Cohomology

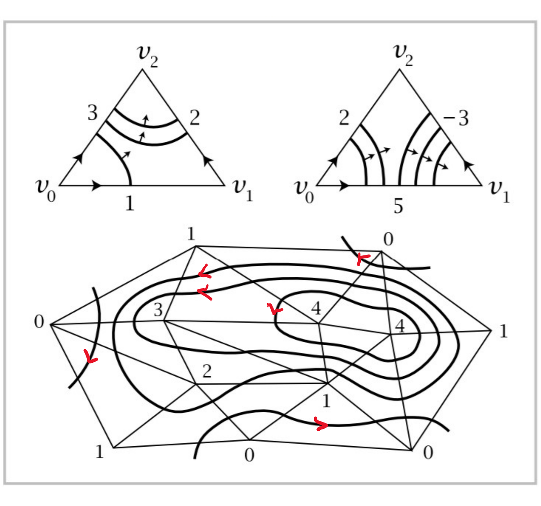
10/19/20 10:12 AM

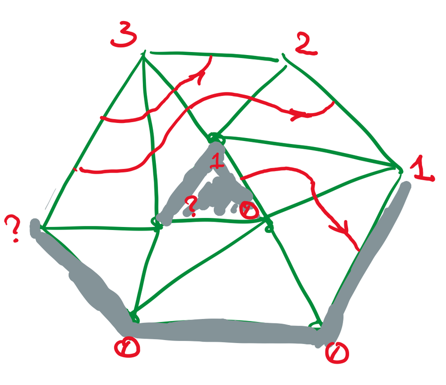
Administrivia.

- HW2 due this Friday (10/23). Maybe an informal OH on Wednesday.
- Project proposal due <u>next</u> Friday (10/30).
  - Each person submit to Canvas individually.
- One last <u>homework 3</u> due Friday two weeks from now (11/06).
- Final project presentation on Finals week (11/30 – 12/04).

Cohomology.

The <u>obstruction</u>/inability to extend a good local fcn to a good global fcn.





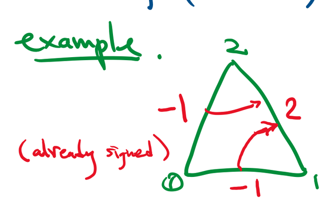
Cochain complex.

$C^k(X;G) := \{$fcns. from $C_k(X)$ (under +) to $G\} =: \mathrm{Hom}(\Delta_k(X), G)$

$\delta^k : C^{k-1}(X;G) \to C^k(X;G)$ by

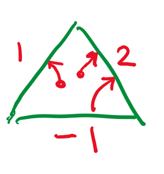
$\delta\psi([0\ldots k]) := \sum_i (-1)^i \psi([0 \ldots \hat{i} \ldots k])$

example.



$G = \mathbb{Z}$.

$\delta\psi([0.1.2]) = 2 - 1 + (-1) = 0$.



- $\delta\psi$ means surplus/deficit. $\delta\psi = 0$: conservation of flow.

Prop. $\delta\psi^i = \psi\partial^i$

pf. $C_i(X) \xrightarrow{\partial} C_{i-1}(X) \xrightarrow{\psi} G$; δ pulling ψ back. (composite: $\delta\psi$)

Prop. $\delta \circ \delta = 0$.

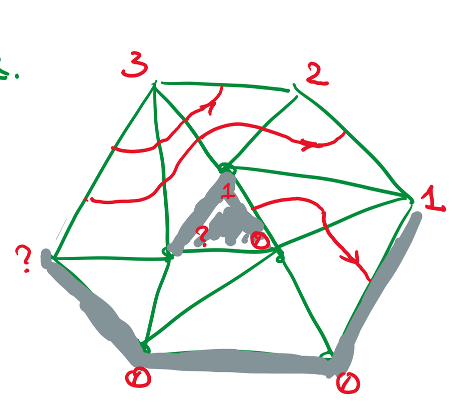
pf. mechanical.

Now $0 \to C^0(X) \xrightarrow{\delta^1} C^1(X) \xrightarrow{\delta^2} C^2(X) \to \cdots$ is a <u>cochain complex</u>

Cohomology group.

