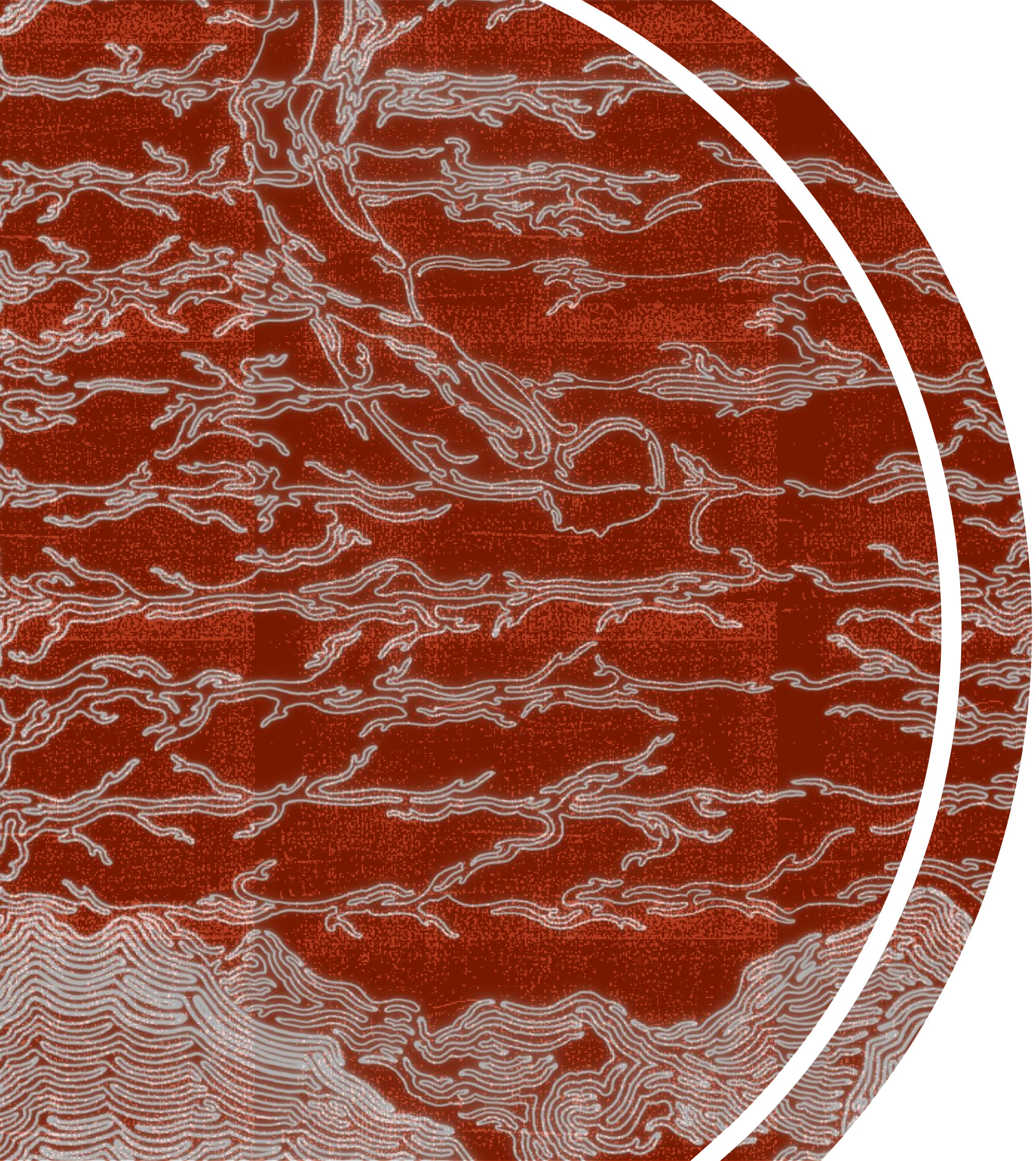


Introduction to Computational Topology

Hsien-Chih Chang

September 14, 2021

Introduce Yourself!

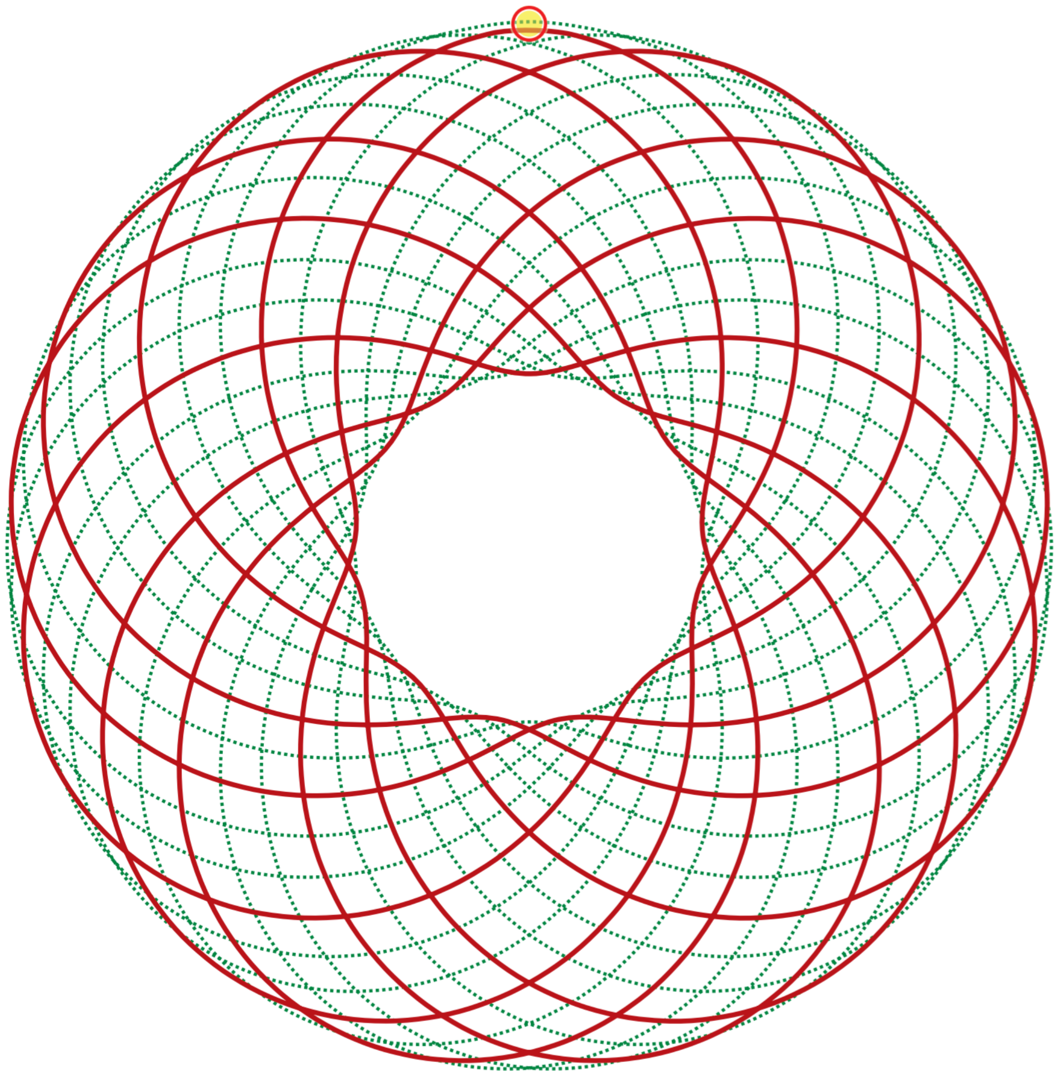
Who are you?

Why do you care about topology?

Shape of the Space
How do we Compute it?

