Duality theorems and friends

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Poincaré Duality.

$$H^\bullet(X;\mathbb{Z}) \times H^\bullet(Y;\mathbb{Z}) \xrightarrow{\smile} H^\bullet(X\times Y;\mathbb{Z})$$
$$a \times b \longmapsto \pi_1^*(a) \times \pi_2^*(b)$$
$\pi_1 : X\times Y \to X$, $\pi_2 : X \times Y \to Y$.

Tensor product $A \otimes B$: $\{a\otimes b \mid a \in A, b \in B\}$ satisfying :

$(a+a')\otimes b = a\otimes b + a'\otimes b$.
$a \otimes (b+b') = a\otimes b + a \otimes b'$
$ra\otimes b = a \otimes rb$

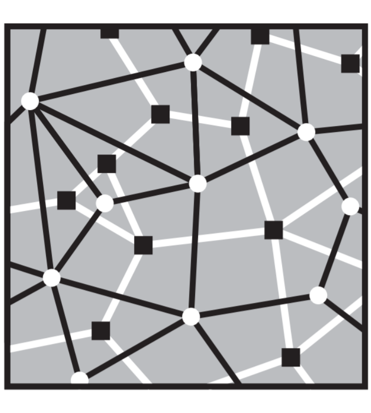
Künneth Formula.
$$H^\bullet(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^\bullet(Y;\mathbb{Z}) \xrightarrow{\sim} H^\bullet(X\times Y;\mathbb{Z})$$

Cor. $\dim H^k(X\times Y) = \sum_{i=0}^{k} \dim H^i(X) \cdot \dim H^{k-i}(Y)$

example: $\dim H^k(\mathbb{T}^n) = ?$ $\dim H^k(\Sigma_g) = [1, 2g, 1]$.
$\dim H^k(\mathbb{T}^n) = \dim H^k(S^1 \times \cdots \times S^1) = \binom{n}{k}$.

- The Betti numbers are palindromic. Coincidence? I think not...

Planar graph & its dual :

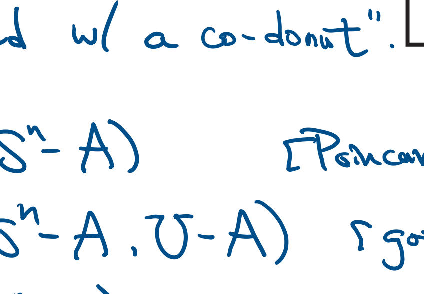


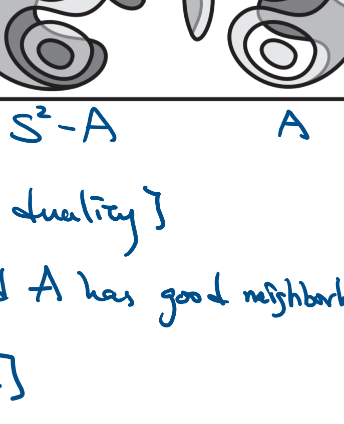
$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$
$$\wr| \quad \wr| \quad \wr|$$
$$0 \to C_0^* \xrightarrow{\delta_1} C_1^* \xrightarrow{\delta_2} C_2^* \to 0$$

$$H_k(\Sigma) = H^{2-k}(\Sigma) = H_{2-k}(\Sigma)$$

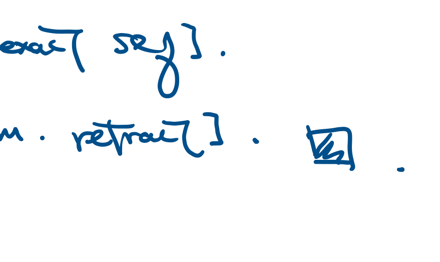
Poincaré Duality.
Let M be an closed R-orientable n-manifold. One has isomorphism
$$H^k(M;R) \xleftarrow{\sim} H_{n-k}(M;R)$$
(Actually, coming from cap product $[M] \frown \varphi$ ($\sigma \frown \varphi$).
$\frown : C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R)$
$\sigma \frown \varphi := \varphi(\sigma[0\ldots l]) \cdot \sigma[l \ldots k]$) (1-cochain; R; 1-chain)

example. surface dual. (over $\mathbb{Z}_2$).
$\alpha_i \rightsquigarrow b_i$
$\beta_i \rightsquigarrow a_i$





Alexander Duality.
Let A be good (→ compact, locally contractible) subspace of S^n. Then
$$\tilde{H}_i(S^n - A) \simeq \tilde{H}^{n-i-1}(A) \quad \forall i$$
"Every donut hole is filled w/ a co-donut".



