Fixed-point theorems
10/12/20 10:34 AM

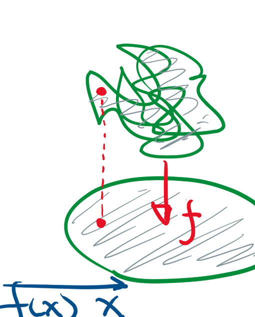
Administrivia.

- HW1 is graded, all feedbacks in a separate pdf.
  - 4 – correct and convincing.
  - 3 – mostly correct modulo minor details.
  - 2 – heading the right direction, but with major gaps.
  - 1 – honest effort but incorrect solution.
  - 0 – missing / not even wrong. (rare)
- HW2 is out, due 10/23 (Fri)
- Project proposal. due 10/30 (Fri).
  - Every person has to submit a 1-2 page proposal.
  - Theoretical, experimental, survey, art, games, etc.
  - I will assemble them and send it to everyone.
  - People join in groups of ≤ 3 on topic you like.
  - Come talk to me in groups. during the 1st week of Nov.
  - Final presentation on exam week (Nov 30 – Dec 4).

—— ◎ ——

Brouwer's Fixed-pt Thm.

Every map $f: D^n \to D^n$ has a fixed pt.

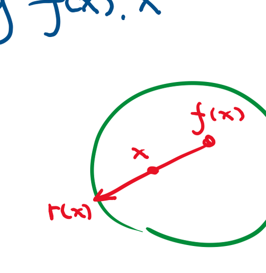


Pf. If f has no fixed pt.

$r(x) :=$ intersection pt between $\partial D^n \approx S^{n-1}$ & ray $\overrightarrow{f(x).x}$

<u>Claim</u>. r is a retract from D^n to ∂D^n

(check that r is fixed on ∂D^n)



then $r \circ i = \mathbb{1}$ for $i: \partial D^n \to D^n$.

$\boxed{H_{n-1}(\partial D^n)} \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$ should be identity $\mathbb{1}_*$

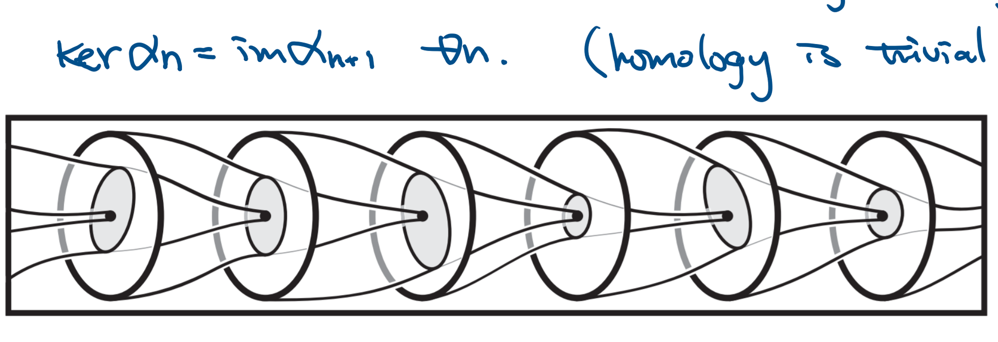
but $\boxed{\mathbb{Z}} \to 0 \to \mathbb{Z}$ ✗ ■

Wait. How do we know $H_{n-1}(S^{n-1}) = \mathbb{Z}$?

Exact sequence

$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$ is <u>exact</u> if

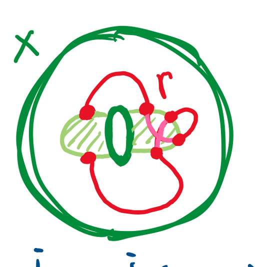
$\ker \alpha_n = \operatorname{im} \alpha_{n+1}$ $\forall n$. (homology is trivial)



Relative Homology.

$C_n(X,A) := C_n(X)/C_n(A)$. $\partial: C_n(X,A) \to C_{n-1}(X,A)$

Define $H_n(X,A)$ for chain complex $(C_\bullet(X,A), \partial)$



$A \xrightarrow{i} X \xrightarrow{j} (X,A)$

<u>connecting map</u> $\partial_*: H_n(X,A) \to H_{n-1}(A)$:

Fix $[\gamma] \in H_n(X,A)$, find β in $C_n(X)$: $j(\beta) = \gamma$.

