Discrete Morse Theory
11/11/20 10:14 AM

Administrivia. HW3 due this Friday (11/13)

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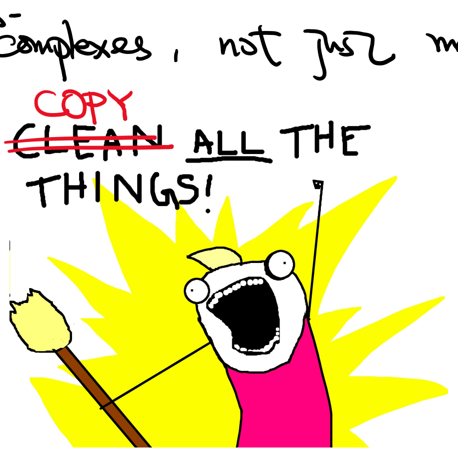
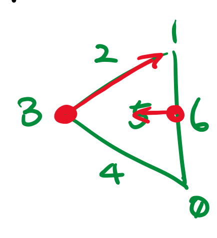
  Off-by-one errors: k-simplex on $k+1$ nodes.
- Final Project signup form.

Today's goal: Introduce a combinatorial version of Morse theory. [Forman]
Benefit: Applies to Δ-complexes, not just manifolds!



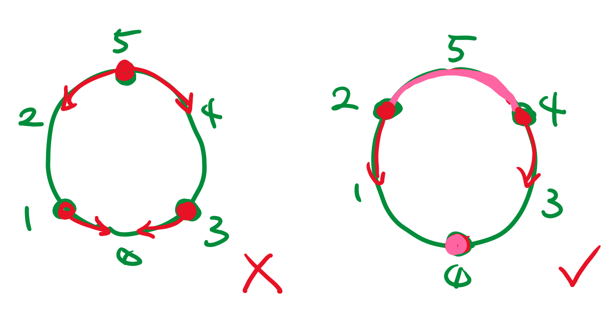
Discrete gradient. $f: K \to \mathbb{R}$. p-simplex α.

- $(p+1)$-simplex β : $\beta > \alpha$, $f(\beta) \le f(\alpha)$
- $(p-1)$-simplex γ : $\gamma < \alpha$, $f(\gamma) \ge f(\alpha)$



Discrete Morse function. $f: K \to \mathbb{R}$ is Morse if
$\forall$ p-simplex α, discrete gradient is unique (if exist).

example.



which one is Morse?

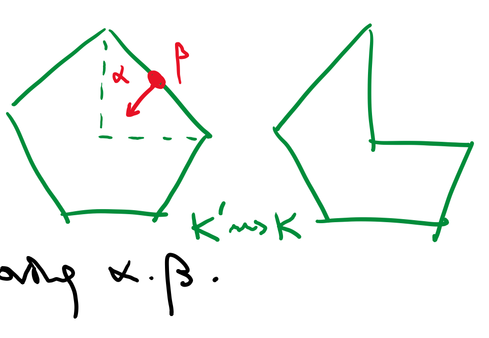
Critical cell. p-simplex α is critical if no discrete gradient.
Lemma. Given a Morse function,
either upward or downward gradient exists, not both.

Sublevel set. $K_{\le c} := \bigcup_{\alpha: f(\alpha) \le c} \bigcup_{\beta \le \alpha} \beta$

Lemma. If $(a, b]$ contains no critical pt. then $K_{\le a} \simeq K_{\le b}$
Pf. "Collapse" all intermediate cells using gradients. ∎

Lemma. If $(a, b]$ contains single p-dim crit. cell α, then
$K_{\le b} \simeq K_{\le a} \sqcup$ p-handle

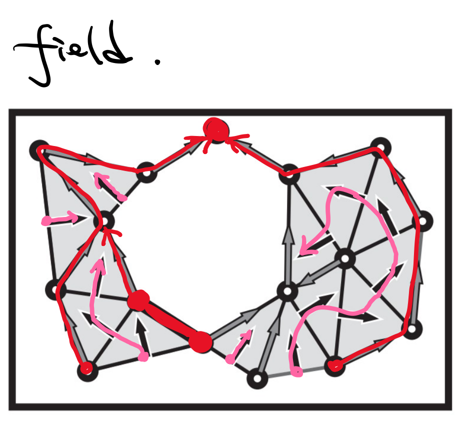
[Whitehead]
Collapse. If $K' = K \cup \{\alpha, \beta\}$, where
β is a face of α, and
not a face of anything else.
Then one can collapse K' into K by removing α, β.



Think about when a critical cell is inserted.
- Must glue all faces of α = attaching a p-handle. ∎

Discrete flow lines.
Intuition: Given K. Morse fcn is NOT important!
We only need the gradient field.