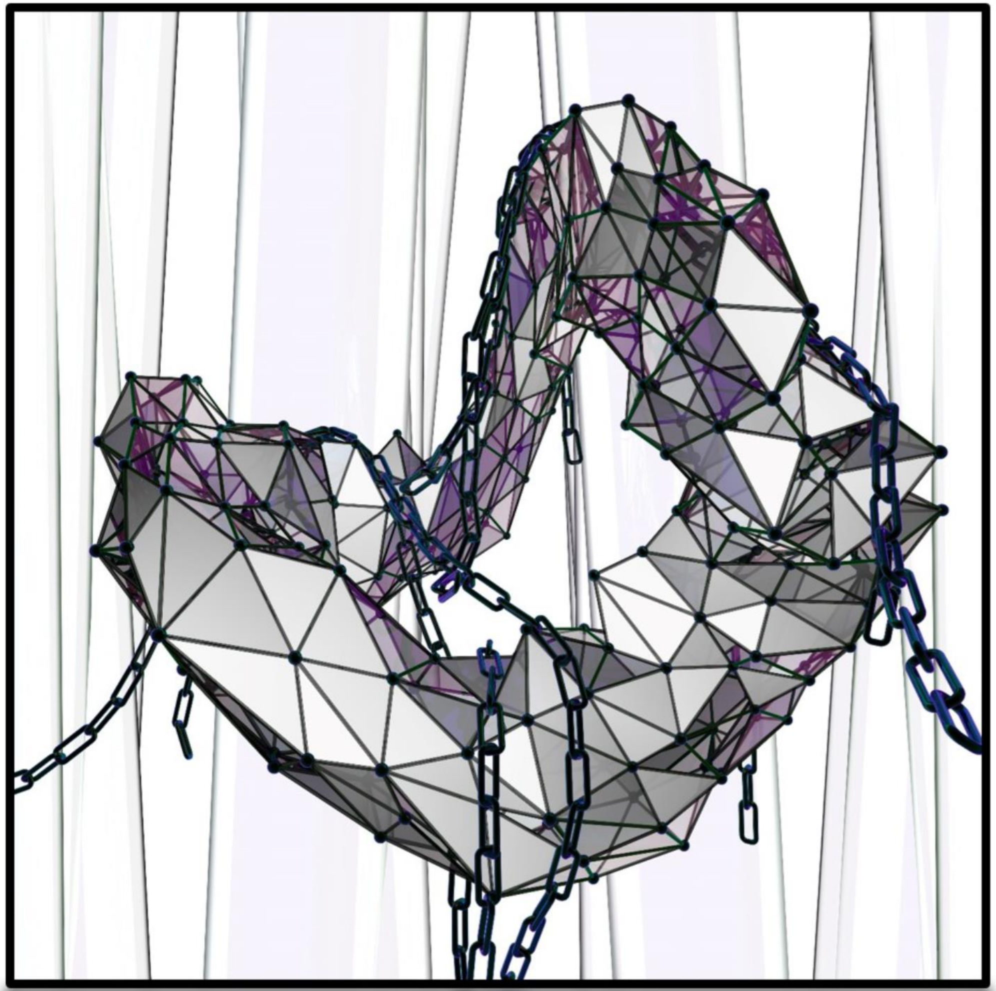
Course Outline

- Theory
  - Curves



Course Outline

- Theory
  - Curves
  - Surfaces/Complexes

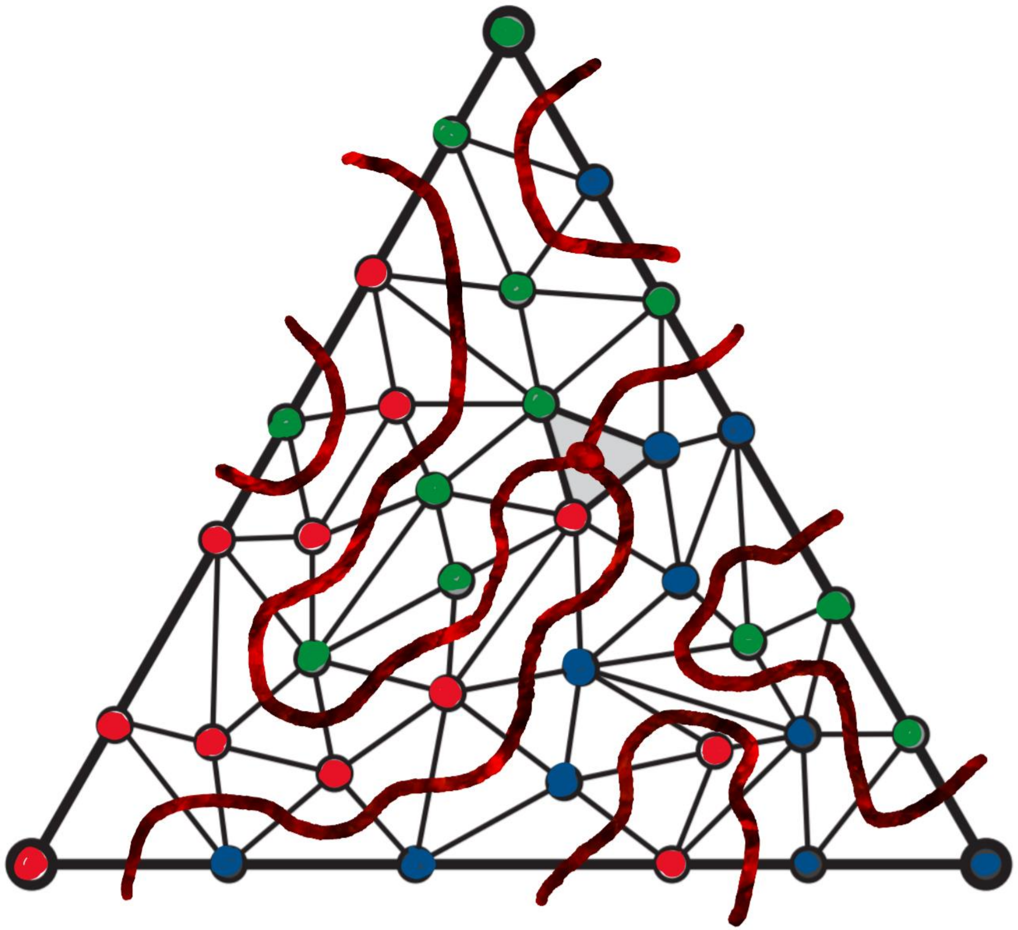


Course Outline

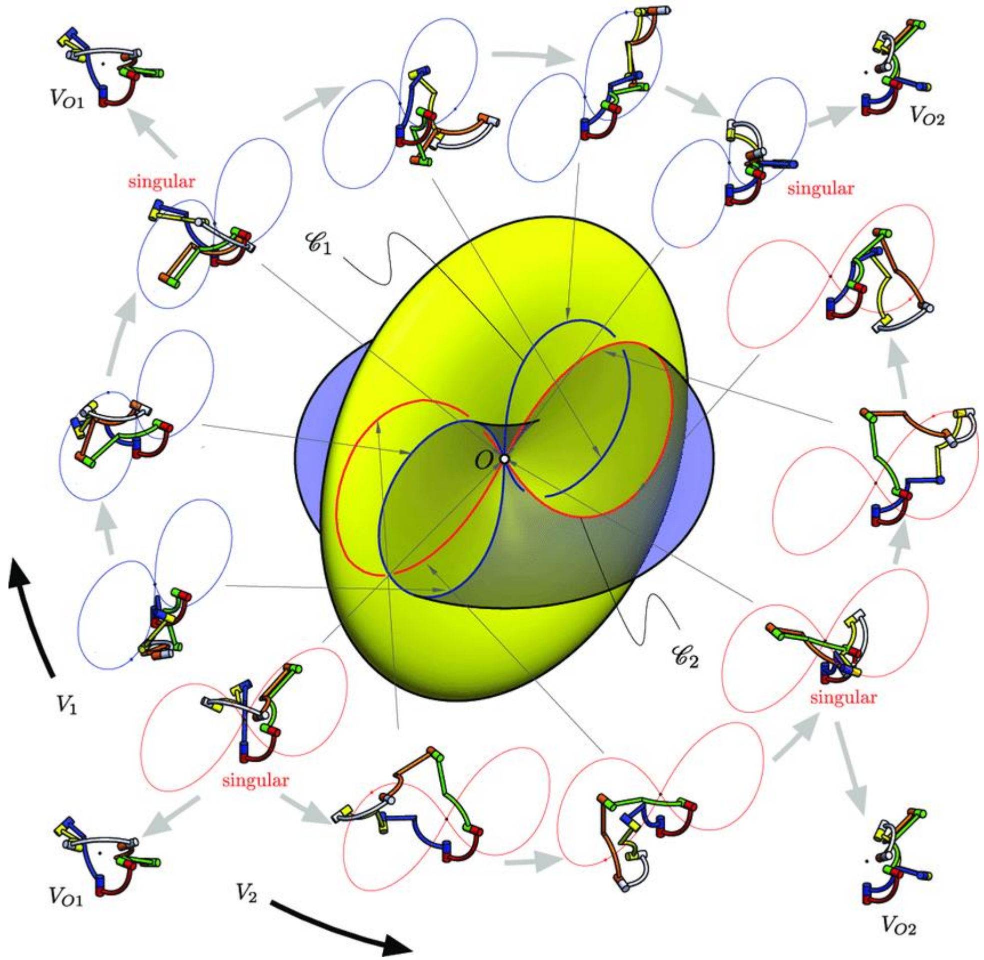
- Theory
  - Curves
  - Surfaces/Complexes
  - Homotopy