$j(\partial\beta) = \partial j(\beta) = \partial\gamma = 0$

Thus $\partial\beta = i(\alpha)$ for some $\alpha \in C_{n-1}(A)$

Define $\boxed{\partial_*[\gamma] := [\alpha] \in H_{n-1}(A)}$

$\ker \partial_* = \operatorname{im} j_*$, $\ker i_* = \operatorname{im} \partial_*$

<u>Excision Thm</u>. $U \subseteq A \subseteq X$ s.t. $\bar{U} \subseteq A^\circ$. Then

$$H_\bullet(X-U, A-U) \simeq H_\bullet(X,A)$$

<u>Cor</u>. If A is a deformation retract of (some neighborhood of) X, then

$H_\bullet(X,A) \simeq \tilde{H}_\bullet(X/A)$. $\tilde{H}_n = H_n$. $\tilde{H}_0 \oplus \mathbb{Z} = H_0$

Short exact sequence on chain complex

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_n(A) & \to & C_n(X) & \to & C_n(X,A) & \to & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \to & C_{n-1}(A) & \to & C_{n-1}(X) & \to & C_n(X,A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

implies long exact sequence on homology groups:

<u>Thm</u>. $\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$ is exact.

<u>example</u>. $\partial D^2 \to D^2 \to (D^2, \partial D^2)$

$\cdots \to \underset{0}{H_2(D^2)} \to H_2(D^2, \partial D^2)$

$\to H_1(\partial D^2) \to \underset{0}{H_1(D^2)} \to H_1(D^2, \partial D^2)$

$\to H_0(\partial D^2) \to H_0(D^2) \to H_0(D^2, \partial D^2) \to 0$

we have:

$0 \to H_2(D^2, \partial D^2) \to H_1(\partial D^2) \to 0$

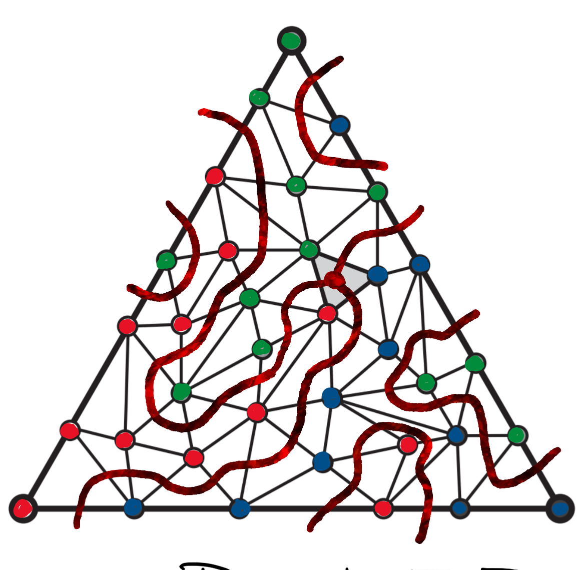
thus $H_2(S^2) \simeq H_1(S^1) \simeq \mathbb{Z}$.

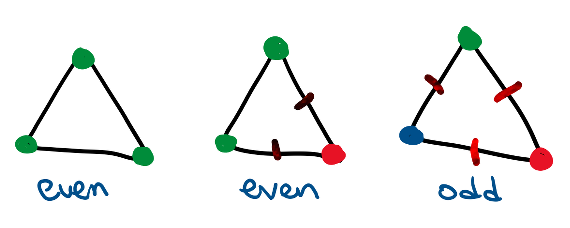
Discrete setting:

<u>Sperner's lemma</u>.

Every properly [d]-labeled simplicial subdivision of Δ^d contains a cell with all labels.

<u>pf by picture</u>:

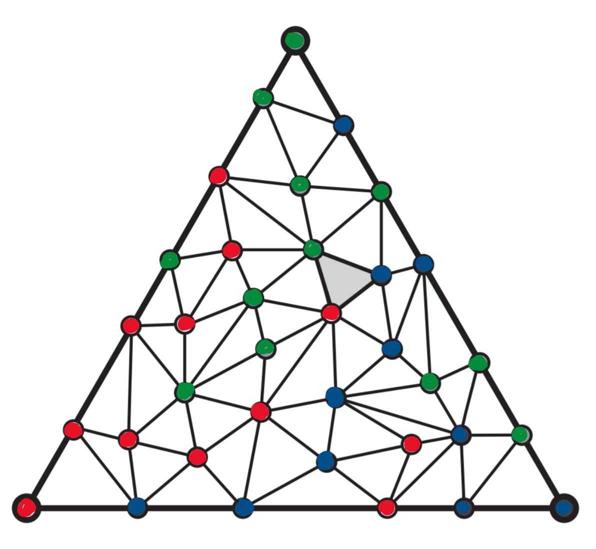




doors on the exterior is odd

⇒ ∃ a room w/ odd # doors.

Connection to Brouwer's FPT.