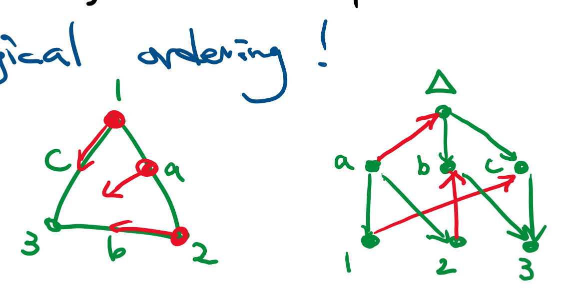
[Duval'94, Stanley'93]
Discrete vect. field V:
pairing of neighboring p- & $(p-1)$-cells that are disjoint. "partial matching"
leftover cells are critical.



Discrete flow line:
$(\alpha_0 < \beta_0) > (\alpha_1 < \beta_1) > \dots > (\alpha_k < \beta_k)$. in V.

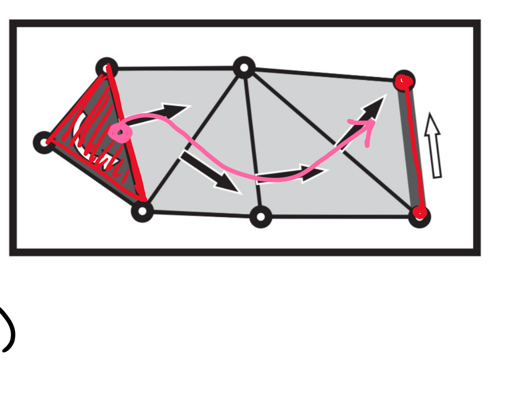
Discrete vect. field is acyclic if no cyclic flow line.

Thm. A discrete vect. field V is the gradient field of a discrete Morse fcn. iff it is acyclic.
"pf". dag iff $\exists$ topological ordering!
Hasse diagram:



∎

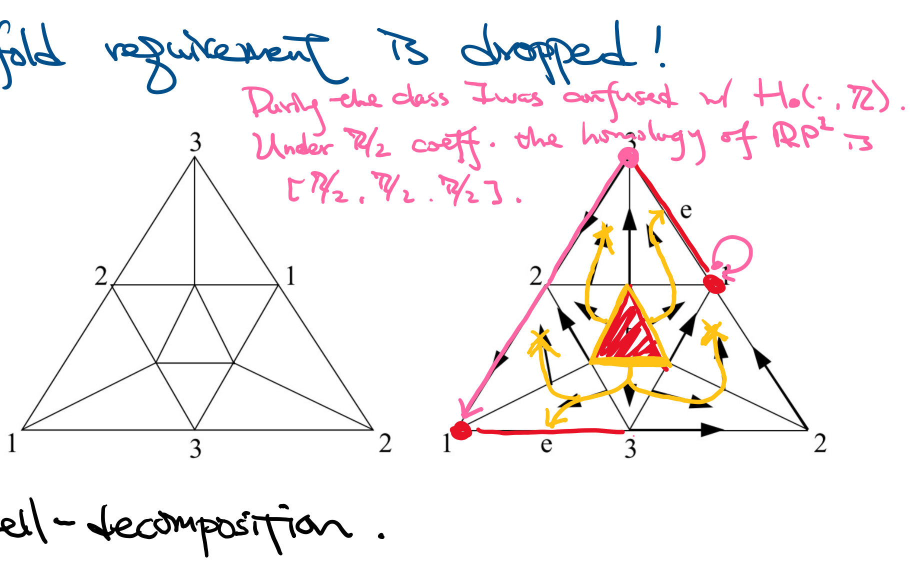
Morse complex $(MC_\bullet, \partial)$:
$MC_k := \{k\text{-dim crit. cells}\}$.
$\partial\alpha :=$ $(p-1)$-simplices by following flow lines. to a neighboring p-coface



[Forman'98]
Morse Homology Thm. $MH_\bullet(V) \simeq H_\bullet(K)$

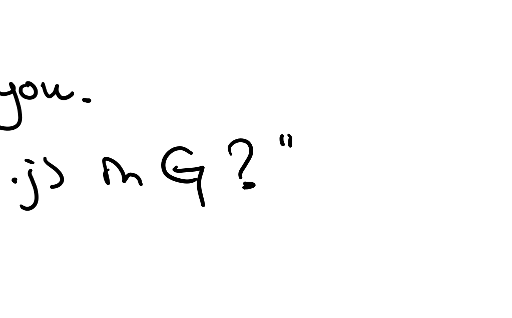
Notice that the manifold requirement is dropped!
example. $\mathbb{RP}^2$
$MC_\bullet = \{[1], [e], [\Delta]\}$
$\partial 1 = 0 \quad \partial e = 1 + 1 = 0$
$\partial\Delta = e + e = 0$
$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0$

During the class I was confused w/ $H_\bullet(\cdot, \mathbb{Z})$. Under $\mathbb{Z}/2$ coeff. the homology of $\mathbb{RP}^2$ is $[\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2]$.