$H^i(X;G) := \ker\delta^{i+1} / \mathrm{im}\,\delta^i$ (ith cocycle / ith coboundary.)

example.



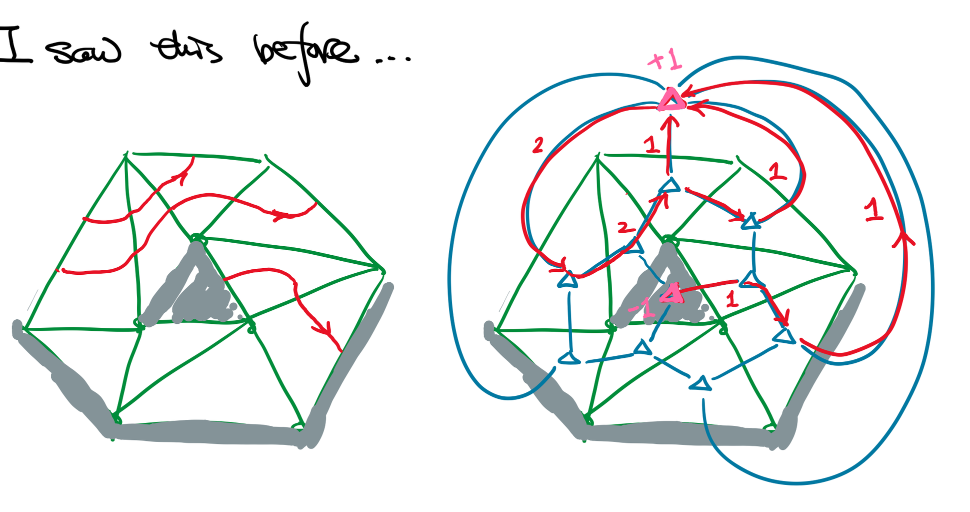
$\psi \in C^1(X;\mathbb{Z})$. $\delta^2\psi = 0$.

$\Rightarrow \psi \in \ker\delta^2$

$\mathrm{im}\,\delta^1 :=$ flows (edge labelings) coming from vertex labels.

$\psi \in H^1(X;\mathbb{Z})$ not zero.

Wait, I saw this before...

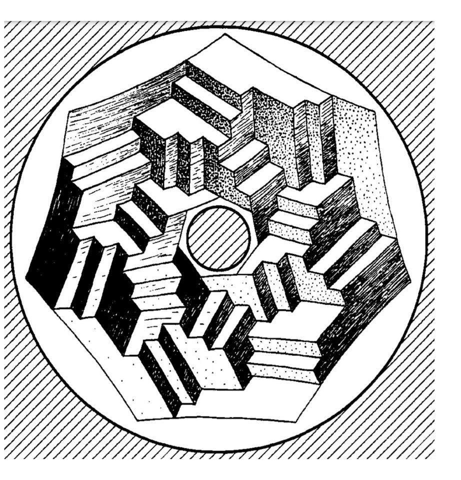


1-cocycle : flow in the dual planar graph G^*.

