy moves to untangle. [Nowik 2009]

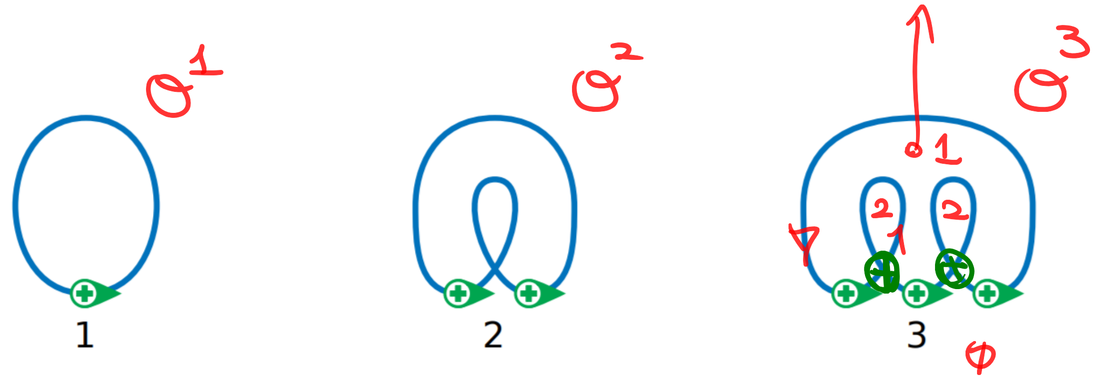


THEOREM. Some generic curves require $\Omega(n^2)$ regular homotopy moves to untangle. [Nowik 2009]

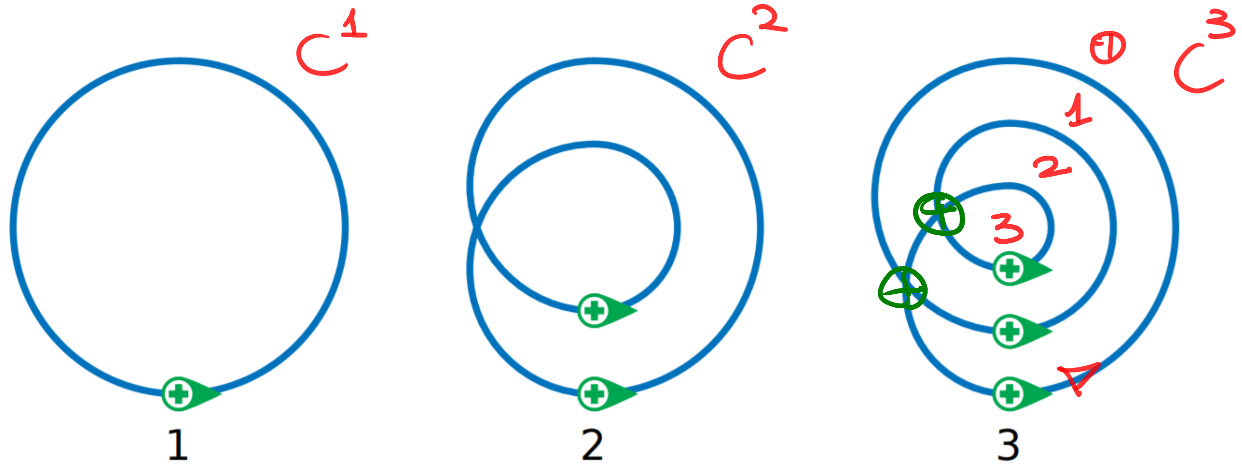


Canonical Curves

- $St(C) = \sum_{\text{vertex } x} sgn(x) \cdot Wind_x(C)$



$St(O^k) = k-1$



$St(C^k) = 1 + \dots + (k-1) = \binom{k}{2}$

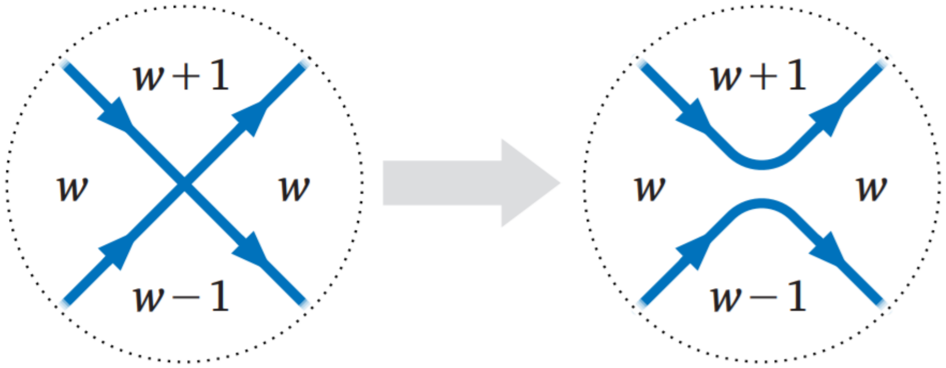
Closing Q. Canonical Curves in Pacman space?