Application: Simplifying cell-decomposition.

Cancellation Thm. f Morse on complex K.
If $\exists!$ flow line from α to β.
Then $\exists g$ Morse s.t. α, β not critical.



Evasiveness.
Say there a graph G hidden from you.
You are allowed to ask "Is edge (i, j) in G?"
Goal: Figure out if G has cycles.

Question. Do you have to check every edge?

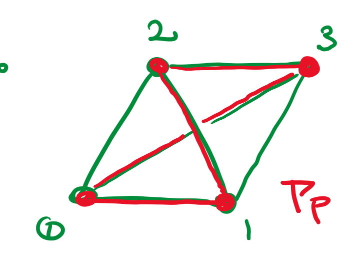
Formulate differently, let $g(x_1, \dots, x_e)$ be a Boolean fcn. one per edge. $X: E(G)$.
- Consider property $P \subseteq 2^X$ s.t. $g(x) = 0$ iff $x \in P$.
  P is monotone if $A \in P$ & $B \subseteq A \Rightarrow B \in P$. (downward-closed).
- Assume hidden $G \subseteq X$. determine if $G \in P$ by asking "Is x_i an element of G"?

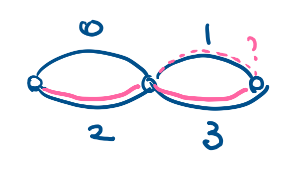
[Aanderaa–Rosenberg'73] [KSS'84]
Evasiveness Conjecture. If P monotone, nontrivial, symmetric, Σ_X preserves P.
then P is evasive, i.e. requires $\#X$ questions.

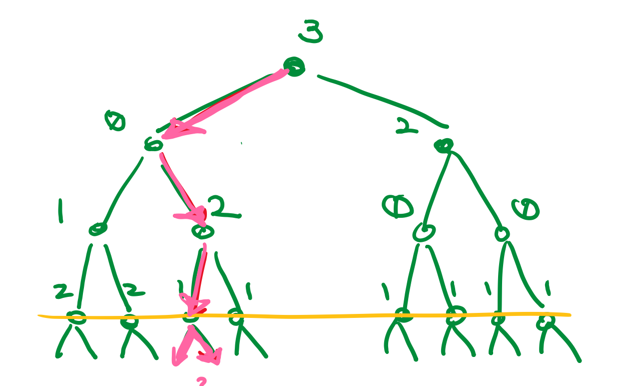
We can construct a complex T_P on X:
- Add cell σ if $\sigma \in P$.

[Forman'00]
Thm. 1. P non-evasive $\Rightarrow$ T_P collapse to a pt.
2. $\tilde{H}_\bullet(T_P) \ne 0 \Rightarrow P$ evasive.
3. $\#\{\text{evasive } G \subseteq X\} \ge 2 \cdot \sum_i \dim \tilde{H}_i(T_P)$.

example.

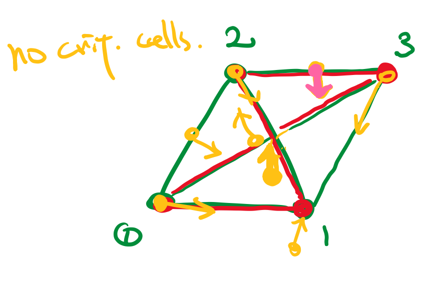






one possible guessing algorithm.

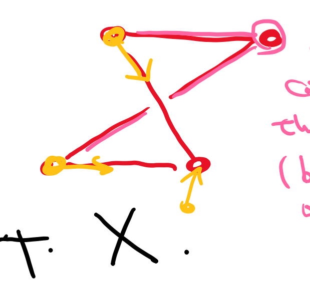
Observation. Guessing algorithm induces discrete gradient field.



$[0123] > [013]$
$[023] > [03]$
$\boxed{[123] > [23]}$
$[13] > [3]$

$[012] > [02]$
$[12] > [2]$
$[01] > [0]$
$[1] > [\,]$.

Restricting gradients to T_P,



crit. cells, corresponding to graph pairs that requires the last query. (because the pair has one graph in T_P and the other not.)

there's no critical cells wrt. X.

Prop. $\left\{\substack{\text{evasive } G \text{ on } X \\ \text{under algorithm}}\right\} \overset{2-1}{\longleftrightarrow} \{\text{critical cells in } T_X\}$

Thm. # evaders under any algorithm $\ge 2 \cdot \sum_i \dim \tilde{H}_i(T_P)$
Pf. by strong Morse ineq.
$\#\text{crit. cells of dim } i \ge \dim \tilde{H}_i(T_P)$. ∎

Therefore the problem reduces to proving T_P topo-nontrivial.