Exterior Calculus and isolation lemma

10/28/20 10:07 AM

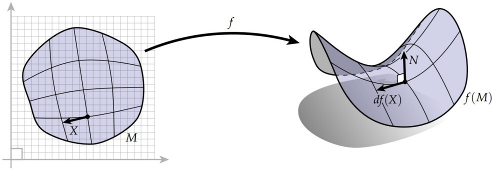
Administrivia.

- Come talk to me about the project if you haven't!
- Midterm feedback form. Help me out please!

* * *

Crash Course on Differential Geometry ¡calculus, linear algebra!

Differential.



$df : X \mapsto df(X)$. a vector.

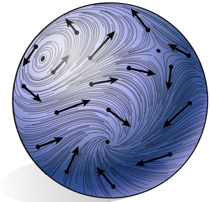
$T_*M \to T_*f(M)$ Tangent bundles.

$T_*(M)$ consists of vector space $T_x(M)$, one for each $x \in M$.

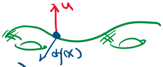
Think about a vector field on M as a section of $T_*(M)$:

$\forall x \in M$, pick a vector from $T_x(M)$,

so that the vectors flow smoothly.



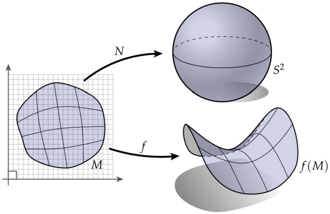
One can define inner product on $f(M)$: $df(X) \cdot df(Y)$
which induces a metric $\langle X, Y \rangle_f$ on M



We can also define normal vectors on $f(M)^{\subseteq \mathbb{R}^3}$: $df(X) \cdot u = 0$

$N: M \to S^2$ Gauss map (!)

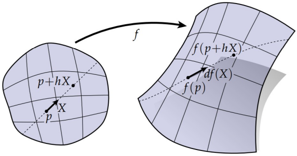
(No time for discussion about curvatures today. Go check out!)



Geometric interpretation of $df(X)$:

$$df_p(X) = \lim_{h \to 0} \frac{f(p+hX) - f(p)}{h}$$

$$= \underbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 \end{bmatrix}}_{\text{Jacobian matrix}}$$



⇒ Treat $df(X)$ as matrix-vector product $df \cdot X$!

In other words, df / Jacobian matrix is a change-of-basis.

Exterior Calculus.

calculus studies the change of scalars.

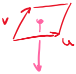
vect. calculus " (tiny) vectors.

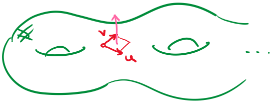
exterior calculus " (tiny) signed volumes.

Wedge product.

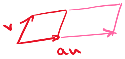
example. signed area.

$$u \wedge v = -v \wedge u$$

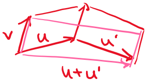




$$(au)\wedge v = a(u \wedge v)$$



$$(u+u')\wedge v = u \wedge v + u' \wedge v$$



example. signed volume.

$$u \wedge v \wedge w = v \wedge w \wedge u = w \wedge u \wedge v$$
$$= -u \wedge w \wedge v = -w \wedge v \wedge u = -v \wedge u \wedge w$$

Forms. Given vector space V.

$\Lambda(V) :=$ forms on V. forming an algebra w/ addition & wedge product.

Let $\{e_i\}$ be basis of V. then generators of $\Lambda(V)$ are $de^i \in V^\vee$ (dual space of V) s.t. $de^i(x) = x^i$
$V \to \mathbb{R}$. the ith coordinate.

Naturally. $\Lambda(V) = \bigoplus_p \Lambda^p(V)$. where

$$\Lambda^p(V) = \langle de^{i_1} \wedge \cdots \wedge de^{i_p} : 1 \le i_1 \le \cdots \le i_p \le \dim V \rangle$$

"eats p vectors and spits out a number"

$\Omega(M) :=$
$\Lambda(T_*M)$ (as collections of $\Lambda(T_xM)$, $\forall x \in M$) :
differential forms on M.

$\Omega^p(M) :=$

p-form: section $\alpha : M \to \Lambda^p(T_*M)$ s.t. α_x flow smoothly.

example. $\int x^2 \boxed{dx} \longrightarrow$ a 1-form on $\mathbb{R}$.

