1. ***The wheels on the bus go round and round.*** Consider an undirected grid graph G: each vertex is labelled with a coordinate (x, y) where x and y are integers ranging from 1 to $\sqrt{n}$, and an edge exists between two vertices (x_1, y_1) and (x_2, y_2) if either $|x_1 - x_2| = 1$ and $y_1 = y_2$, or $x_1 = x_2$ and $|y_1 - y_2| = 1$. Also, each edge of G is associated with a positive *length*. In addition, we are given:

   - a vertex v_0 in G called the *endpoint*, and
   - a fixed integer k.

   Describe and analyze an efficient algorithm to compute a *shortest* closed walk P with respect to the edge-lengths that starts and ends at the given endpoint v_0, and the *rotation number* of P (as defined for the polygonal curves) has to be equal to k.

   [Hint: Any shortest path in G has to be spur-free; that is, no subpath is of the form x–y–x. Yes, the closed walk P might visit the same vertex more than once.]

2. ***Mixing microbial colonies.*** Imagine we have two microbial colonies of the same origin, but dyed with different colors. Assuming that the given microbial colonies are immortal by providing enough food and space, and they grow in circular phases parametrized by real numbers in $[0, 1)$; each colony has grown independently for a while so the first colony is in phase α and the second is in phase β. Now we mix the two colonies together, and let's denote the common phase they evolved into as $f(\alpha, \beta)$. In other words, f is a function from $S^1 \times S^1$ to S^1. We made the following assumptions:

   - $f(\cdot, \cdot)$ is *continuous*: a small change to the input phases of the two colonies only results in small change to the final common phase.
   - $f(\cdot, \cdot)$ remains *identity* on the diagonal: when the two input colonies have the same phase θ, the result mixed colony also has phase θ.
   - $f(\cdot, \cdot)$ is *symmetric*: swapping the dye between the two input colonies does not change the result.

   (a) Prove that the function $f(\cdot, \cdot)$ gives rise to a well-defined retraction from the Möbius band M to its boundary circle S^1.

   (b) Prove that there are no retractions from the Möbius band M to its boundary circle S^1.

   (c) What happened? How do you explain the contradiction between item (a) and (b)?