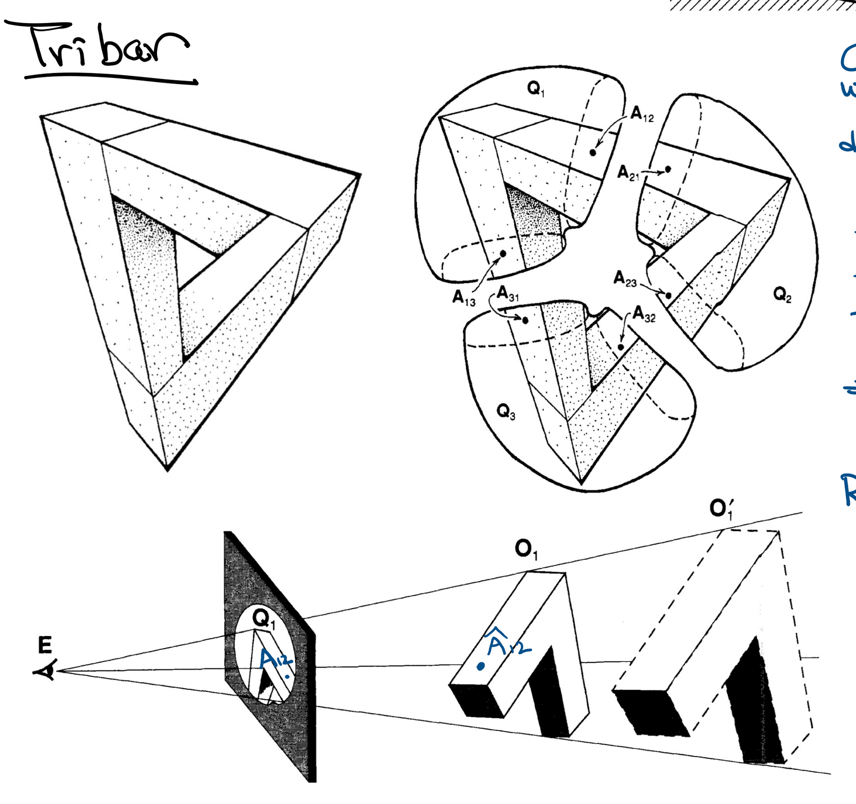
1-coboundary = <u>circulation</u> induced by vertex potential in G / face potential in G^*.

Why? Seems redundant.

Exhibit A : Impossible objects.



Tribar



Choose 3D realization O_i of Q_i.

we can define

$d_{ij} := \dfrac{d(E, \hat{A}_{ij}) \text{ on } O_i}{d(E, \hat{A}_{ij}) \text{ on } O_j}$

- if $d_{ij} = 1$, the whole $O_i \cap O_j$ matches.
- $d_{ij} = d_{ji}^{-1}$
- scaling O_i by ξ_i : $d_{ij} \mapsto \xi_i d_{ij}$, $d_{ji} \mapsto d_{ij}/\xi_i$

$\Rightarrow \{d_{ij}\}$ are 1-cochain / $\{\xi_i\}$ are 0-cochain in Čech cohomology. mapping $O_i \cap O_j$ to d_{ij}, O_i to ξ_i. w/ coefficient in $(\mathbb{R}^+, \times)$-ring! not $(\mathbb{R}, +)$-field.

Realizable if by rescaling O_i by ξ_i, all ratios are 1.

$\Rightarrow \exists \xi_i$ s.t. $\dfrac{\xi_i d(E, \hat{A}_{ij})}{\xi_j d(E, \hat{A}_{ji})} = \dfrac{\xi_i}{\xi_j} d_{ij} = 1$

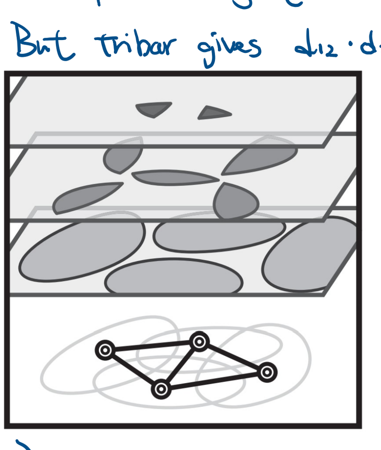
in other words, $\{d_{ij}\}$ are 1-cobdry. $\{d_{ij}\}(Q_i \cap Q_j) = \delta\{\xi_i\}(Q_i \cap Q_j) = \{\xi_i\}\partial(Q_i \cap Q_j) = \{\xi_i\}Q_j / \{\xi_i\}Q_i = \xi_j/\xi_i$

if $\{d_{ij}\} \in \check{H}^1(Q; \mathbb{R}^+)$ not zero. (S^1, Annulus)

$\Rightarrow$ impossible object.

But tribar gives $d_{12} \cdot d_{23} \cdot d_{31} \neq 1$, $\{d_{ij}\}$ not zero.

Čech complex and Nerve Thm. redux



point set P, balls $B(x_i, \varepsilon)$ covering $X := \bigcup_i B(x_i, \varepsilon)$

$\check{C}_n(\mathcal{U}) := \langle$common intersections of $n+1$ subsets in $\mathcal{U}\rangle =: \langle U_J \rangle$

$\check{\partial}(U_J) := \sum_i (-1)^i U_{J \setminus i}$

Čech homology $\check{H}_\bullet$ defined on $(\check{C}_\bullet(\mathcal{U}), \check{\partial})$.