Consider Δ^d as probability fcn:

$\Delta = \{(\alpha_0 \ldots \alpha_d) : \sum \alpha_i = 1\}$.

vertex $(\alpha_0 \ldots \alpha_d)$ has color i if

$f(\alpha)_i < \alpha_i$ (well-def when no-fixed pt.)

Refine triangulation and take x^* to be limit of full-color Δs.

Intuition: pt w/ color i is moving <u>away</u> from corner i

Complexity to find a fixed pt.

Algorithm: Just follow the paths!

But the triangulation can be <u>huge</u>.

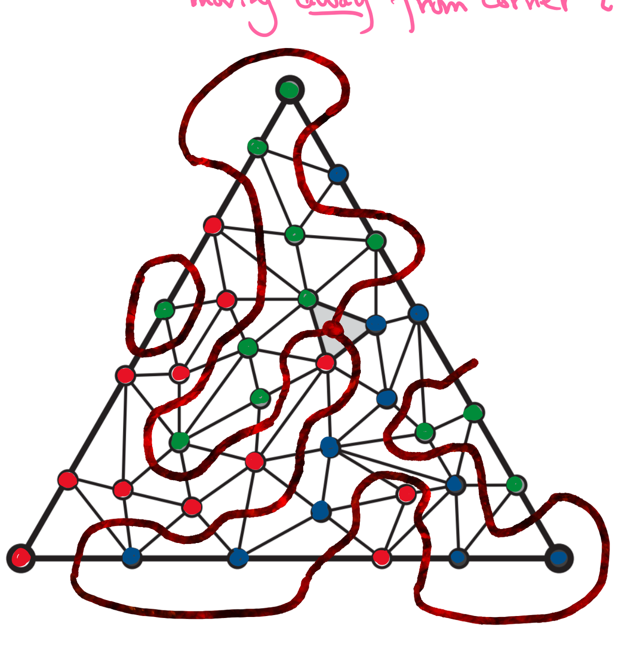
(say we want error $|x - x^*| < \varepsilon$, then the input "ε" is $\log 1/\varepsilon$ bits.)

PPAD (poly-time parity argument, directed)

Given implicit repn. of <u>directed</u> graph, with in-/out-deg 1 (efficient query)

Start at an unbalanced vertex.

<u>Goal</u>: find another unbalanced vertex.



[Papadimitriou94, Chen-Deng09]

<u>Thm</u>. Sperner's problem is PPAD-complete.

[Daskalakis et al. 06, Chen-Deng09]

<u>Implication</u>: Nash equilibrium exists; finding one is PPAD-complete.

<u>Prisoner's Dilemma</u>

P1 \ P2	coop	betray
coop	-1, -1	-10, 0
betray	0, -10	-5, -5

<u>Coin Matching</u>

P= \ P≠	head	tail
head	1, -1	-1, 1
tail	-1, 1	1, -1

Braess's paradox.

Adding a road slows down overall traffic.

—— ◎ ——

Borsuk-Ulam Theorem.

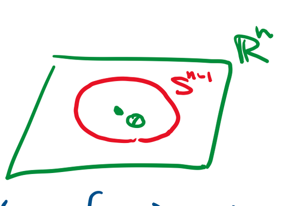
Every map $f: S^n \to \mathbb{R}^n$ that is <u>antipodal</u>: really a statement about $\mathbb{RP}^n$.

$-f(x) = f(-x)$ $\forall x \in S^n$

must have $f(x) = 0$ for some $x \in S^n$.

<u>Cor</u>. $\exists x \in S^n$: $h(x) = h(-x)$ for any $h: S^n \to \mathbb{R}^n$.

Pf. $f(x) = h(x) - h(-x)$. f is antipodal.

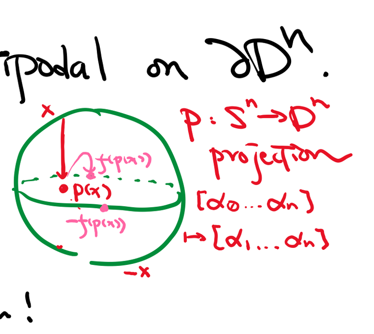


<u>Cor</u>. There's no antipodal map $f: S^n \to S^{n-1}$.

Pf. If there's one, it's also $f: S^n \to \mathbb{R}^n$ w/o $f(x) = 0$.

<u>Cor</u>. There's no map $f: D^n \to S^{n-1}$ that is antipodal on ∂D^n.

Pf. such f extends back to $\hat{f}: S^n \to S^{n-1}$ by mapping $\hat{f}(x) = f(p(x))$ and $\hat{f}(-x) = -f(p(x))$



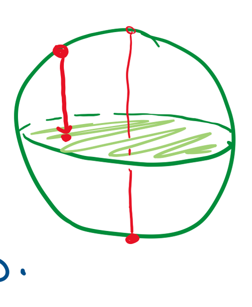
We can prove Brouwer's FPT from Borsuk-Ulam!