$\int e^{-(x^2+y^2)} \boxed{dxdy} \longrightarrow dx \wedge dy$. a 2-form on $\mathbb{R}^2$.

$\int d\theta = \int \boxed{\frac{xdy - ydx}{x^2+y^2}} \longrightarrow$ a 1-form on $\mathbb{R}^2 \setminus 0$

Test your intuition: $\overset{\Omega(M)}{\Lambda(T_*M)} = \bigoplus_p \overset{\Omega^p(M)}{\Lambda^p(T_*M)}$ w/ wedge product

$\alpha \in \Omega^p,\ \beta \in \Omega^q \Rightarrow \alpha \wedge \beta \in \Omega^{p+q}$.

What are we trying to do?

- Linear algebra view on p-form:

$de^i \cdot X = x^i$ $\qquad$ $de^i = ?$

$[\qquad] \cdot \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = x^i$ $\qquad$ $[0 \dots \overset{i}{1} \dots 0]$

More generally.

1-vector X $\qquad$ $X = \sum x^i e_i = x^1 \frac{\partial}{\partial e^1} + \dots + x^n \frac{\partial}{\partial e^n}$

1-form α $\qquad$ $\alpha = \sum \alpha_i de^i = \alpha_1 de^1 + \dots + \alpha_n de^n$

$de^i \cdot \frac{\partial}{\partial e^j} = [i=j]$

Now $\alpha(X) = \alpha \cdot X = \sum_i \alpha_i de^i \left(\sum_j x^j \frac{\partial}{\partial e^j} \right) = \sum_i \alpha_i x^i$

- Wedge product using linear algebra:

Consider vectors u, v. 1-forms α, β.

$u' = \begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix}$, $v' = \begin{bmatrix} \alpha(v) \\ \beta(v) \end{bmatrix}$ project u along α, β.

Then, $u \wedge v = \det[u', v'] = \det \begin{bmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{bmatrix}$.

satisfying all properties of wedge product (check!)

Exterior Derivative.

$d : \Omega^p \to \Omega^{p+1}$

$\alpha \mapsto d\alpha$ s.t.

(1) linear to $+$ and $\cdot$: $d(\alpha+\beta) = d\alpha + d\beta$
$d(c\cdot\alpha) = c\cdot d\alpha$

(2) $d(\overset{\nearrow k\text{-form}}{\alpha} \wedge \beta) = d\alpha\wedge\beta + d\beta\wedge\alpha = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$

(3) df is the normal differential on $\overset{\mathbb{R}\text{-valued}}{\text{fcn}}$/0-form f.

(4) $d(df)=0$ $\forall$ 0-form f.

Test your intuition again : what are trying to do?

example.

- Take $f: \mathbb{R}^3 \to \mathbb{R}.^{=\Omega^0}$ 0-form.

$$df := \frac{\partial f}{\partial e^1}de^1 + \frac{\partial f}{\partial e^2}de^2 + \frac{\partial f}{\partial e^3}de^3 \qquad \nabla f = (df)^\# \text{ gradient.}$$

- Take $\alpha: \mathbb{R}^3 \to \Omega^1(\mathbb{R}^3) \simeq \mathbb{R}^3$ 1-form.

$$d\alpha = d\left(\alpha_1 de^1 + \alpha_2 de^2 + \alpha_3 de^3\right)$$

$$d(\alpha_j \wedge de^j) = d\alpha_j \wedge de^j + \alpha_j \wedge \cancel{dde^j}^{0} = \sum_i \frac{\partial \alpha_j}{\partial e^i} de^i \wedge de^j$$

after simplification using $\alpha\wedge\beta = -\beta\wedge\alpha$. one has

$$d\alpha = \left(\frac{\partial\alpha_3}{\partial e^2} - \frac{\partial\alpha_2}{\partial e^3}\right)de^2\wedge de^3 + \left(\frac{\partial\alpha_1}{\partial e^3} - \frac{\partial\alpha_3}{\partial e^1}\right)de^3\wedge de^1 + \left(\frac{\partial\alpha_2}{\partial e^1} - \frac{\partial\alpha_1}{\partial e^2}\right)de^1\wedge de^2$$

