Complexes

10/5/20 10:50 AM

Administrivia :

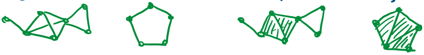
- Sorry no office hour today due to emergencies. Will make it up.
- HW1 due this Friday (no late submissions).

——— ② ———

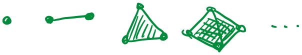
Complexes.

Idea : Topological spaces more general than surfaces/manifolds.

graphs : curves :: complexes : surfaces.



simplex :



facets : subsimplicies.

Complex : simplex glued together at sub-simplicies.

Example. Delaunay triangulation/complex.

Simplicial Complex.

- Every face of each simplex is also a simplex.
- Any two simplicies σ & τ has $\sigma \cap \tau$ a common facet (or empty).

Δ-Complex.

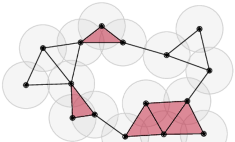
- multi-graph ver. of simp. complex.

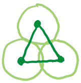
k-skeleton : collection of $\leq k$-simplices.

Čech Complex.

¡Geometry!

$\check{C}_\varepsilon(X)$: Form a d-simplex on $(x_0 \dots x_d)$ if $\bigcap_i B(x_i, \varepsilon/2) \neq \phi$.





Nerves.

Let S be finite collection of subsets. (say balls)

$$\mathrm{Nrv}(S) := \{ T \subseteq S \text{ non-empty} \mid \bigcap_{s \in T} s \neq \phi \}$$

Nerve of S naturally forms a complex.

Nerve Thm. $\mathrm{Nrv}(S)$ & $\bigcup_{s \in S} s$ are homotopically equivalent if subsets in S are closed & triangulable, and all common intersections are contractible.

[Borsuk 1948]
[Alexandroff 1928]

Helly Thm. Let Y be collection of convex subsets in $\mathbb{R}^d$.
If any $d+1$ (or fewer) subsets in Y intersects, then all subsets in Y intersects.

By Helly Thm!. balls in $\mathbb{R}^d$ satisfy assumptions for Nerve Thm.

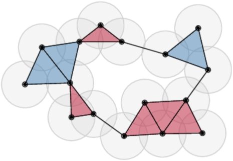
Cor. $\check{C}_\varepsilon(X) \simeq \bigcup_{x \in X} B(x, \varepsilon/2)$

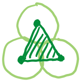
Cor. Delaunay complex is always a triangulation.

High complexity in general! $\Theta(n^{d+1})$

Vietoris-Rips Complex.

$VR_\varepsilon(X)$: Form a d-simplex on $(x_0 \ldots x_d)$ if the balls pairwise intersects.





Decided by 1-skeleton. $O(n^2)$ info.

But may not be the same homotopy type.

VR Lemma. $\check{C}_\varepsilon(X) \subseteq VR_\varepsilon(X) \subseteq \check{C}_{2\varepsilon}(X)$.

Pf. $\check{C}_\varepsilon(X) \subseteq VR_\varepsilon(X)$ easy.

$VR_\varepsilon(X) \subseteq \check{C}_{2\varepsilon}(X)$: consider simplex $(x_0 \dots x_d)$ in $VR_\varepsilon(X)$.

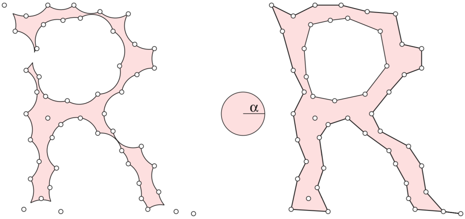
$d(x_0, x_i) \leq \varepsilon$ by VR_ε. thus $B(x_i, {}^{2\varepsilon}/_{2}) \ni x_0$ $\forall i$.

$\Rightarrow (x_0 \dots x_d)$ is also a simplex in $\check{C}_{2\varepsilon}(X)$. ▨

Alpha shape. [EKS 1983]

Given finite point set X, remove all empty balls of radius α.

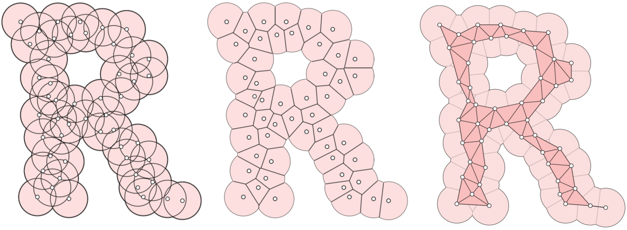
Draw straight edges between two pts if they are on the same circular arc.