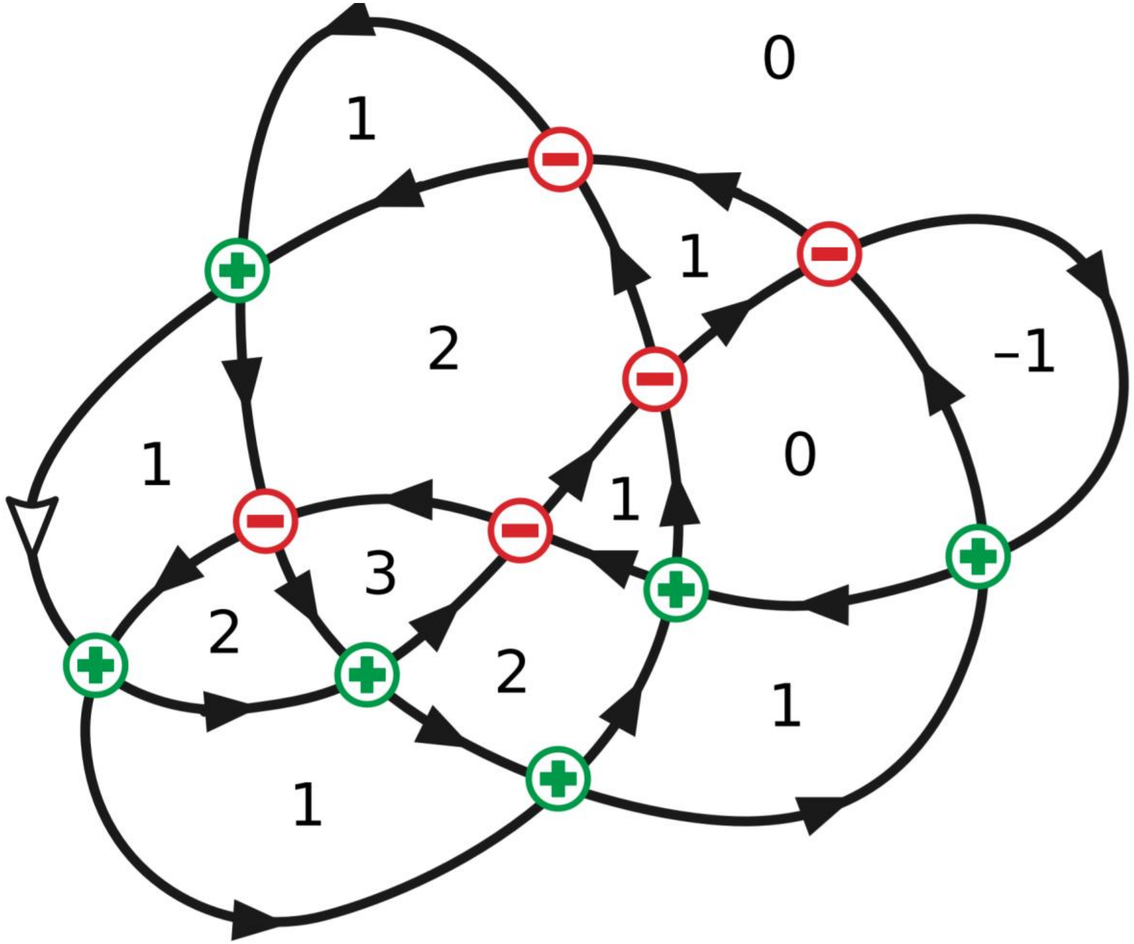
Coming up next week.

Going upwards, one-dimension Higher.

Smoothing and Seifert Decomposition

[Seifert 1931]
[Gauss ~1823]





Smoothing and Seifert Decomposition

[Seifert 1931]
[Gauss ~1823]