Course Outline

- Theory
  - Curves
  - Surfaces/Complexes
  - Homotopy
  - Homology

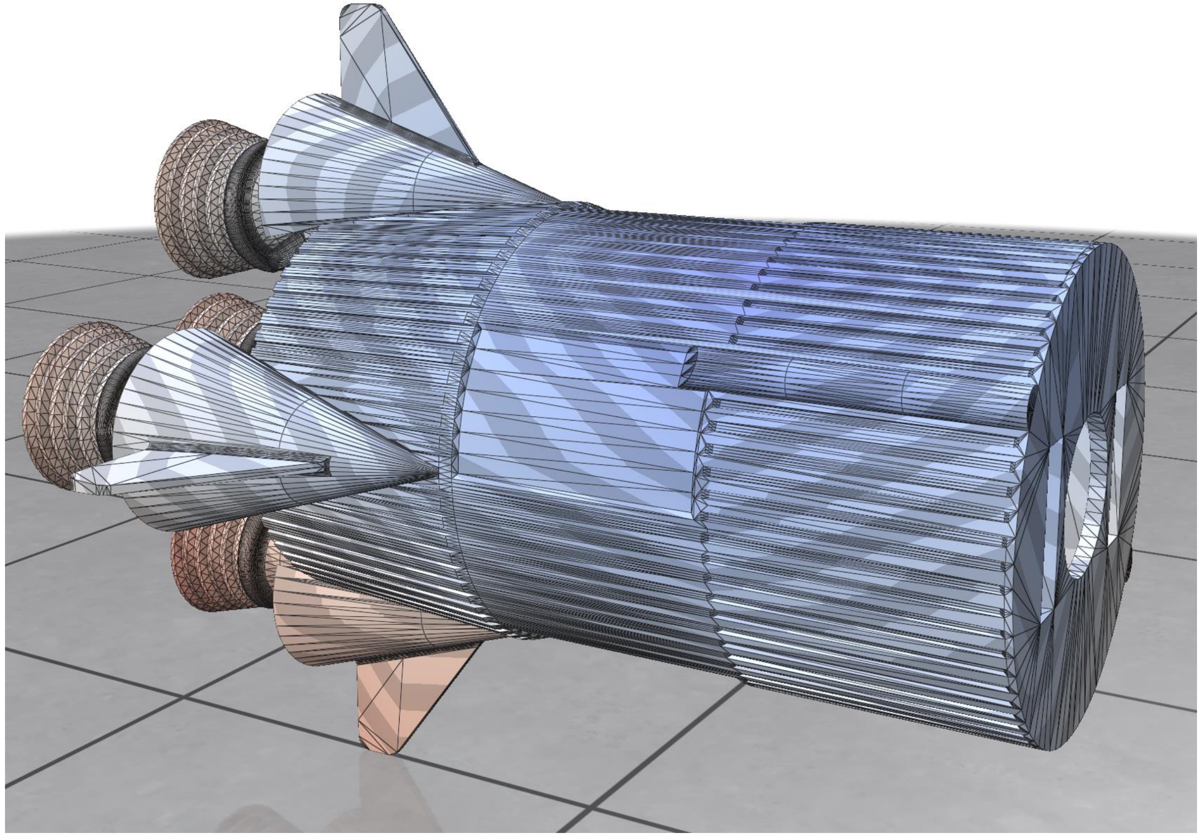


Course Outline



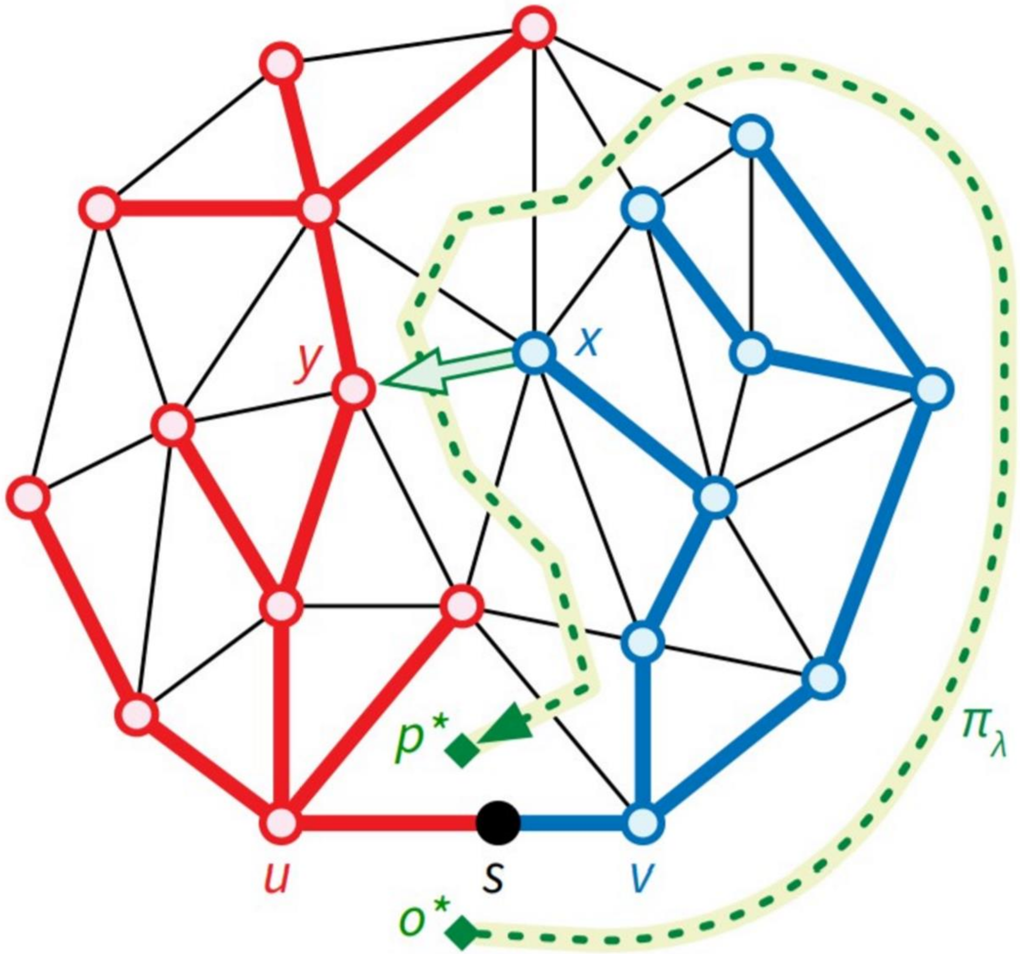
- Applications
  - Linkage and Folding

Course Outline



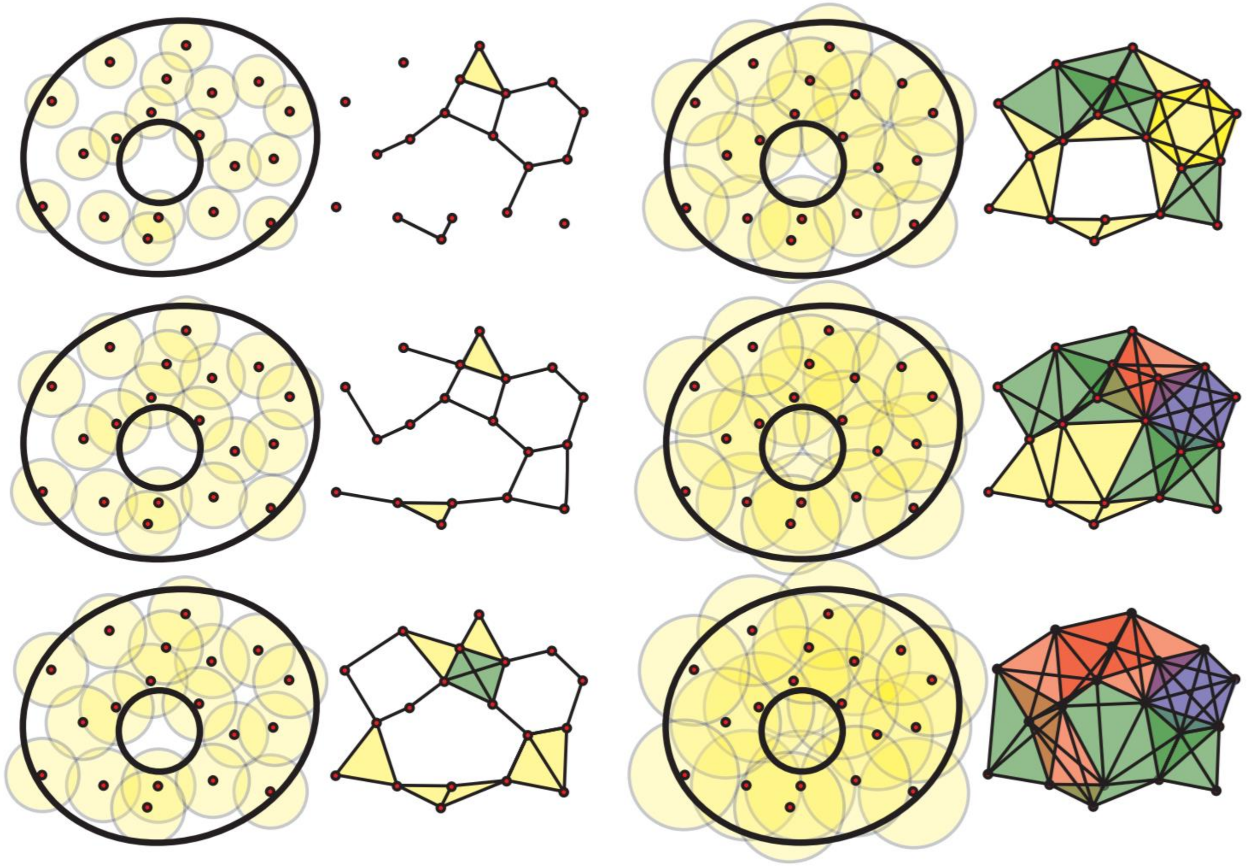
- Applications
  - Linkage and Folding
  - Mesh Generation

Course Outline



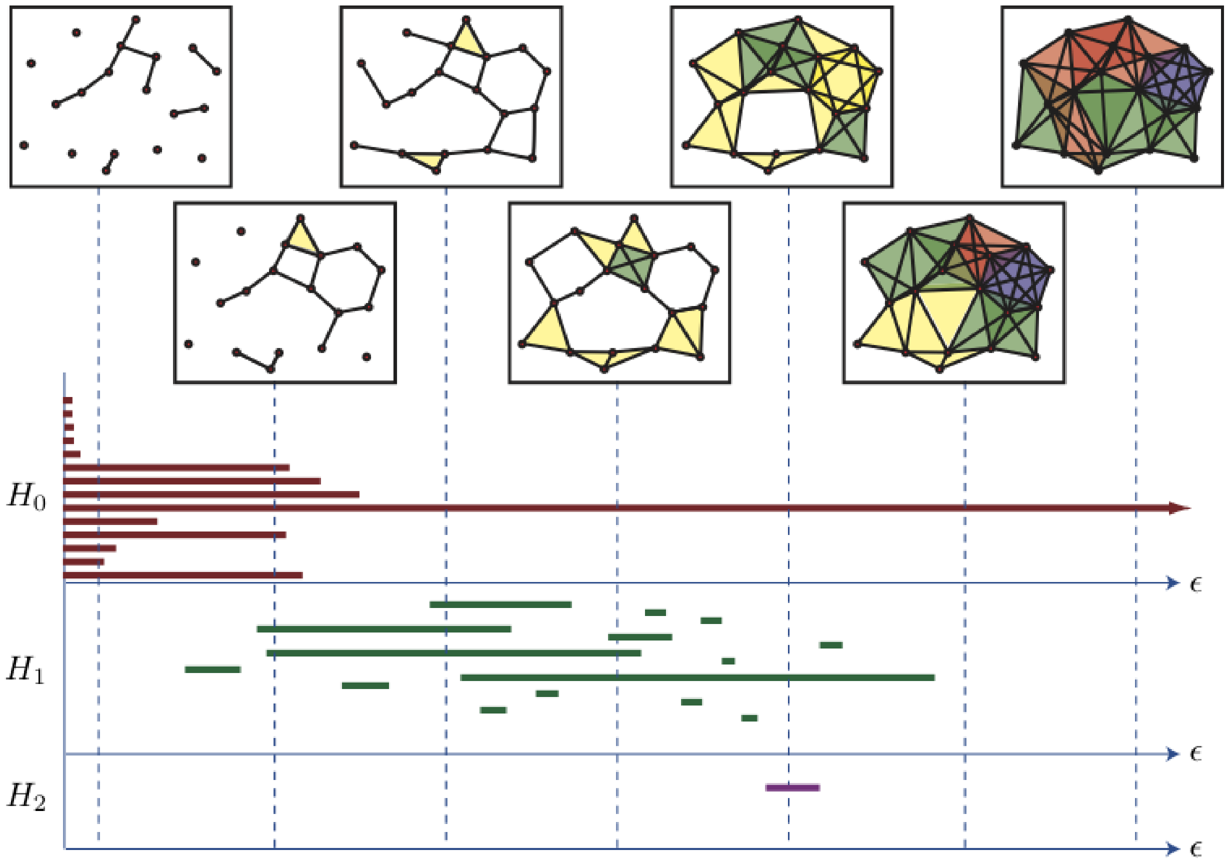
- Applications
  - Linkage and Folding
  - Mesh Generation
  - Optimization

Course Outline



- Applications
  - Linkage and Folding
  - Mesh Generation
  - Optimization
  - Topological Data Analysis

Course Outline



- Applications
  - Linkage and Folding
  - Mesh Generation
  - Optimization
  - Topological Data Analysis

Course Outline

- Theory
  - Curves
  - Surfaces/Complexes
  - Homotopy
  - Homology
- Applications
  - Linkage and Folding
  - Mesh Generation
  - Optimization
  - Topological Data Analysis

Logistics

- Course webpage
- Lectures
- Office Hours
- Discussion forum
- Homework
- Project

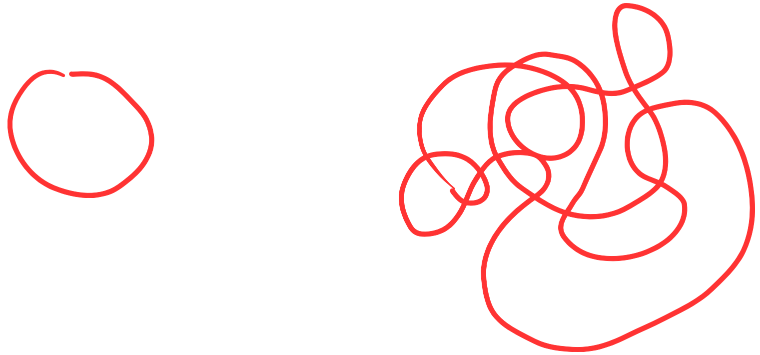
STOP me if you are LOST

Simple Planar Curves

Formal Definitions

- Planar curve

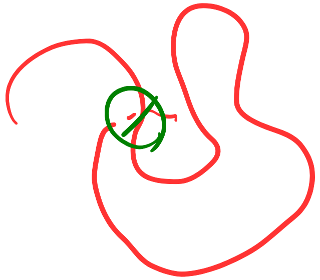
$\gamma : S^1 \longrightarrow \mathbb{R}^2$