Pf. $r(x) :=$ intersection between ∂D^n & ray $\overrightarrow{f(x).x}$.

"Fake" proof to Borsuk-Ulam: [Bárány 80], [MW 79], …

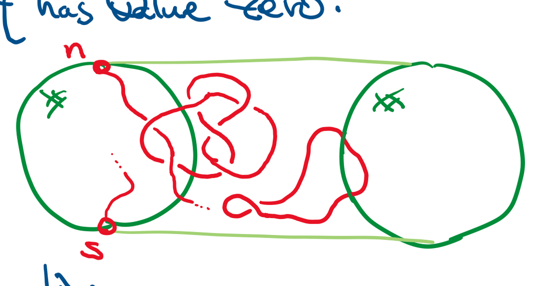
Consider $g: S^n \to D^n$ to be the projection map to the disk at equator:

⇒ g has <u>exactly one</u> antipodal pair w/ $g(x) = 0$.



The idea is to perform homotopy from f to g. while preserving the parity of pairs that has value zero.

$F(x,t) := (1-t) \cdot g(x) + t \cdot f(x)$ (must be antipodal)



Consider zero set $Z := F^{-1}(0)$.

If f is sufficiently generic, Z is 1-dim.

The path in Z connecting n to s is invariant under $(x,t) \mapsto (-x,t)$

⇒ no way the two ends can meet! ■?

Actual proof: Homology; discrete ver. (Tucker's lemma) [Lefschetz49]; Cohomology (!)

<u>Applications</u>. Fair divisions.

<u>Ham Sandwich Thm</u>. [Stone-Tukey] A ham sandwich has a straight cut that divides ham, cheese, and bread evenly.

Pf. Imagine each cut as a hyperplane h.

Every hyperplane in $\mathbb{R}^3$ corresponds to a point on S^3:

$\{h: h_1x_1 + h_2x_2 + h_3x_3 \le h_0\} \longleftrightarrow \{(h_0, h_1, h_2, h_3)\}$

Define $f: S^3 \to \mathbb{R}^3$ by

$f(h) :=$ (amount of ham, cheese, bread on one side of h)

f is cont., by Borsuk-Ulam, $\exists h \in S^3$: $f(h) = f(-h)$. ■

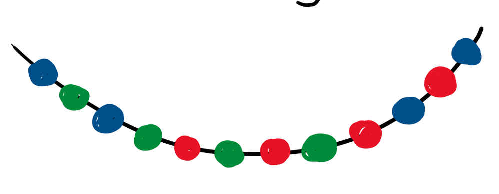
<u>Ham sandwich thm for points</u>. Let $A_1 \ldots A_d$ be pt sets in $\mathbb{R}^d$.

There's a hyperplane dividing all A_i's <u>evenly</u>. $\lfloor \frac{|A_i|}{2} \rfloor$ or $\lceil \frac{|A_i|}{2} \rceil$

<u>Necklace Thm</u>. Every (open) necklace w/ d colors of stones can be divided between two thieves using $\le d$ cuts.

Pf. place the necklace along the moment curve in $\mathbb{R}^d$:

$\gamma(t) := (t, t^2, t^3, \ldots, t^d)$



Define $A_i := \{\gamma(t) : t^{\text{th}}$ stone has color $i\}$

By ham sandwich theorem, ∃ fair division hyperplane, which intersects γ at most d times! ■

Powerful generalizations:

<u>Spicy Chicken Theorem</u>. [KHA14, BZ14] A spicy (convex) fat chicken can be cut into n (p^m) chunks, each w/ equal amount of meat & spice.

<u>Thm</u>. Any convex body in $\mathbb{R}^d$ w/ $(d-1)$ cont. $\mathbb{R}$-valued fcns f_k. There's a partition into p^m (∀p.m.) convex bodies $K_1 \ldots K_n$ s.t.

$\mu(K_i) = \mu(K_j)$, $f_k(K_i) = f_k(K_j)$ $\forall i,j,k$.

<u>Application</u>. Fair clustering.

—— ◎ ——

Complexity of Borsuk-Ulam.

<u>Tucker's Lemma</u>. Every symmetrical triangulation of Δ^d w/ $[\pm 1, \ldots, \pm d]$ labeling that is sym. on $\partial\Delta^d$ has an edge w/ opposite label.

i.e. there is no simplicial map

$\Delta^d \to \Diamond^{d-1} \simeq S^{d-1}$