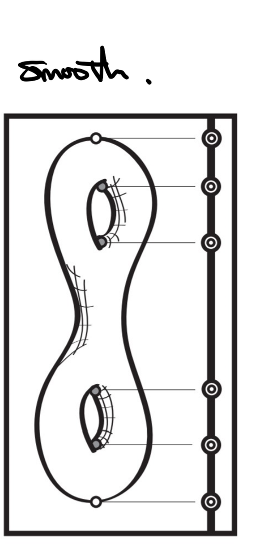
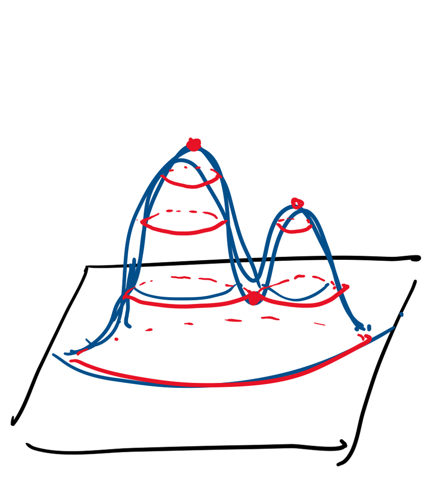
Morse theory and shape analysis

11/9/20 10:18 AM

Morse Theory : Put Geometry Back to Topology.

• Topology can be useful even when the surface is a terrain!





Critical pts. Let M be a manifold. $h: M \to \mathbb{R}$ smooth. height fcn.

Consider sublevel set $M_{\le a} := h^{-1}(-\infty, a]$
$= \{x \in M : h(x) \le a\}$

"Rising water"

As the water rises, there are critical events where the topology changes. Critical pts.

Geometric pt of view : i geometry. linear algebra!

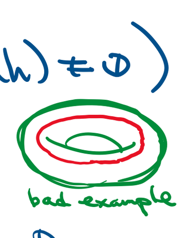
Consider derivative $\nabla h = \left[\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \dots, \frac{\partial h}{\partial x_d}\right]$ in tangent bundle T_*M

Pt x is critical if $\nabla h(x) = [0, 0, \dots, 0]$ in tangent space TM_x

$h(x)$ is a critical value.

Assume 1. all critical pts are non-degenerate ($\det$ Hessian$(h) \neq 0$) $\left[\frac{\partial^2 h}{\partial x_i \partial x_j}\right]$

and 2. having different critical values. Morse fcn.



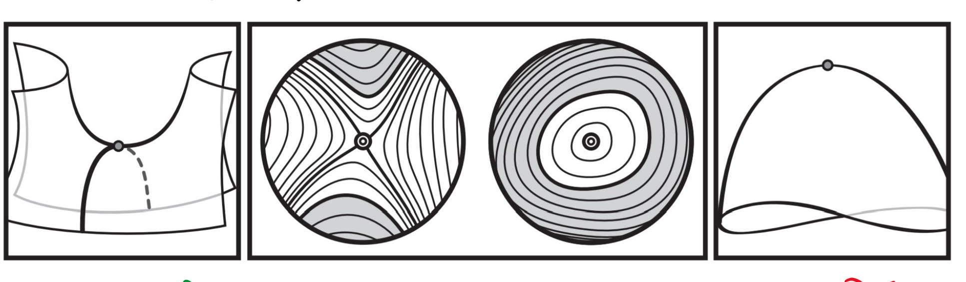
bad example.

Morse index $\mu(p)$ of crit. pt. p := # negative eigenvalues of Hessian(h) at p.

