Homology

10/7/20 10:41 AM

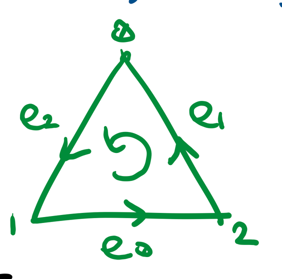
Chain complex $(C_\bullet, \partial_\bullet)$

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Boundary Map: $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\partial_n(\sigma[0,\dots,n]) = \sum_i (-1)^i \cdot \sigma[0 \dots \hat{i} \dots n]$$

example. $w \xrightarrow{e} v$ $\quad \partial_1 e = v - w$



$$\partial_2 f = [1.2] - [0.2] + [0.1] = e_0 + e_1 + e_2$$

Lemma. $\partial_{n-1} \circ \partial_n = 0$ [Fundamental Lemma of Homology]

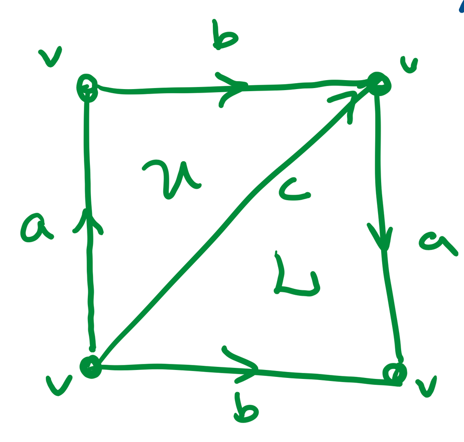
Cycles. chain σ s.t. $\partial_n \sigma = 0$; $\ker \partial_n$.

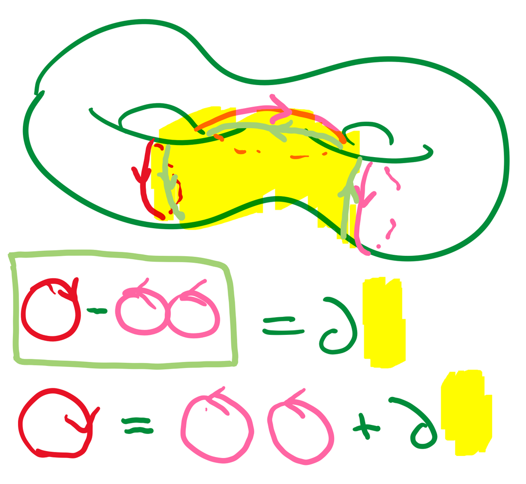
Boundaries. chain σ of the form $\partial_{n+1}\tau$; $\operatorname{im} \partial_{n+1}$.

Simplicial Homology Groups $H_\bullet^\Delta(X)$.

$$H_n^\Delta(X) := \ker \partial_n / \operatorname{im} \partial_{n+1} = n\text{-Cycles} / n\text{-Boundaries}$$

example.





$C_0 = \langle v \rangle, C_1 = \langle a.b.c \rangle, C_2 = \langle U.L \rangle$

$\partial_1(a.b.c) = v - v = 0$. $\quad H_0^\Delta(K) = \ker\partial_0 / \operatorname{im}\partial_1 = \langle v \rangle / 0 \simeq \mathbb{Z}$

$\partial_2 U = -a - b + c$

$\partial_2 L = -a + b - c$ $\quad H_1^\Delta(K) = \ker\partial_1 / \operatorname{im}\partial_2 = \langle a.b.\cancel{c}^{\,-a-b+c} \rangle / \langle -a-b+c, -a\cancel{+b-c}^{\,-2a} \rangle$

$= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

Compute Homology using linear algebra:

$$\partial_2 \begin{bmatrix} U \\ L \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \ker\partial_n \left[\overbrace{\begin{matrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{matrix}}^{\operatorname{im}\partial_n} \quad 0 \right] \quad H_n^\Delta = \mathbb{Z}/\alpha_1 \oplus \cdots \oplus \mathbb{Z}/\alpha_k$$

$$\left[\begin{array}{c|ccc|c} U & -1 & -1 & 1 & a \\ L-U & 0 & 2 & -2 & b \\ & & & & c \end{array}\right] \rightsquigarrow \left[\begin{array}{c|ccc|c} U & -1 & 0 & 0 & a \\ L-U & 0 & 2 & -2 & b-a \\ & & & & c+a \end{array}\right] \rightsquigarrow \left[\begin{array}{c|ccc|c} U & -1 & 0 & 0 & a \\ L-U & 0 & 2 & 0 & b-a \\ & & & & c+b \end{array}\right]$$

$$H_1^\Delta(K) = \langle a, b-a, c+b \rangle / \langle -a, 2(b-a) \rangle \simeq \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

Great! What's missing compare to $\pi_1(X)$?

Christmas Wishlist:

(0) Functoriality.

$f: X \to Y \Rightarrow f_*: H_n(X) \to H_n(Y) \quad \forall n \in \mathbb{N}_0$.

s.t. $(f \circ g)_* = f_* \circ g_*$, $1_* = 1$

(1) Invariance.

$f, g: X \to Y$ homotopic $\Rightarrow f_* = g_*: H_n(X) \to H_n(Y)$

With additional (technical) assumptions $\Rightarrow$ Homology Theor(ies)

To get them we need to work hard. (seriously.)

Puzzle time! SO^3 (all rotations in $\mathbb{R}^3$) think about unit ball gluing antipodal pts.

what is $H_\bullet(SO^3)$? $\quad H_\bullet(SO^3) = [\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}]$

$C_0 = \langle v \rangle \quad \partial_1 e = 0$

$C_1 = \langle e \rangle \quad \partial_2 f = 2e \quad H_1(SO^3) = \langle e \rangle / \langle 2e \rangle = \mathbb{Z}/2\mathbb{Z}$.

$C_2 = \langle f \rangle \quad \partial_3 b = \underbrace{f - f}_{=0}! \quad H_2(SO^3) = 0$

$C_3 = \langle b \rangle$

Robot arms: kinetic map $\kappa: \mathbb{T}^N \to SO^3$

can robot arms move continuously to achieve rotations?

Prop. There's no global solution to inverse kinetic map problem.

i.e. no map $s: SO^3 \to \mathbb{T}^N$ s.t. $\kappa \circ s = 1$.

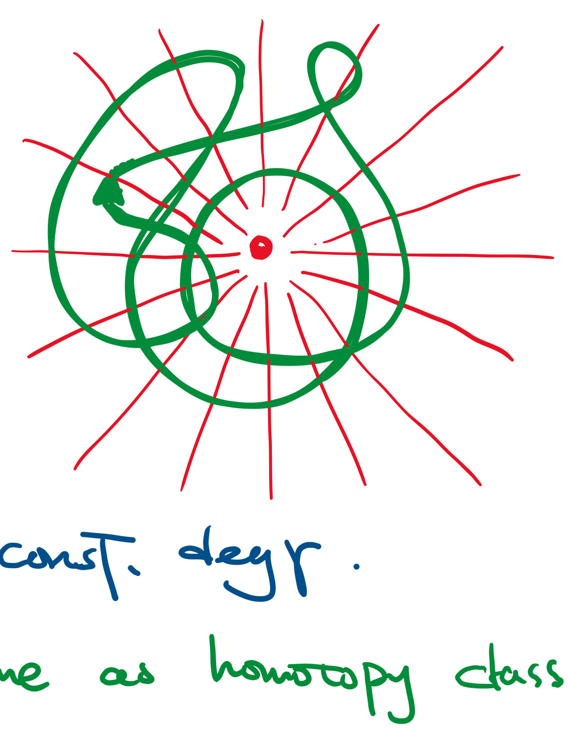
pf. $SO^3 \xrightarrow{s} \mathbb{T}^N \xrightarrow{\kappa} SO^3$

$\mathbb{Z}_2 \xrightarrow{s_*} \mathbb{Z}^N \xrightarrow{\kappa_*} \mathbb{Z}_2 \qquad \kappa_* \circ s_* = 1_*$ ■

Winding numbers redux.