$S^2 - A$ A

pf. $H_k(S^n - A) \simeq H^{n-k}(S^n - A)$ [Poincaré duality]
$\simeq H^{n-k}(S^n - A, U - A)$ [good A has good neighborhood]
$\simeq H^{n-k}(S^n, U)$ [excision]
$\simeq \tilde{H}^{n-k-1}(U)$ [long exact seq].
$\simeq \tilde{H}^{n-k-1}(A)$ [deform. retract]. ☒.

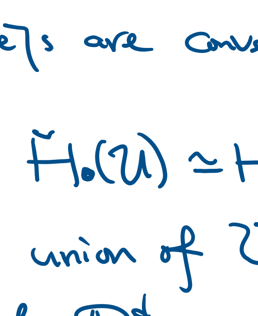
Excision Thm. $A \subseteq U \subseteq X$ s.t. $\bar{A} \subseteq U^\circ$. Then
$$H_\bullet(X-A, U-A) \simeq H_\bullet(X, U)$$

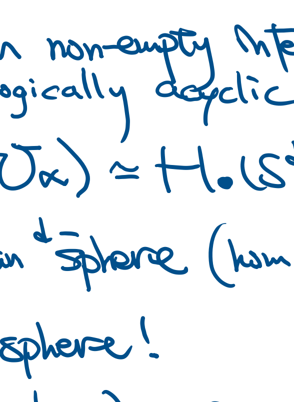
example. solid-donut hole in $\mathbb{R}^3$ is also a solid-donut (minus a pt).
$\tilde{H}_1(S^3 - ◎) \simeq \tilde{H}^1(◎) \simeq \mathbb{Z}$ I incorrectly wrote $\mathbb{Z}\times\mathbb{Z}$ here; the donut is solid.
but so does the complement of a knot.

Application. Helly Thm (in discrete/comp. geometry)

Helly Thm. Let $\mathcal{U} = \{U_\alpha\}$ be a collection of $N \geq d+2$ convex subsets of $\mathbb{R}^d$, s.t. every $d+1$ subsets in $\mathcal{U}$ has common intersection.
Then, intersection of all subsets in $\mathcal{U}$ is non-empty.

example. rectangles in $\mathbb{R}^2$.
$N \geq 4$. every triple has non-empty intersection.





pf. Induction on # subsets.
Base: $N = d+2$.
The nerve $N(\mathcal{U})$ is a $(d+1)$-simplex $[0\ldots d+1]$ w/ all n-faces.
Assume no common intersection : $N(\mathcal{U}) \simeq \partial\Delta^{n+1} \simeq S^n$.
Because all subsets are convex, each non-empty intersection is homologically acyclic.

By Nerve Thm. $\tilde{H}_\bullet(\mathcal{U}) \simeq H_\bullet(\bigcup_\alpha U_\alpha) \simeq H_\bullet(S^d)$.
In other words, union of U_α is an d-sphere (hom. speaking)
But no subset of $\mathbb{R}^d$ can be d-sphere!
$\because \tilde{H}_d(A) \simeq \tilde{H}^{d-d-1}(S^d - A) \simeq \tilde{H}^{-1}(S^d - A) = 0$. ⨯.
Induction Step: Same argument w/ IH. ☒

Graph Laplacian. (Application in spectral graph theory & linear system solvers) ; linear algebra!

Given an undirected graph G, w/ vertex set V, edge set E.
also, assume it is unweighted.

Adjacency matrix : $A_G = [A_{ij}] := [[ij \in E]]$ $[w_{ij}]$
Degree matrix : $D_G = [D_{ij}] := [[i=j] \cdot d_i]$ $[\sum_l w_{il} \cdot [i=j]]$

Graph Laplacian $L_G := D_G - A_G$.

Prop. $L_G = \sum_e L_e$ where $L_e = \begin{matrix} & i & j \\ i & 1 & -1 \\ j & -1 & 1 \end{matrix}$ $\begin{bmatrix} w_{ij} & -w_{ij} \\ -w_{ij} & w_{ij} \end{bmatrix}$

Graph as electrical (resistor) network.
incidence matrix
$$B = [B_{e,i}] = \left[\begin{cases} 1 \ (w_e) & \text{if } e \to i \\ -1 \ (-w_e) & \text{if } i \to e \end{cases}\right]. \quad w_e = 1/r_e \text{ resistence}$$
Prop. $B^T B = L$.