Morse Lemma. Let $L_p := B \cap M_{\le h(p)-\varepsilon}$ (lower, ball). U_p = closure$(B - L)$ (upper)

Then, 0. $f(x)$ locally looks like $-x_1^2 \dots - x_{\mu(p)}^2 + x_{\mu(p)+1}^2 \dots + x_d^2$

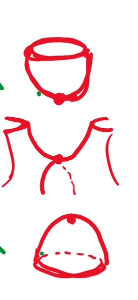
1. $U \simeq D^n$ for $n = \dim M$.
2. $L \simeq D^{n-\mu(p)+1} \times S^{\mu(p)-1}$
3. $L \cap U \simeq D^{n-\mu(p)} \times S^{\mu(p)-1}$



example: M surface. Say $p = (0 \dots 0)$.)($\to$ ⌣⌢

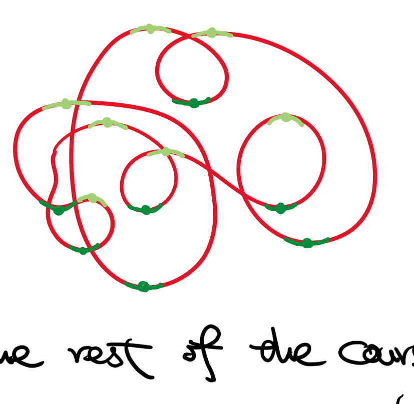
$f(x) = f(0) - x_1^2 \dots - x_\mu^2 + x_{\mu+1}^2 \dots + x_d^2$

• $\mu = 0$: $f(x) = f(0) + x_1^2 + x_2^2$. $L = \emptyset$. local min
• $\mu = 1$: $f(x) = f(0) - x_1^2 + x_2^2$. $L = D^2 \times S^0$ saddle.
• $\mu = 2$: $f(x) = f(0) - x_1^2 - x_2^2$. $L = D^1 \times S^1$ local max



example: rotation number redux.

⌣ : $\mu = 0$
⌢ : $\mu = 1$



Two main routes to take (for the rest of the course):

1. Build Morse homology $(MC_\bullet, \partial)$ on crit pts $(Cr(h), \mu)$.
2. Build efficient representation (Reeb graphs) to summarize M.

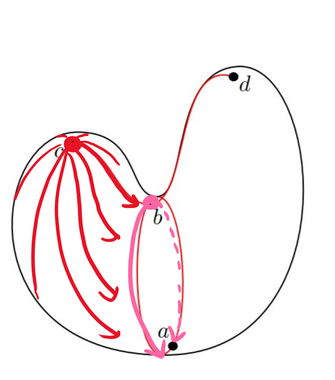
—— ⓪ ——

Morse Homology.

Flow lines.

Gradient field ∇h defines flow lines from crit. pts to crit. pts.

(solve diff. eq. $\gamma' = \nabla h(\gamma)$ w/ $\gamma(0) = p$)



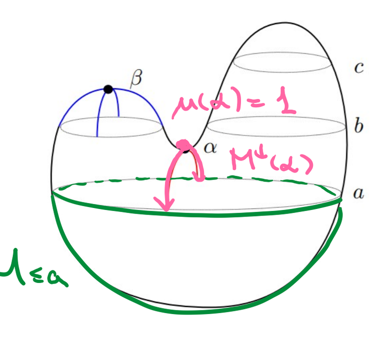
Manifold M is decomposed into flow lines.

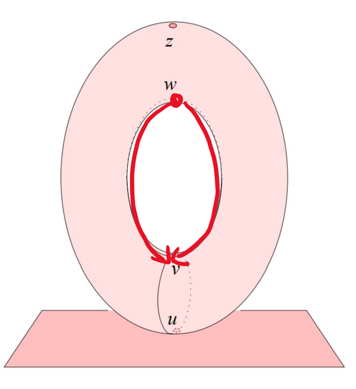
Consider flow lines originated from crit. pt p. $M^\downarrow(p)$ decending manifold. (unstable)

Prop. $M^\downarrow(p)$ has dimension $\mu(p)$.

Prop. If h has no crit. pt in $f^{-1}[a,b]$, then $M_a \simeq M_b$. (level set; diffeomorphic)

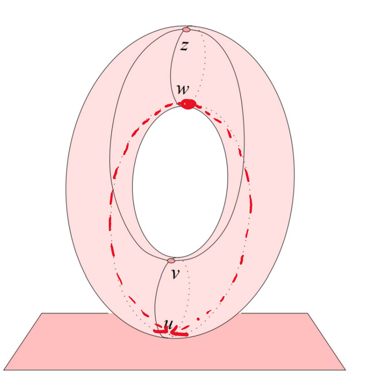
Prop. If $f^{-1}[a,b]$ contains exactly one crit. pt. α of index k.
Then $M_{\le b} \simeq M_{\le a} \cup M^\downarrow(\alpha)$
"attaching one k-handle"





flow lines from 1-crit pt to 1-crit pt.