$\gamma: S^1 \to \mathbb{R}^2 \setminus p$

$$H(\gamma): \underset{\mathbb{Z}}{\underset{\wr}{H_1(S^1)}} \to \underset{\mathbb{Z}}{\underset{\wr}{H_1(\mathbb{R}^2 \setminus p)}}$$



Thus $H(\gamma)$ is multiplication by const. $\deg \gamma$.

$\operatorname{wind}(\gamma) := \deg \gamma \qquad$ same as homotopy class $[\gamma]$.

Euler characteristic redux.

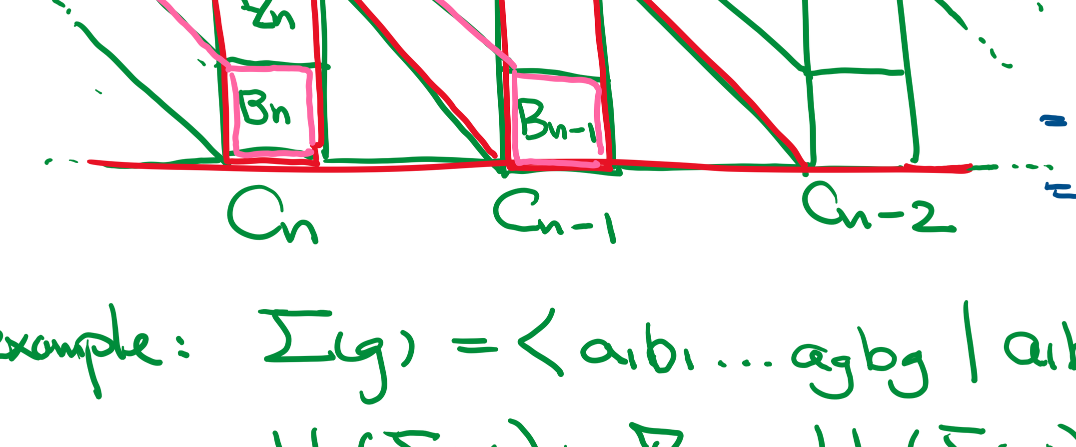
Betti numbers: $\dim H_n(X)$.

Euler characteristic: $V - E + F = \sum_n (-1)^n \dim C_n(X) = \chi(X)$

Euler–Poincaré Thm. $\chi(X) = \sum_n (-1)^n \cdot \dim H_n(X)$

pf. telescope sum: $\dim C_n = \dim \overset{Z_n}{\text{Cycles}_n} + \dim \overset{B_{n-1}}{\text{Boundaries}_{n-1}}$

$\dim \overset{Z_n}{\text{Cycles}_n} = \dim \overset{B_n}{\text{Boundaries}_n} + \dim H_n$.



$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

$\operatorname{im} i = \ker \partial_n$

$\sum_n (-1)^n \dim C_n$

$= \sum_n (-1)^n (\dim H_n + \dim B_n + \dim B_{n-1})$

$= \sum_n (-1)^n \dim H_n + \underbrace{\cancel{\dim B_{-1}}}_{0} + \underbrace{\cancel{\dim B_n}}_{0}$ ■

example: $\Sigma(g) = \langle a_1 b_1 \dots a_g b_g \mid a_1 b_1 \bar{a}_1 \bar{b}_1 \cdots a_g b_g \bar{a}_g \bar{b}_g = 1 \rangle$

$H_0(\Sigma(g)) = \mathbb{Z}$. $\quad H_1(\Sigma(g)) = \mathbb{Z}^{2g}$. $\quad H_2(\Sigma(g)) = \mathbb{Z}$.

$\chi(\Sigma(g)) = 1 - 2g + 1 = 2 - 2g$!