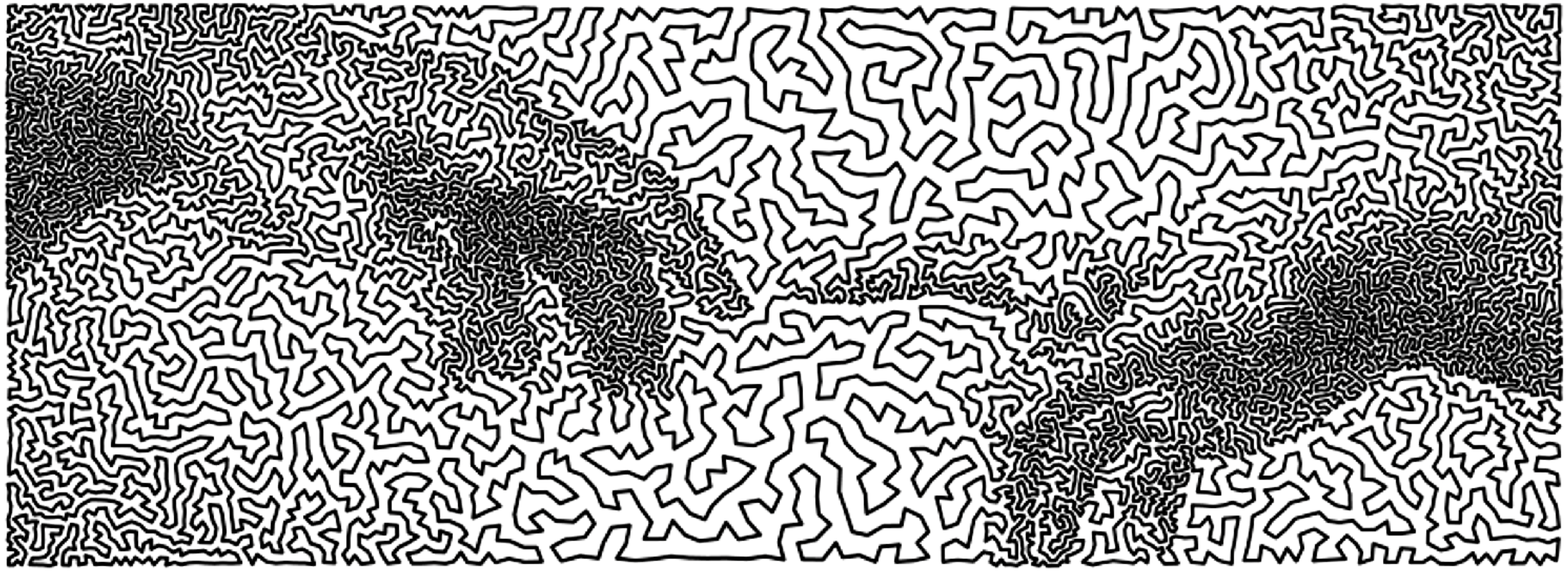
- Simple

γ embedding

injective



no self-crossings



Simple Planar Curves

Jordan curve

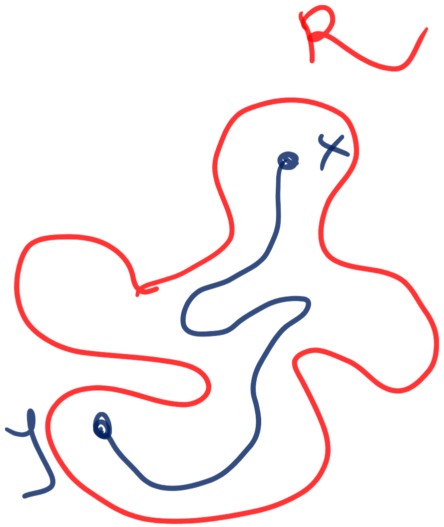
[Jordan 1887]

Formal Definitions

- Connected

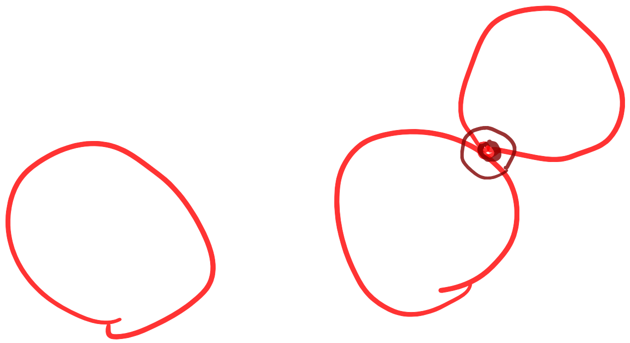
$x \sim y$ if $\exists \pi : [0,1] \to R$, $\pi(0) = x$, $\pi(1) = y$

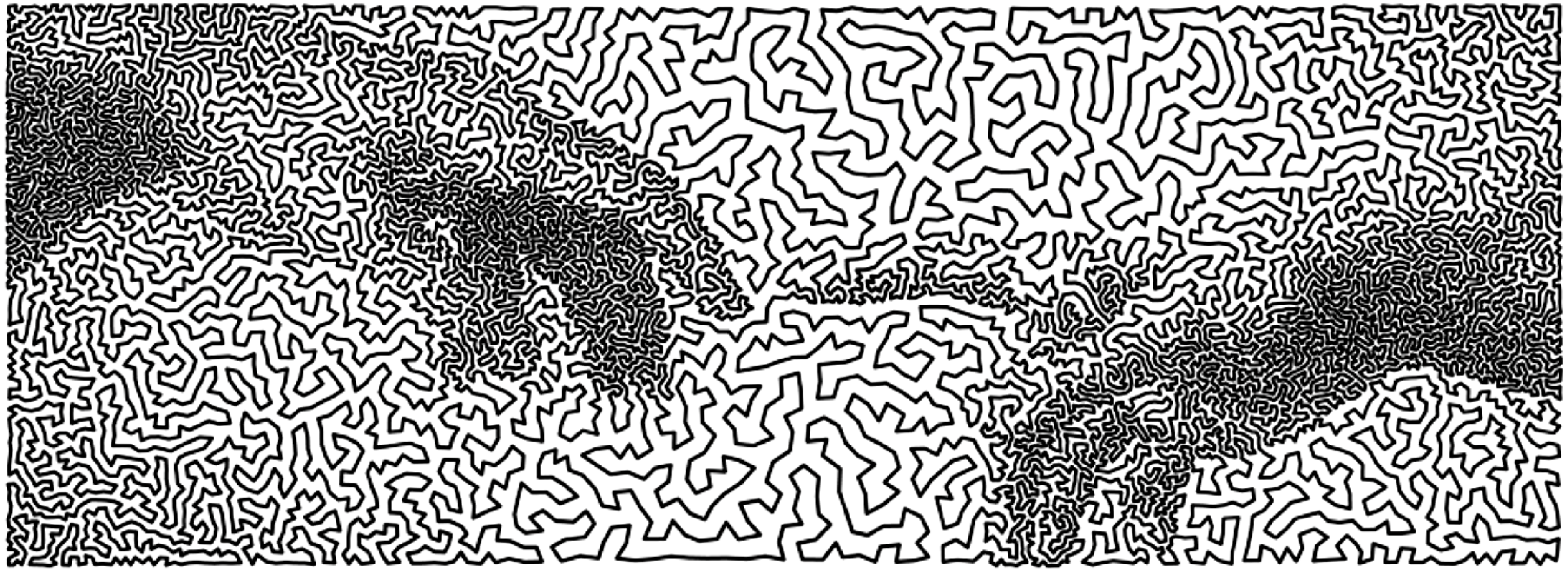
$R \subseteq \mathbb{R}^2$ connected if $\forall x, y \in R$, $x \sim y$



- Connected component

maximal subregions that are connected.

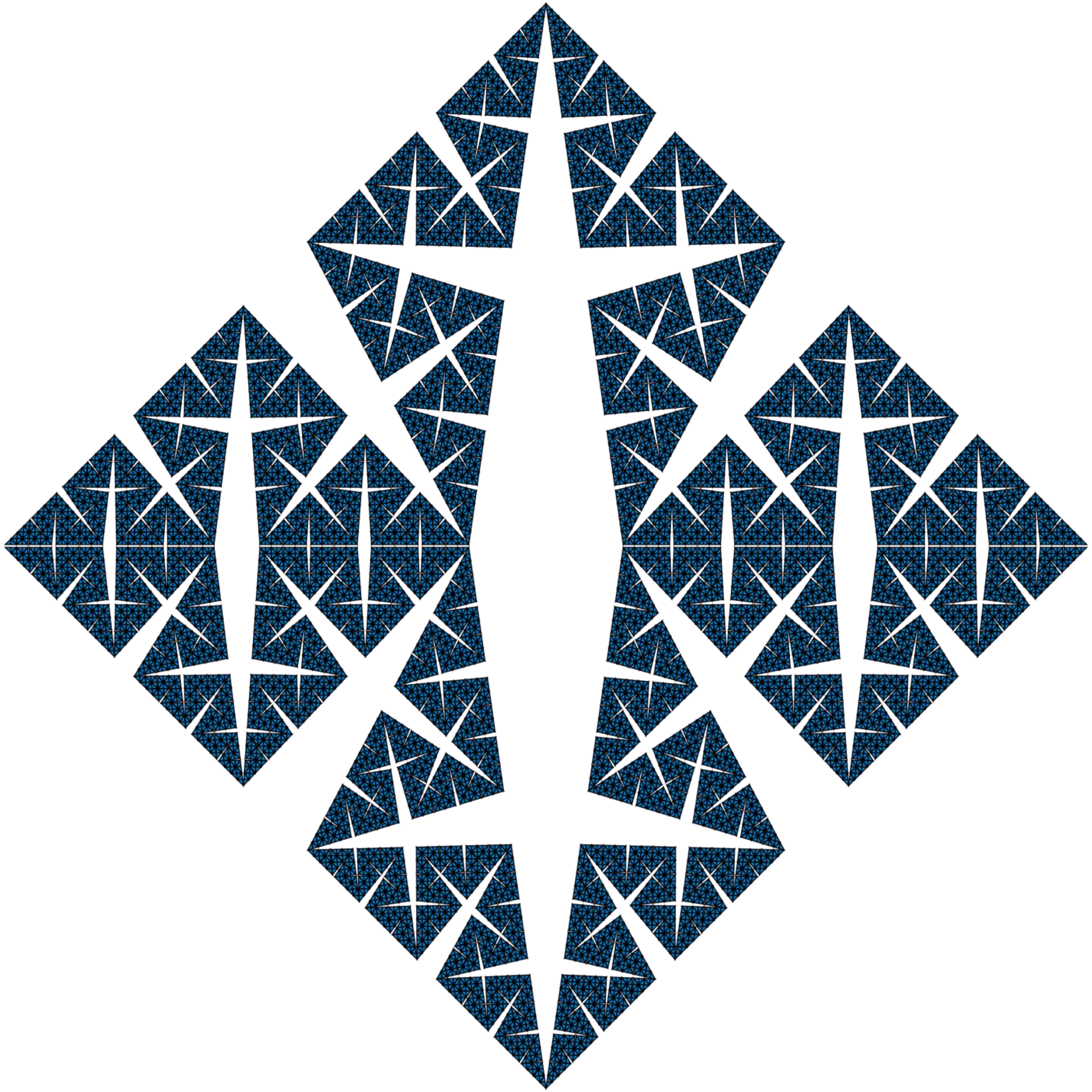




Jordan Curve Theorem

[Bolzano ~1800s] [Jordan 1887]

Any simple closed curve C separates $R^2 \setminus C$ into exactly two connected components