Thm. $\check{C}_n(\mathcal{U}) \hookrightarrow C_n(X)$ induces isomorphism $\check{H}_n(\mathcal{U}) \simeq H_n(X)$.

if $\tilde{H}_\bullet(U_J) = 0$ for all nonempty U_J (acyclic)

"pf": $\check{H}_\bullet(\mathcal{U}) := H_\bullet(\check{C}(\mathcal{U})) = H_\bullet(X)$ (Čech complex.)

Čech cohomology : $\check{H}^\bullet(\mathcal{U}) := H^\bullet(\check{C}(\mathcal{U}))$

$C^0(\mathcal{U};\mathbb{R}) := \{\mathbb{R}$-valued fcn on open sets in $\mathcal{U}\}$ $\{\xi_i\}$

$C^1(\mathcal{U};\mathbb{R}) := \{\mathbb{R}$-valued fcn on pairwise intersections$\}$ $\{d_{ij}\}$

Puzzling evidence :

[Ophoven, Belgium]

My explanation in class that these item fail:

3. scaling O_i by ξ_i : $d_{ij} \mapsto \xi_i d_{ij}$, $d_{ji} \mapsto d_{ij}/\xi_i$

1. d_{ij} indep. to A_{ij}

is INCORRECT!

(3) is false but not required. (1) is true even w/ curvy bars.

The actual reason uses the bars being straight in <u>some</u> way : one has

$d_{12} \cdot d_{23} \cdot d_{31} \neq 1$.

How does cohomology relates to homology?

Thm. Let $C_\bullet(X;F)$ be chain complex and

$C^\bullet(X;F) := \mathrm{Hom}(C_\bullet(X;F), F)$ be cochain complex.

Then $\dim H^i(X;F) = \dim H_i(X;F)$

pf. $[\delta^i] = [\partial^i]^T$ as matrix. and

$\dim C^i = \dim \mathrm{Hom}(C_i, F) = \dim C_i$.

Thus $\dim H^i(X;F) = \dim\ker\delta^{i+1} - \dim\mathrm{im}\,\delta^i$

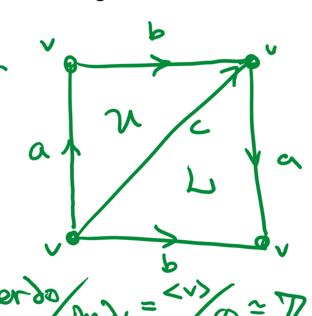
$= \dim C^i - \mathrm{rank}\,\delta^{i+1} - \mathrm{rank}\,\delta^i$ $\quad C^{i+1} \xleftarrow{\delta^{i+1}} C^i \xleftarrow{\delta^i} C^{i-1}$

$= \dim C_i - \mathrm{rank}\,\partial^i - \mathrm{rank}\,\partial^{i+1}$ $\quad C_{i+1} \xrightarrow{\partial^{i+1}} C_i \xrightarrow{\partial^i} C_{i-1}$

$= \dim H_i(X;F)$ ∎

More generally, $H^i(X;\mathbb{Z}) \simeq (H_i(X)/\mathrm{Tor}_i(X)) \oplus \mathrm{Tor}_{i-1}(X)$.

example.



$C_0 = \langle v \rangle$, $C_1 = \langle a, b, c \rangle$, $C_2 = \langle U, L \rangle$

$\partial_1(a, b, c) = v - v = 0$.

$\partial_2 U = -a - b + c$

$\partial_2 L = -a + b - c$

$C^0 = \langle v^* \rangle$ $C^1 = \langle a^*, b^*, c^* \rangle$, $C^2 = \langle U^*, L^* \rangle$. $(x^*(x) = n_x \in \mathbb{Z})$

$H_0^\Delta(K) = \ker\partial_0 / \mathrm{im}\,\partial_1 = \langle v \rangle / 0 \simeq \mathbb{Z}$

$H_1^\Delta(K) = \ker\partial_1 / \mathrm{im}\,\partial_2 = \langle a, b, c \rangle / \langle -a-b+c, -a+b-c \rangle = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ (corrections: $-a-b+c$, $-2a$) $\quad H_2^\Delta(K) = 0$.