Not good!



Morse-Smale fcn. to make sure flow line goes from k to $k-1$ dimension.

$(M^\downarrow(x) \pitchfork M^\uparrow(y))$.

Intuition Test: We have flow lines sending objects from dim k to dim $k-1$. What are we trying to do?

Morse complex. k-skeleton MC_k: vect space w/ basis on crit. pts w/ Morse index k.

boundary ∂: # flow lines connecting k-crit. pt to $(k-1)$-crit. pts.

With assumptions. ∂ is well-def & $\partial^2 = 0$.

Now. build Morse homology $MH_\bullet(h)$. (because $(MC_\bullet, \partial)$ is a chain complex)

Morse Homology Thm. M compact. h Morse. then
$MH_\bullet(h) \simeq H_\bullet(M)$ (indep. of h).

"pf." build cell structure based on decending manifolds.

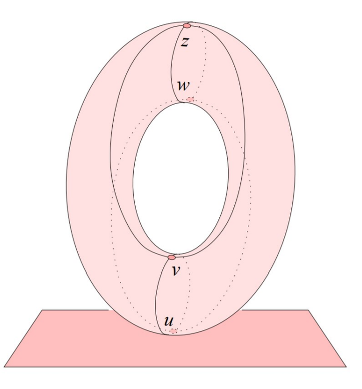
Benefit of using Morse homology: Only crit. pts matter.

example. tilted torus.

$0 \to \mathbb{Z}/2 \xrightarrow{0} (\mathbb{Z}/2)^2 \xrightarrow{0} \mathbb{Z}/2 \to 0$

$\partial z = 2v + 2w = 0$.
$\partial w = \partial v = 2u = 0$.

$M_h(t) = P(t) = t^2 + 2t^1 + t^0$



Strong Morse Ineq. $M_h(t) = P(t) + (1+t)Q(t)$

$M_h(t) = \sum_{p \text{ crit.}} t^{\mu(p)}$, $P(t) = \sum_i \beta_i(M) \cdot t^i$, $Q(t) \in \mathbb{N}[t]$. polynomial in t w/ nonneg. integer coeff.

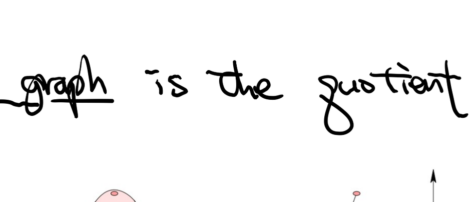
"pf". Similar to Euler-Poincaré.

Cor. [Weak Morse Ineq.]. $m_i :=$ # i-crit. pts $\ge \beta_i := \dim H_i$

Cor. $\chi(M) = \sum_{p \text{ crit.}} (-1)^{\mu(p)}$ pf. plug in $t = -1$ ∎

example. p-source, q-sink flow in the plane must have $p+q-2$ saddle pts.

(sink ⊖ at ∞)



$p = 2$. $q = 1$.