Hodge star $\star : \Omega^p \to \Omega^{n-p}$
$\star(de^{i_1}\wedge\cdots\wedge de^{i_k}) = de^{i_{k+1}}\wedge\cdots\wedge de^{i_n}$
$(i_1, \ldots, i_n)$ even perm.

$$\nabla \times X = (\star d X^\flat)^\# \text{ curl.}$$
↳ cross product

- Take $\beta: \mathbb{R}^3 \to \Omega^2(\mathbb{R}^3) \simeq \mathbb{R}^3$ 2-form.

$$d\beta = d\left(\beta_1 de^2\wedge de^3 + \beta_2 de^3\wedge de^1 + \beta_3 de^1\wedge de^2\right) = \left(\frac{\partial\beta_1}{\partial e^1} + \frac{\partial\beta_2}{\partial e^2} + \frac{\partial\beta_3}{\partial e^3}\right)de^1\wedge de^2\wedge de^3$$

$$\nabla\cdot X = \star d \star X^\flat \text{ div.}$$

$$\begin{tikzcd}
C^\infty \arrow[r, "\nabla"] \arrow[d, "s"'] & X \arrow[r, "\nabla\times"] \arrow[d, shift left, "\flat"] & X \arrow[r, "\nabla\cdot"] \arrow[d, shift left, "\flat"] & C^\infty \arrow[d, "s"'] \\
\Omega^0 \arrow[d, dash, "s"'] & \Omega^1 \arrow[u, shift left, "\#"] \arrow[d, dash, "s"'] & \Omega^1 \arrow[u, shift left, "\#"] \arrow[d, dash, "\star"'] & \Omega^0 \arrow[d, dash, "\star"'] \\
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3
\end{tikzcd}$$

de Rham cohomology.

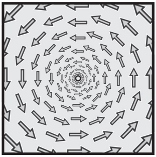
Consider cochain complex $\mathcal{C}_{dR}(\Omega^\bullet, d)$.

define the de Rham cohomology $H^\bullet_{dR}(M)$:

cocycle/closed p-form $[\alpha]$: $d[\alpha] = 0$.

coboundary/exact p-form $[\alpha]$: $\exists (p-1)$-form β : $d\beta = \alpha$.

example. $d\theta = \dfrac{x\,dy - y\,dx}{x^2+y^2}$



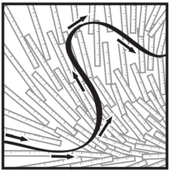
is a closed form (check!)

but not exact: θ is not a 0-form!

($\theta = 2\pi$, wraps around)

Stokes' Thm. α : compacted-supported p-form.

Σ : oriented $(p+1)$-manifold,



Then $$\int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha$$

example. [FTC] : $\int_{[a,b]} df = \int_{\partial[a,b]} f = f(b) - f(a)$

$\Sigma \subseteq \mathbb{R}$

[Green] : $\int_\Sigma (\nabla \times X)\, dA = \int_\Sigma \left(\frac{\partial X_2}{\partial e^1} - \frac{\partial X_1}{\partial e^2}\right) de^1 \wedge de^2$

$\Sigma \subseteq \mathbb{R}^2$

$$= \int_\Sigma d(X_1 de^1 + X_2 de^2)$$

$$= \int_{\partial\Sigma} (X_1 de^1 + X_2 de^2) = \int_{\partial\Sigma} X \cdot d\ell$$

[Stokes'], [Divergence], ...

Wedge product turns $H^\bullet_{dR}(M)$ into a ring. (check!)

Thm. [de Rham isomorphism] For M manifold, $H^\bullet_{dR}(M) \simeq H^\bullet(M;\mathbb{R})$

Let the given manifold be discrete (a complex).

Integrate a p-form α over a p-simplex σ : $\hat{\alpha}_\sigma = \int_\sigma \alpha$

Now $\hat{\alpha}$ is a discrete p-form.

example. Say Δ a triange (2-simplex). w/ edges e_0, e_1, e_2.

α discrete 1-form (on edges).

$$\int_\Delta d\alpha = \int_{\partial\Delta} \alpha = \sum_i \int_{e_i} \alpha = \sum_i \hat{\alpha}_{e_i}$$

now we can store $\int_\Delta d\alpha$ on $\Delta \to d\hat{\alpha}$.

Identical to cohomology on our familiar cochain complex!