$\delta^1 v^*(a, b, c) := v^*(\partial^1(a, b, c)) = v^*(v) - v^*(v) = 0$.

$\delta^2 a^*(U) := -a^*(a) - a^*(b) + a^*(c) = -n_a = \delta^2 a^*(L)$

$\delta^2 b^*(U) = -n_b$. $\delta^2 b^*(L) = n_b$.

$\delta^2 c^*(U) = n_c$. $\delta^2 c^*(L) = -n_c$.

$H^0(K;\mathbb{Z}) = \ker\delta^1 = \langle v^* \rangle \simeq \mathbb{Z}$

$H^1(K;\mathbb{Z}) = \ker\delta^2 / \mathrm{im}\,\delta^1 = \langle \frac{b^*}{n_b} + \frac{c^*}{n_c} \rangle / 0 \simeq \mathbb{Z}$.

$H^2(K;\mathbb{Z}) = \ker\delta^3 / \mathrm{im}\,\delta^2 = \langle U^*, L^* \rangle / \langle U^* + L^*, U^* - L^* \rangle \simeq \mathbb{Z}/\mathbb{Z}_2$

$H_\bullet : (\overset{0}{\mathbb{Z}}, \overset{1}{\mathbb{Z} \oplus \mathbb{Z}_2}, \overset{2}{0})$

$H^\bullet : (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2)$

Why are you torturing me w/ math again?

Because cohomology is <u>better</u>; it has a product!

Cup product. $\varphi \in C^k(X;\mathbb{Z})$. $\psi \in C^\ell(X;\mathbb{Z})$.

$(\varphi \smile \psi)(\sigma) := \varphi(\sigma[0 \ldots k])\,\psi(\sigma[k \ldots k+\ell]) \in C^{k+\ell}(X;\mathbb{Z})$

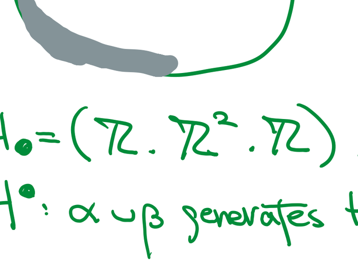
Prop. $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$. maps cocycle to cocycle, cobdry to cobdry.

Now cup product $\smile$ induces product in cohomology $H^\bullet(X;\mathbb{Z})$

$H^i(X;\mathbb{Z}) \times H^j(X;\mathbb{Z}) \xrightarrow{\smile} H^{i+j}(X;\mathbb{Z})$

example.

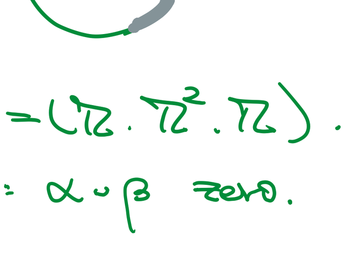
Torus $S^1 \times S^1$:



$H_\bullet = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$.

$H^\bullet$: $\alpha \smile \beta$ generates H^2.

Sphere w/ 2 circles $S^2 \vee S^1 \vee S^1$



$H_\bullet = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$.

$H^\bullet$: $\alpha \smile \beta$ zero.

Moral : cohomology can tell two spaces apart!

Borsuk-Ulam Thm. redux.

There's no antipodal map $f : S^n \to S^{n-1}$ (corrected to $\mathbb{RP}^n \to \mathbb{RP}^{n-1}$). really a statement about $\mathbb{RP}^n$.

pf. $H^\bullet(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$. (!) deg-n polynomials of α over $\mathbb{Z}_2$.

$f^* : H^\bullet(\mathbb{RP}^{n-1}; \mathbb{Z}_2) \to H^\bullet(\mathbb{RP}^n; \mathbb{Z}_2)$

Take $[\alpha] \in H^1(\mathbb{RP}^{n-1}) \longmapsto [\beta] \in H^1(\mathbb{RP}^n)$

α^n is zero, but $\beta^n = f^*(\alpha^n)$ is not. ✗ ∎