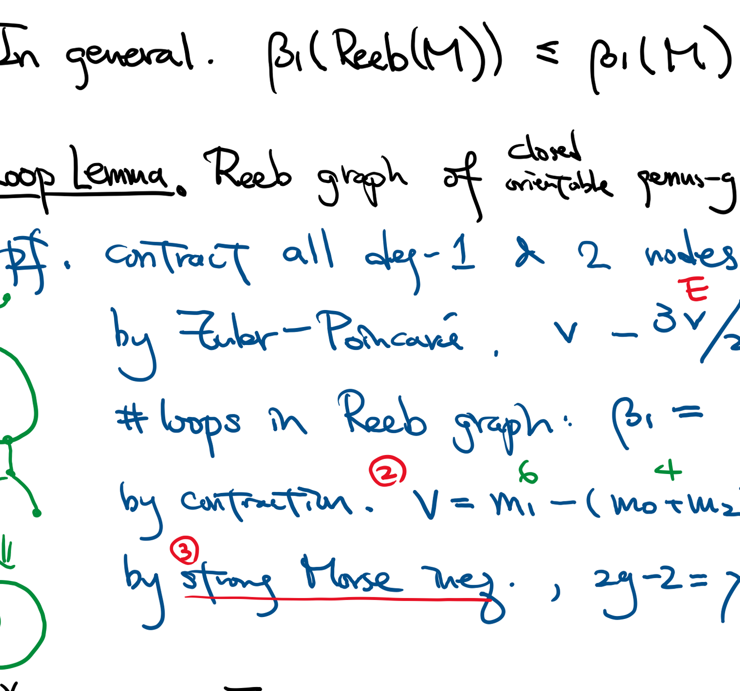
example. a torus must have 2 saddle pts.
a sphere must have a local max & a local min.

—— ⓪ ——

Reeb graphs.

Consider the components of $M_a := \{x \in M : h(x) = a\}$, called contours.

The Reeb graph is the quotient space of M by identifying pts in the same component.

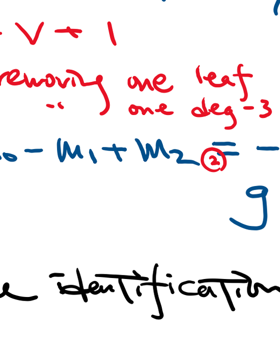


u is a node if u is in image of a crit. pt.

$\Rightarrow$ bijection between nodes & crit pts.

Always 1-dim object.

In general. $\beta_1(\mathrm{Reeb}(M)) \le \beta_1(M)$.



Loop Lemma. Reeb graph of closed orientable genus-g surface has g loops.

pf. contract all deg-1 & 2 nodes iteratively. deg-3 node $\Leftrightarrow$ crit pt of index 1.

by Euler-Poincaré, $V - \overbrace{3V/2}^{E} = \overbrace{\beta_0}^{1} - \beta_1$

loops in Reeb graph: $\beta_1 = 3V/2 - V + 1 = V/2 + 1$ ①
$E - V + 1$

by contraction. ② $V = m_1 - (m_0 + m_2)$. removing one leaf $\Leftrightarrow$ one deg-3 node.

by ③ strong Morse ineq., $2g - 2 = \chi \overset{③}{=} m_0 - m_1 + m_2 \overset{②}{=} -V \overset{①}{=} 2(\beta_1 - 1)$
$g = \beta_1$ ∎.

Application. Imagine segmentation & feature identification.

[Dey-Wang-Wang'18-19]

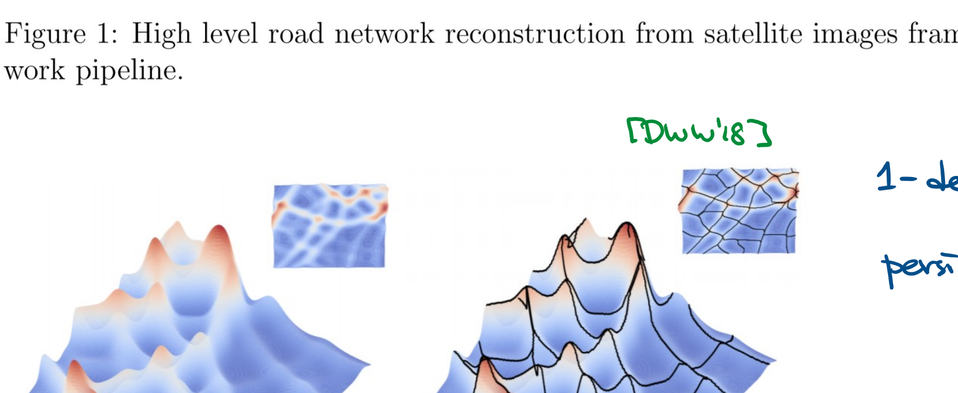


Figure 1: High level road network reconstruction from satellite images framework pipeline.

[DWW'18]

1-desending manifolds + persistent homology.

efficient: $O(n \log n)$ time.