Let c be the current vector at vertices,
which induces voltages at vertices : v
current at edges : i

Kirchoff's law : $B^T i = c$ [conservation of electrical flow].
Ohm's law : $i = Bv$ [current = Δvoltage/resistence]

Together. $Lv = B^T B v = c$
⇒ Given vertex current c, find vertex voltage v that induces c.

Application:
Computing effective resistence:
Send unit current from j to i : $c = e_i - e_j$
Test voltage difference : $(e_i - e_j)^T v = (e_i - e_j)^T L^{-1} c$ ($L^\dagger$ pseudo-inverse)
$= (e_i - e_j)^T L^{-1}(e_i - e_j)$.
$R_{eff}(i,j) := (e_i - e_j)^T L^{-1} (e_i - e_j)$.

Homological explanation. (using Poincaré duality)
Assume G is planar.

$$0 \leftarrow C^2 \xleftarrow{\partial_2} C^1 \,(i) \xleftarrow{\partial_1} C^0 \,(v) \leftarrow 0$$
$$\wr| \quad \wr| \quad \wr|$$
$$0 \to C_0^* \xrightarrow{\delta_1} C_1^* \xrightarrow{\delta_2} C_2^* \to 0$$
$$\wr| \quad \wr|\, i \quad \wr|\, c$$
$$0 \to C_2 \to C_1 \to C_0 \to 0$$

$B = [B_{e,v}]$: 1-coboundary map ∂_1 (voltage induces currents)
$B^T = [B^T_{f,e}]$: 2-coboundary map δ_2. (conservation of flow at vertices of G / faces of G^*)

We want flow i s.t. $\partial_1 v = Bv = i$ and $\delta_2 i = B^T i = c$.
$H^1(G) = 0$! ⇒ $\ker \partial_2 = \operatorname{im} \partial_1$; if $\partial_2 i = 0$ then $\exists v : \partial_1 v = i$.
- Given G & c, find flow i w/ excess c & circulation in G^*.

Lemma. All eigenvalues of L_G are real
pf. Because L_G is real & symmetric.
$v^* A v = v^* A^* v = (Av)^* v = (\lambda v)^* v = \lambda^* v^* v$.
thus $\lambda \|v\|^2 = \lambda^* \|v\|^2$. $\lambda = \lambda^*$ real. ☒.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be eigenvalues of L_G.
Lemma. $L_G = \sum_i \lambda_i u_i u_i^T$.
pf. $\sum_i \lambda_i u_i u_i^T u_j = \lambda_i u_j u_j^T u_j = \lambda_i u_j = A u_j$ ☒.

Lemma. L_G is positive semi-definite (psd) : $\lambda_1 \geq 0$. $L_G \succeq 0$.
(Thm [Courant-Fischer] M real symmetric.
$$\lambda_1(M) = \min_{v \in \mathbb{R}^n - 0} \frac{v^T M v}{v^T v}.\text{)}$$

pf. $v^T L_G v = \sum_e v^T L_e v$.
$= \sum_e (v_i - v_j)^2 \geq 0$.

Lemma. $\lambda_1 = 0$.
pf. $L_G \mathbb{1} = 0$ ☒.

Lemma. $\lambda_2 > 0$ iff G connected.
pf. ⇒ easy.
⇐: $\lambda_2 = u_2^T L u_2 = \sum_{ij \in E} (u_2[i] - u_2[j])^2 = 0$.
if G connected, $u_2 = u_2[1] \cdot \mathbb{1}$. ⨯. ☒

Woah. What are we doing?
λ_2 is a connectivity measure.
Thm. $$\lambda_2(\mathcal{L}) = \min_{y \perp D^{1/2}\mathbb{1}} \frac{y^T \mathcal{L} y}{y^T y} = \min_S \frac{|E(S, \bar{S})|}{\operatorname{vol}(S) \cdot \operatorname{vol}(\bar{S})}$$ conductance
$(\mathcal{L} = D^{-1/2} L D^{-1/2})$

$\lambda_2(\mathcal{L}) = \Theta(1)$: expander.
Many topological tools sneak in :
- $\lambda_2(\text{planar graph}) = O(\frac{1}{n})$ Tutte embedding.
- Construct optimal expander iterated 2-lift.

[ST04, KOSZ13]
Thm. Laplacian solver returns x to $Lx = c$:
$$\|x - L^{-1}c\|_L \leq \varepsilon \|L^{-1}c\|_2 \quad (\|b\|_L := \sqrt{b^T L b})$$
in $\tilde{O}(m \log 1/\varepsilon)$ time. $m :=$ # non-zeros in L