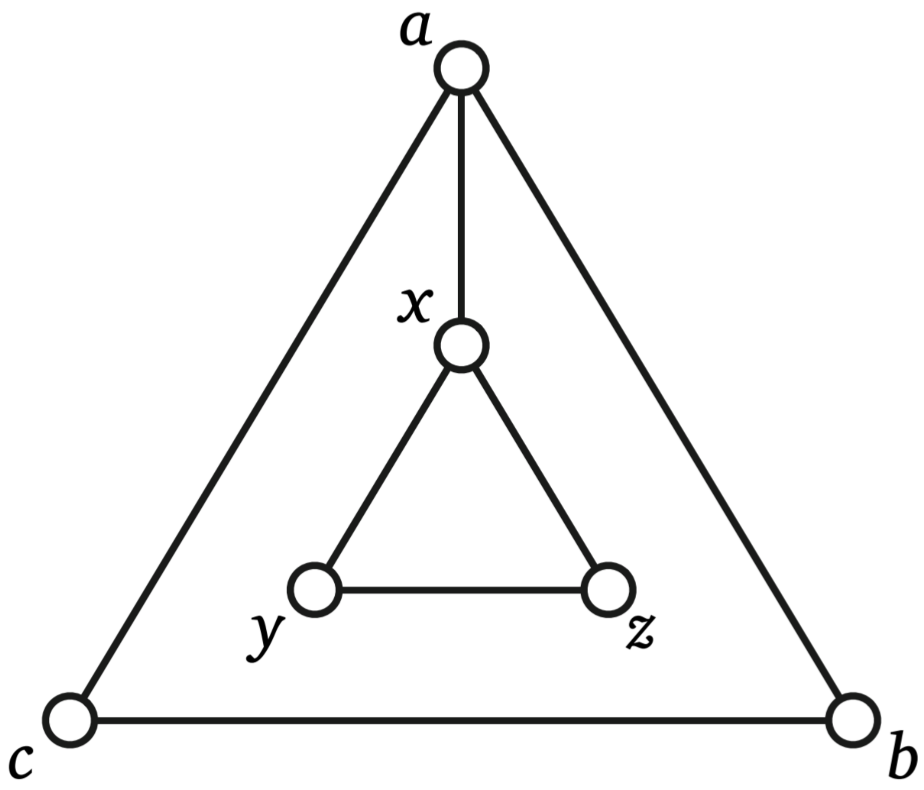
Simple Planar Curves

Osgood curve

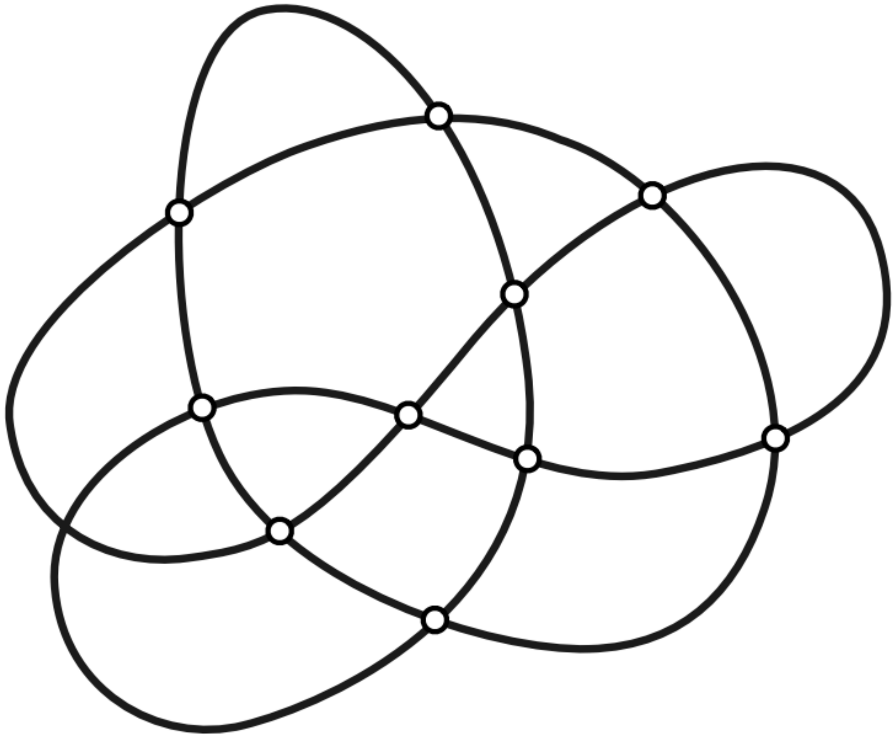
[Osgood 1903]