Thm. [Poincaré Duality] M oriented n-manifold.

$$\int_M \blacksquare \wedge \square : H^p_{dR}(M) \xrightarrow{\simeq} H^{n-p}_{dR}(M)^\vee$$

$$\int_{\blacksquare} \square : H_p(M;\mathbb{R}) \xrightarrow{\simeq} H^p_{dR}(M)^\vee$$

(integration works at the homology/cohomology level because

$$\int_{\Sigma+\partial\Pi} (\alpha + d\beta) = \int_\Sigma \alpha + \int_{\partial\Sigma} \cancel{d}\beta + \int_{\cancel{\partial}\Pi} d\alpha + \int_{\partial\Pi} d\beta = \int_\Sigma \alpha$$

by Stokes' Thm.

Thus $\int_{[\Sigma]} [\alpha]$ is well-defined.)

Laplacian and Hodge Theorem.

Define Laplacian $L := \overset{(-1)^{p(n-p)}}{(d{\star}d{\star} + {\star}d{\star}d)} \left(= \nabla \cdot \nabla \times \right)$ ∇^2 or Δ

$\exists\ \delta : \Omega^p \to \Omega^{p-1}$; then $L = d\delta + \delta d$.

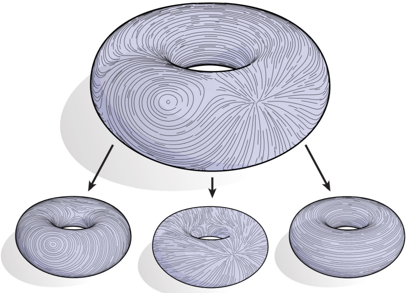
$\cdot \mapsto (-1)^{p(n-p)} {\star} d {\star}$!

Harmonic form $H^\bullet_L(M) := \ker L$.

Hodge Thm. M compact oriented manifold, $\Omega^p(M)$ has decomposition

$$\Omega^p = d\Omega^{p-1} \oplus H_L \oplus \delta\Omega^{p+1}.$$

in other words, every form has d-exact, harmonic, & δ-exact components.



circulation. gradient/co-circulation bi-circulation

Cor. $H_\Delta^\bullet(M) \simeq H_{dR}^\bullet(M)$.

furthermore, Hodge star $\star : H_\Delta^k(M) \longrightarrow H_\Delta^{n-k}(M)$ induces Poincaré duality.

Pf. $\Delta(\star\alpha) = (-1)^{p(n-p)}(d\star d\star + \star d\star d)\star\alpha = d\star d\alpha + \star d\delta\alpha = 0$

$\star\alpha$ is harmonic.

Moral. Every cohomology class has a harmonic representative.

* * *

Application: Isolation lemma in planar graphs.

[MVV'87]

Isolation Lemma. Let F be collection of subsets of $[m]$. If each element in $[m]$ is assigned w/ unif. indep. weight in $[2m]$. Then, w/ pr. $\geq 1/2$, min-weight subset in F is unique.

example. Think about graph w/ m edges, and paths between two vertices.

Derandomizing isolation lemma is a major open problem.

Question. Can we get deterministic isolation on planar graphs?

[DKR'08] [BTV'09]

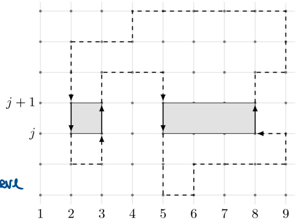
Thm. ~~Any~~ Bipartite planar graph has $O(\log n)$-bit. logspace isolation scheme! for ~~shortest path~~ perfect matching

Pf. Idea. Use Green's Thm!

$$w(e) = \begin{cases} n^4 & e \text{ horizontal} \\ i + n^4 & e \text{ vertical } \uparrow \\ -i + n^4 & \quad '' \quad \downarrow \end{cases}$$

dw is a 2-form. positive everywhere

$$\int_C w = \int_{\partial D} w = \int_D dw > 0.$$



[BTV'09]

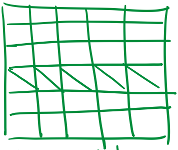
Cor. Planar Reachability $\in$ UL.

Bipartite Planar Matching $\in$ SPL

$\in$ UL [DKT'11]

Open Question. Do we have deterministic logspace isolation lemma for perfect matching on general planar graph?

If so. UL = NL. [K'09]



near-grid