Representation of Curves

- Polygonal

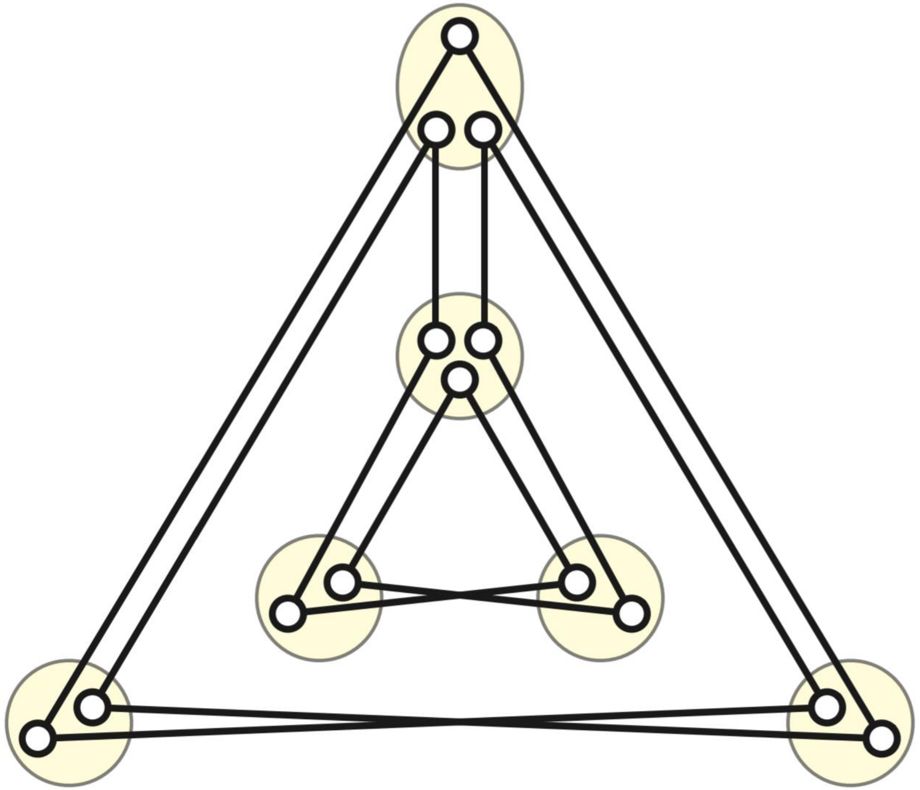


- Generic

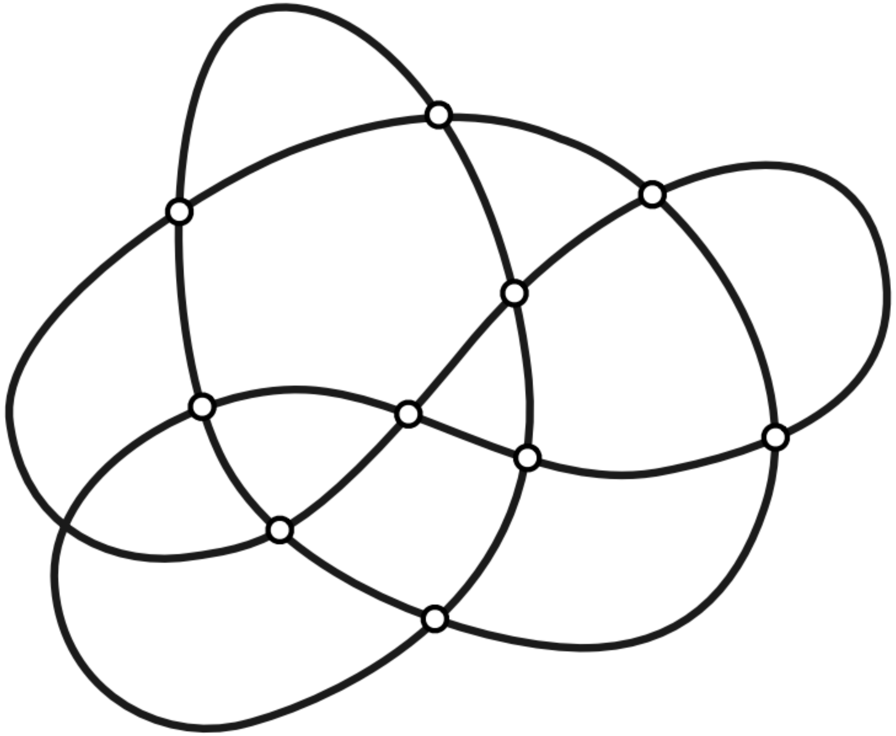


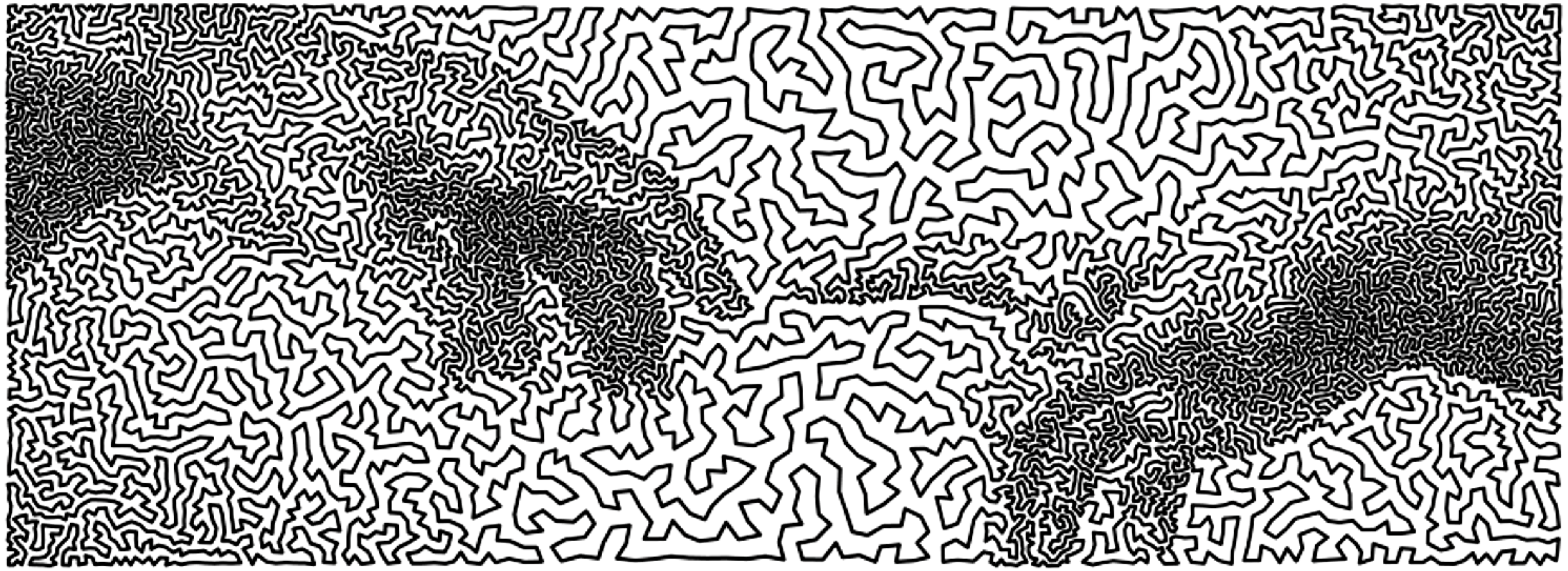
Representation of Curves

- Polygonal



- Generic



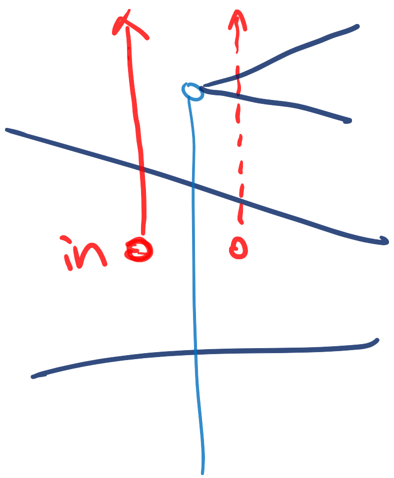


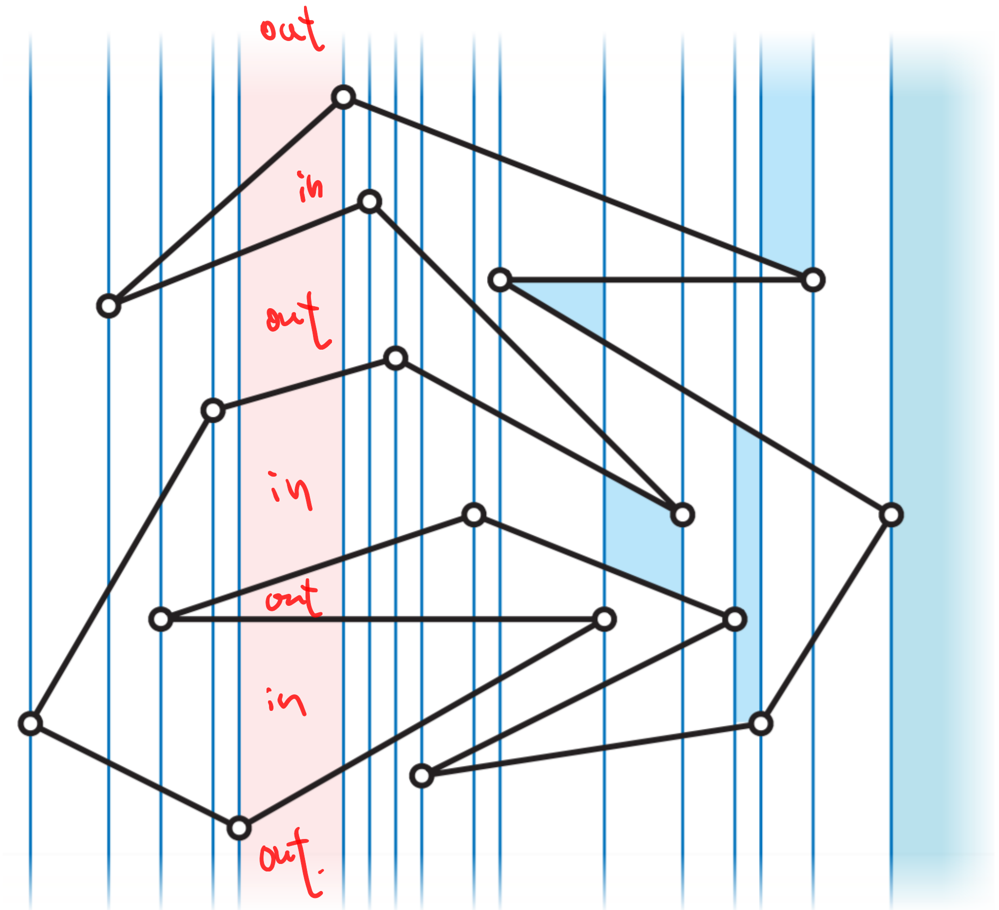
Jordan Polygon Theorem [Dehn 1899]

Any simple polygon P separates $R^2 \setminus P$ into exactly two connected components

Proof of Jordan Polygon Theorem

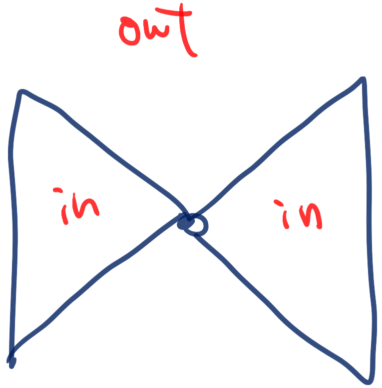
- Intuition: Parity argument

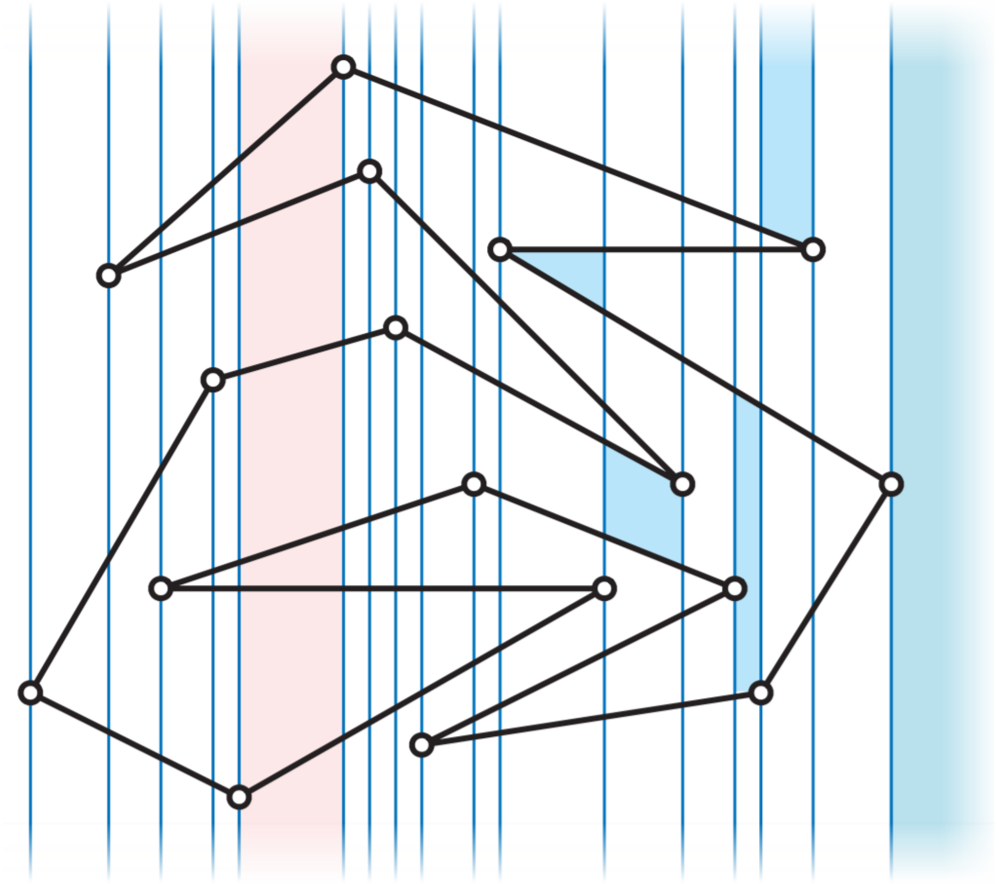




Proof of Jordan Polygon Theorem

- Lemma ≥2
  - Parity argument

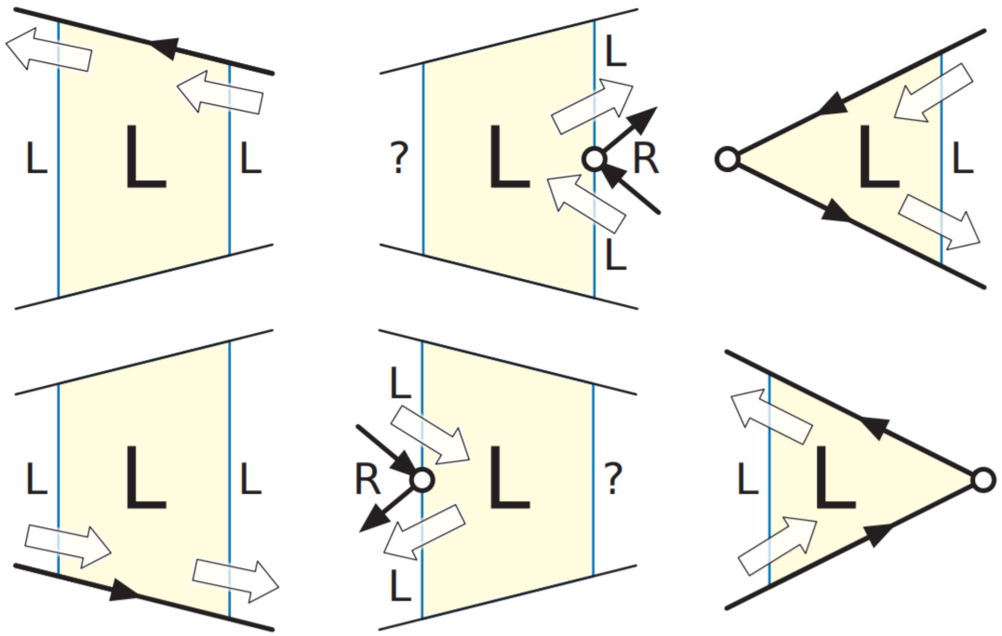


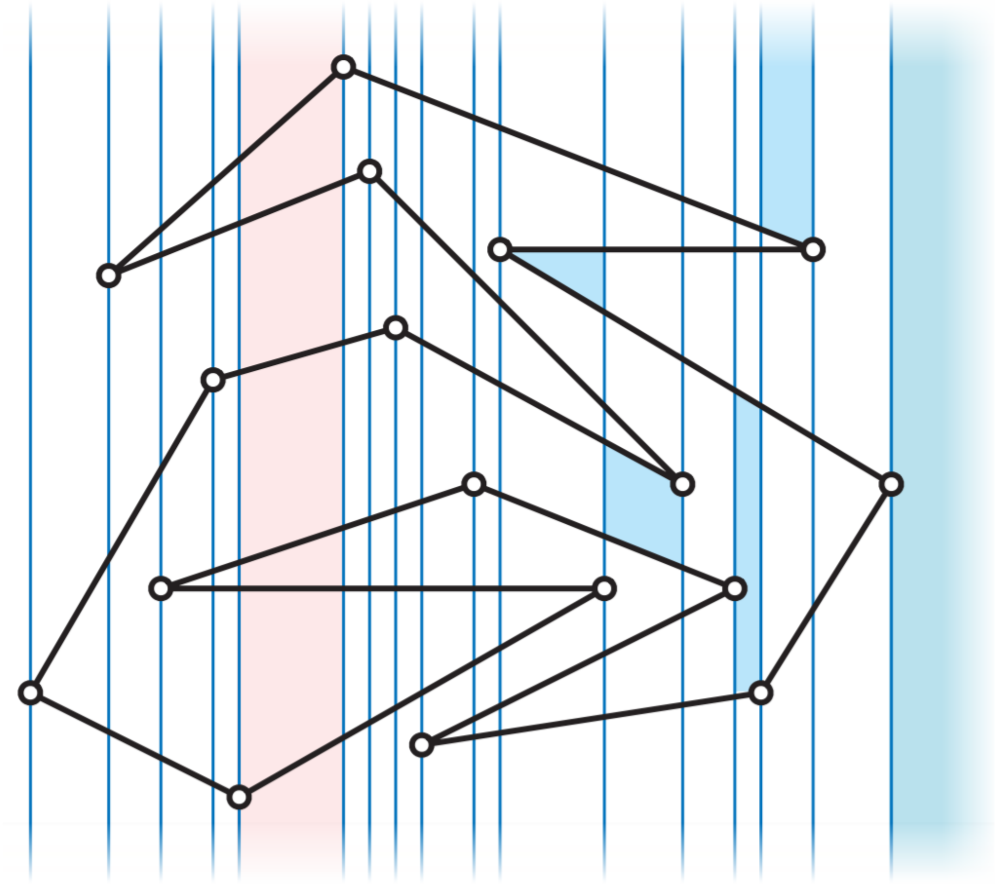


- Lemma ≤2?

Proof of Jordan Polygon Theorem

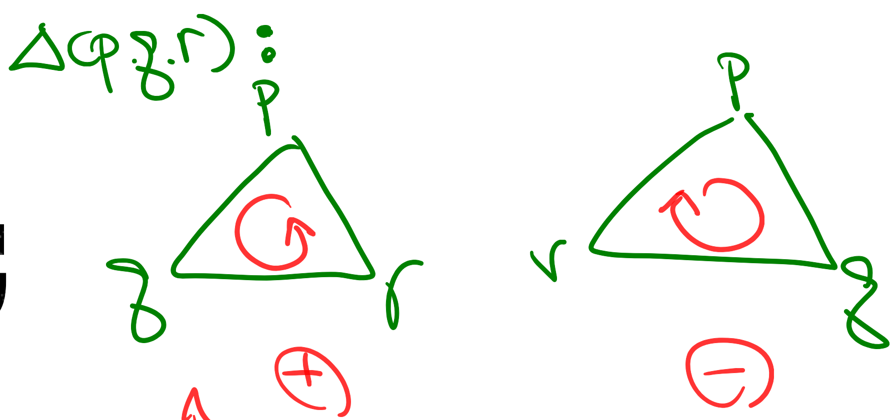
- Lemma ≤2





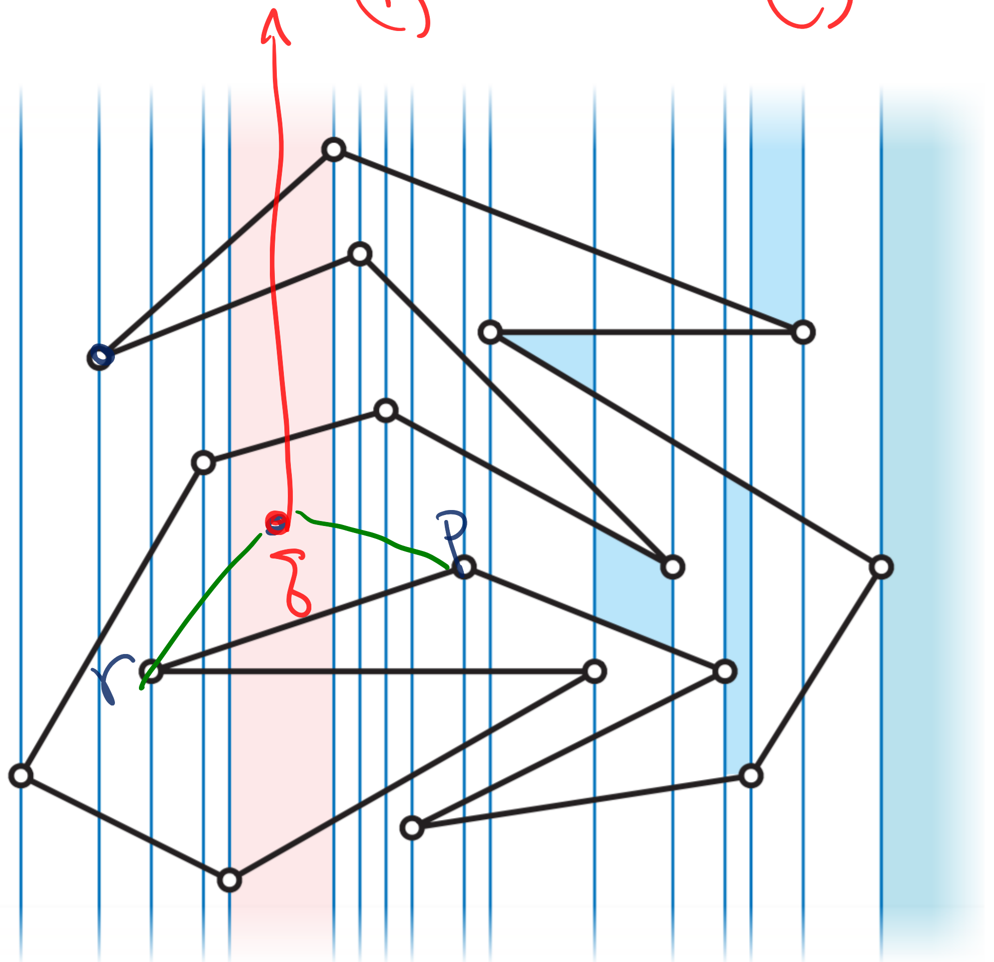
$$\Delta(p,q,r) = \begin{vmatrix} 1 & p.x & p.y \\ 1 & q.x & q.y \\ 1 & r.x & r.y \end{vmatrix}$$

Inside-Polygon Testing



InsidePolygon?(P, q):

- sign ← ∅
- for each segment $\overrightarrow{pr}$:
  - Δ ← Δ(p,q,r) — sign of triangle (p,q,r)
  - if p.x ≤ q.x < r.x:
    - sign ← −Δ · sign — crossing above red
  - if r.x ≤ q.x < p.x:
    - sign ← Δ · sign — crossing below red
- return sign

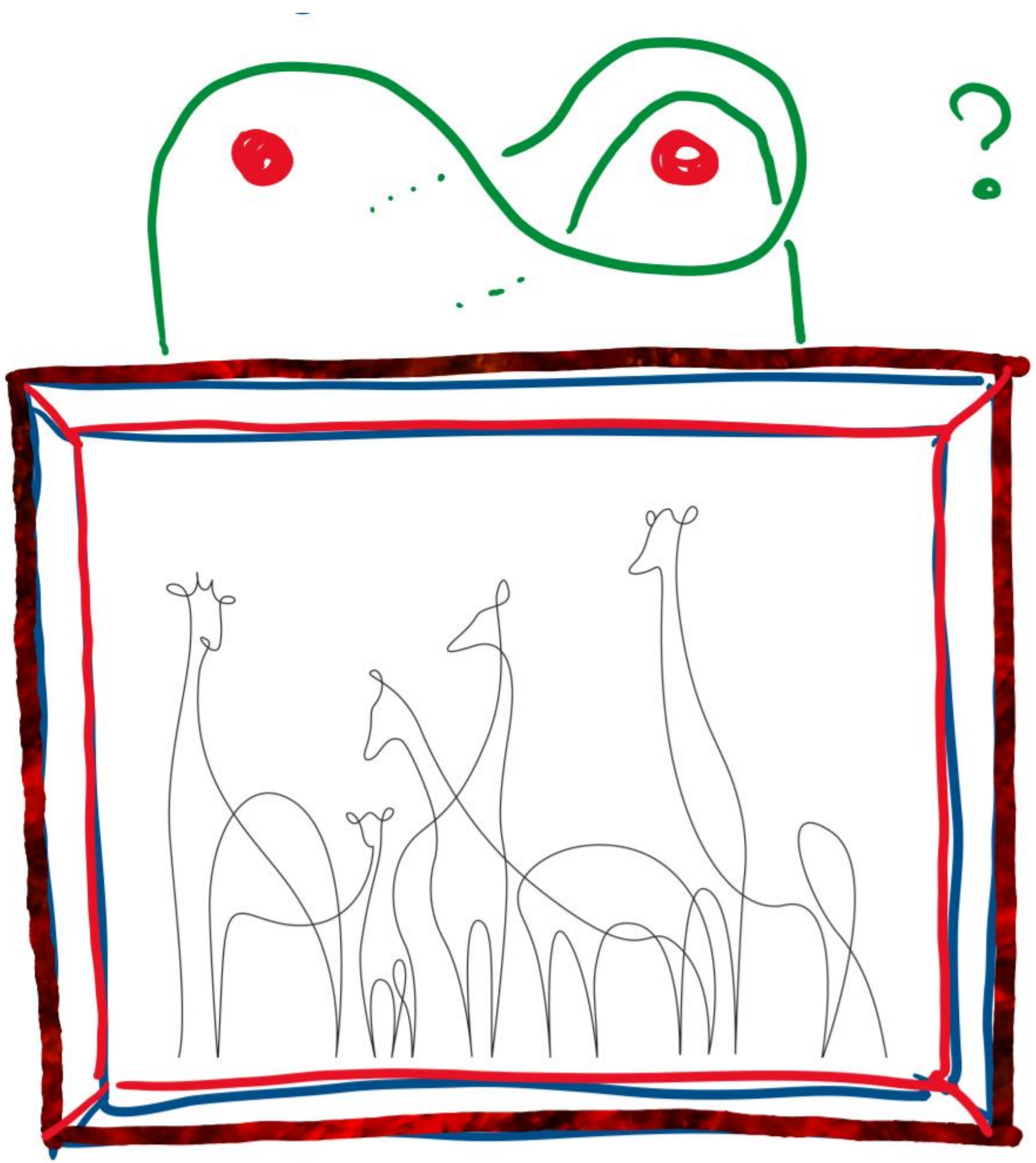


Intermission

Food for thought.

How to compute the area of a simple polygon?

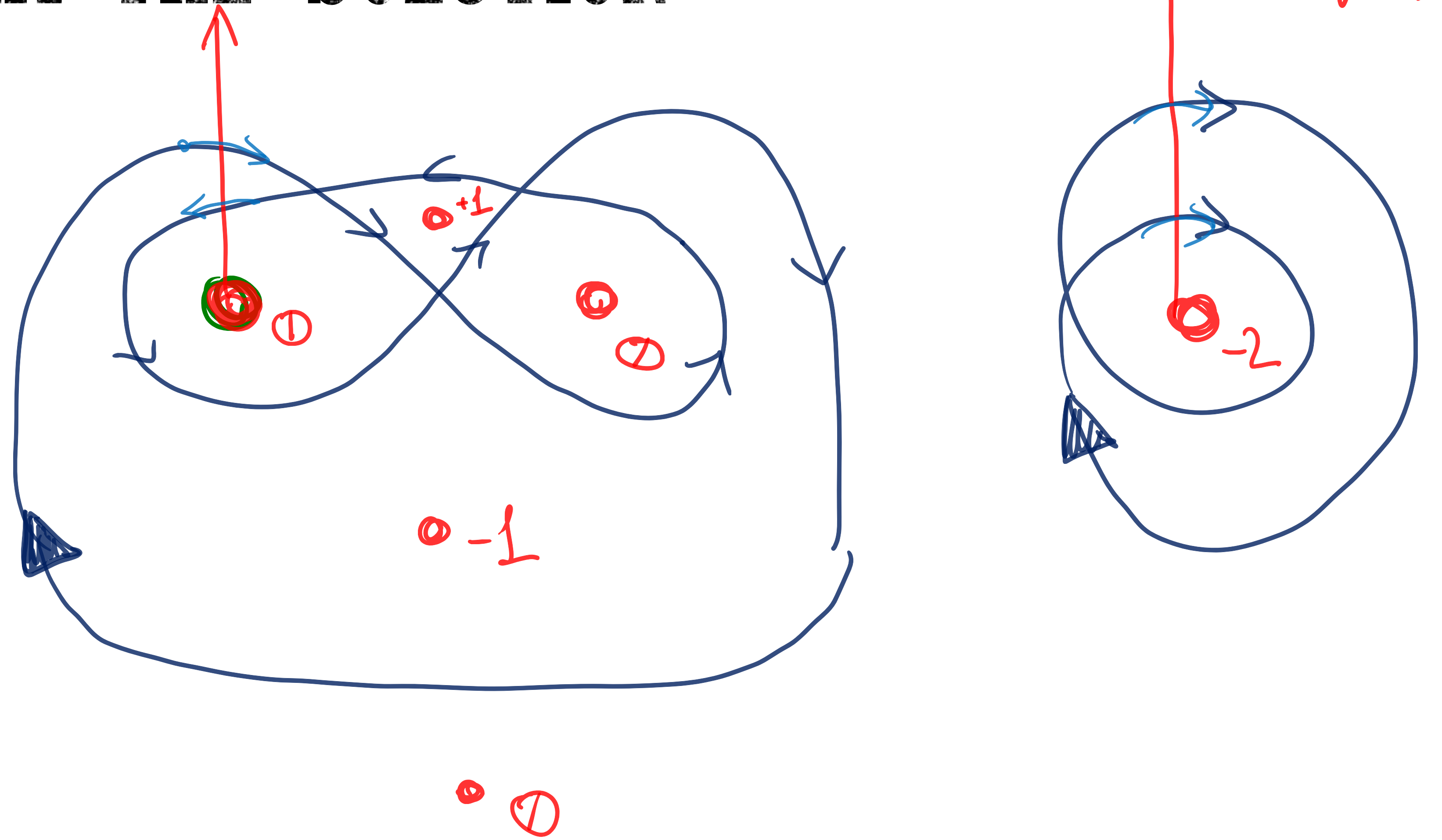
Winding Number



Picture-Hanging Puzzle

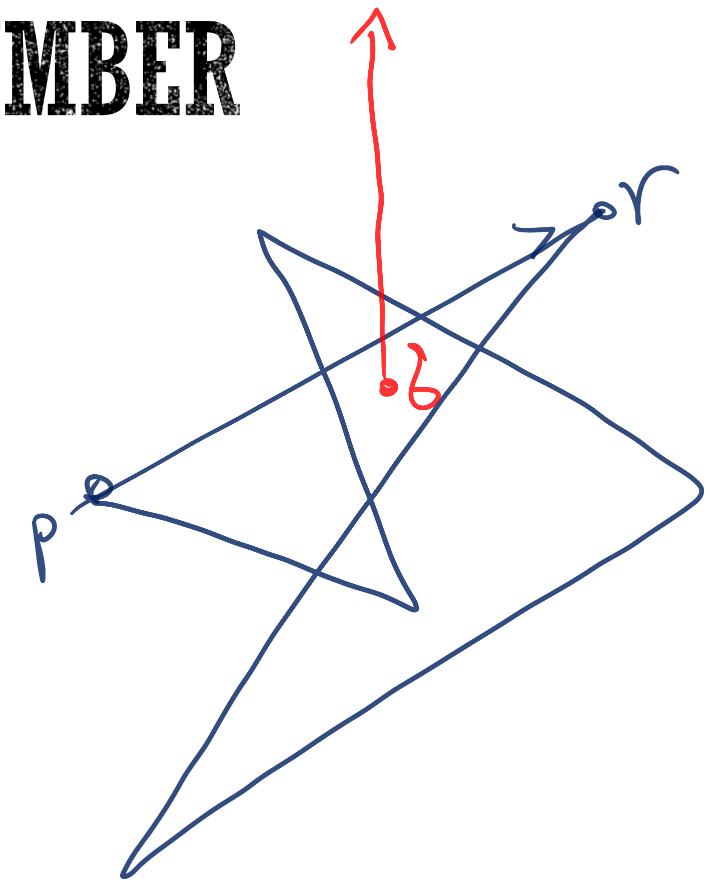
Can you hang a picture with two nails, such that the picture falls if **either** nail is pulled?

Look at the Solution



Computing Winding Number

```
WindingNumber(P, q):
    wind ← 0
    for each segment pr:
        δ ← Δ(p, q, r)
        if p.x ≤ q.x < r.x
        and δ = +1:
            wind--
        if r.x ≤ q.x < p.x
        and δ = -1:
            wind++
    return wind
```



Wind$_q$(P) Invariant under Morphing

- Homotopy

Curve Morphing

Curve Morphing

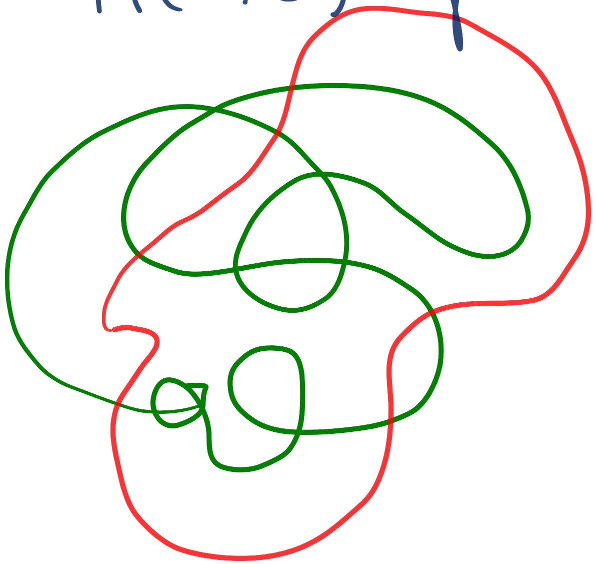
$\text{Wind}_q(P)$ Invariant under Morphing

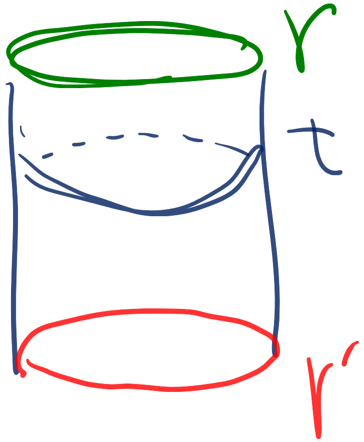
- **Homotopy**

cont. change from γ to γ'

$$H: S^1 \times [0,1] \to \mathbb{R}^2$$

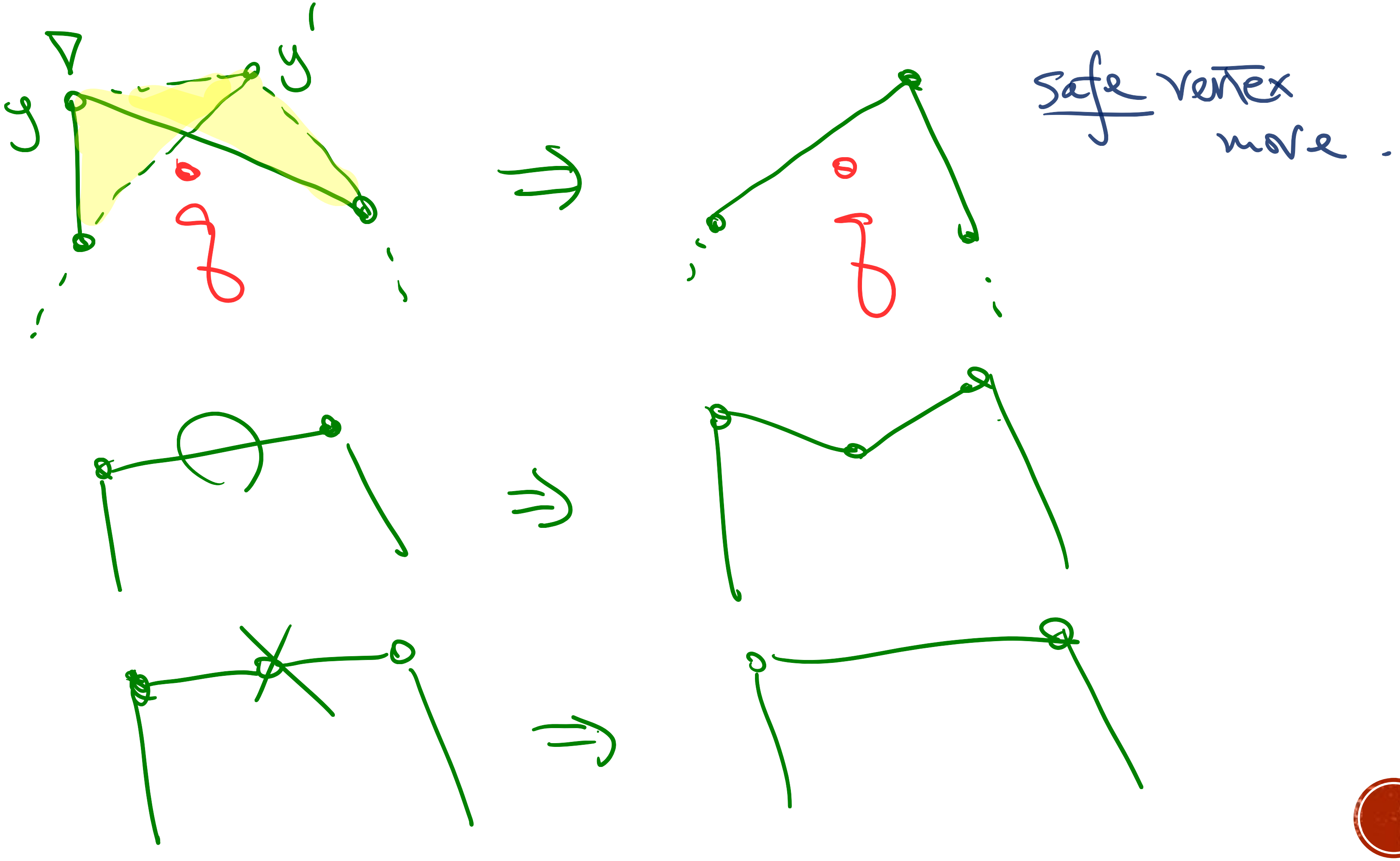
$$H(\cdot, 0) = \gamma \qquad H(\cdot, 1) = \gamma'$$





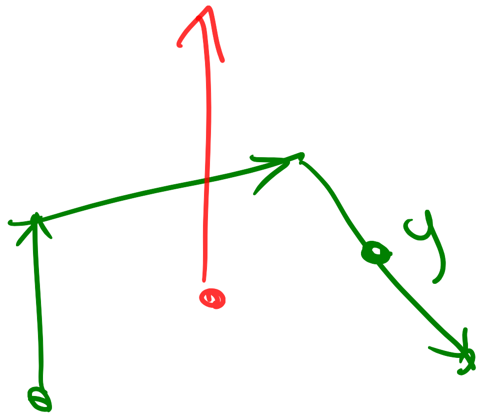
$\text{Wind}_q(P)$ Invariant under Morphing

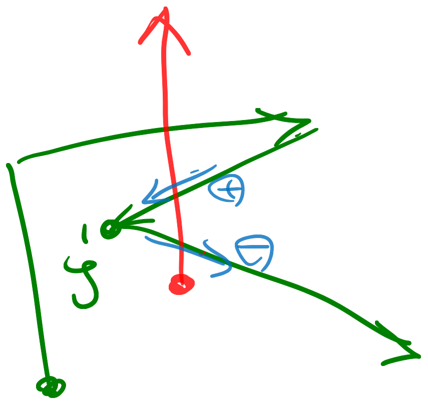
- Homotopy
- Vertex move

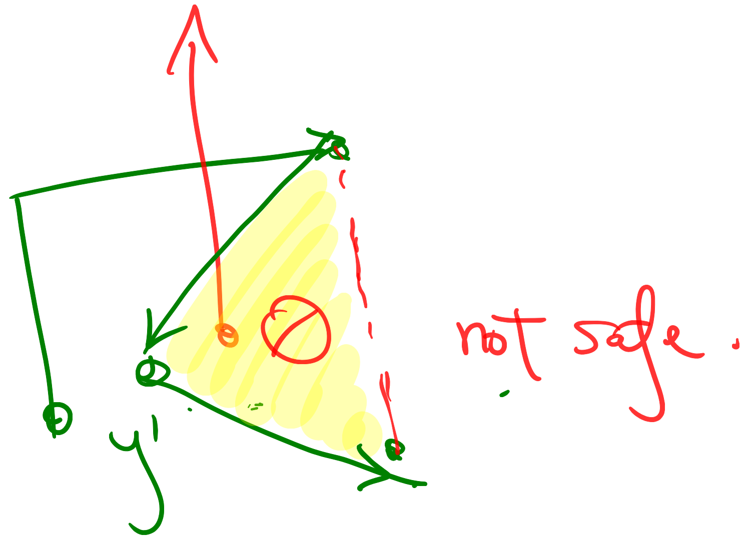


Theorem. $\text{Wind}_q(P)$ is invariant under safe vertex moves.

pf sketch.







Theorem. Two polygons P and Q are homotopic in $R^2 \setminus q$ if and only if they have the same $Wind_q$. [Hopf 1935]

Wind_q is a Complete Homotopic Invariant!

Takeaway.

Planar curve can be described by how many times it goes around reference points.