Alternatively:

Čech + Delaunay (nerve of Voronoi regions)

Lower complexity than Čech!

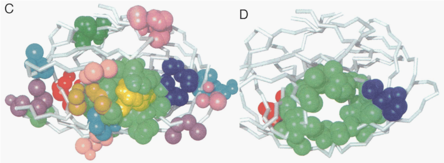


Prop. $A_\varepsilon(X) \simeq \bigcup_{x \in X} B(x, \varepsilon)$

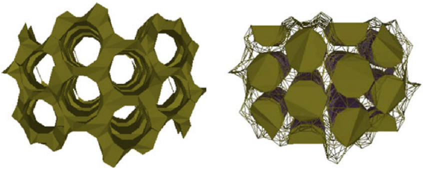
$A_\varepsilon(X) \subseteq Del(X)$

$|A_\varepsilon(X)| = O(n)$!

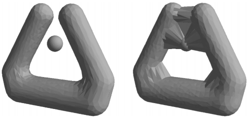
applications: protein structures [Richard 1971]

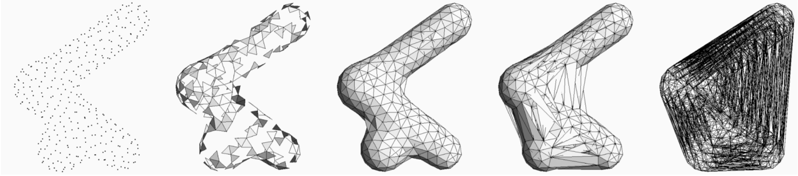


HIV [LEW1998]

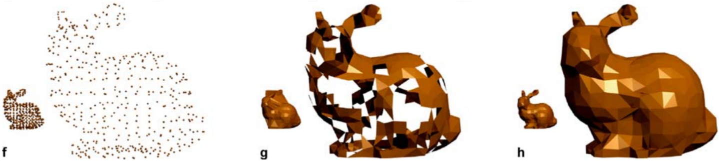


Zoelite





[TC1998]



[CGPZ2006]

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AlphaShape(X, ε):
    Compute Del(X)                                          $O(n^2)$
    for all simplex Δ in Del(X):
        if radius(circ Δ) < ε:                              // α-test.
            add Δ & all subcomplexes to $A_ε(X)$            can be carefully done
    Return boundary of $A_ε(X)$
```

* * *

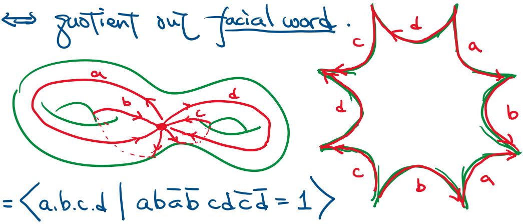
Attempt to find invariants for complexes.

- We could define fundamental group. ⇒ Uncomputable!

Focus on 2-complexes for now.

Intuition: Build 1-skeleton & free group.

Add faces ⇔ quotient out facial word.



$$\pi_1(\Sigma(2,0)) = \langle a,b,c,d \mid ab\bar{a}\bar{b}\,cd\bar{c}\bar{d} = 1 \rangle$$

$\mathbb{Z}$-module : $r\in\mathbb{Z}$ (not a field)

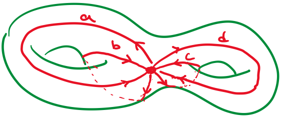
What if we build a vector space ...?

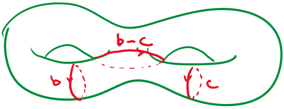
$x+y\in V$ $r\cdot x\in V$. $r(x+y)=rx+ry$.

Naive Idea:

$$\mathbb{Z}^4 \simeq \mathbb{Z}\langle a,b,c,d\rangle / \langle a+b+\bar a+\bar b+\underset{\underset{0}{\wr}}{c}+d+\bar c+\bar d\rangle$$

Now $a+b=b+a$ (commutative : no ordering)





also, $b=(b-c)+c$. Need to consider set of cycles.

Chain complex.

Given a surface (rotation system), associate

0-complex : vect. space over vertices

1-complex : 〃 (directed) edges.

2-complex : 〃 (oriented) faces.

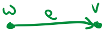
$C_n(X) := \left\{ \sum r_i x_i \;\middle|\; r_i\in\mathbb{Z},\ n\text{-facet } x_i \text{ of } X \right\}$

linear-sum of facets.

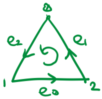
Boundary Map : $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\partial_n(\sigma[0,\dots,n]) = \sum_i (-1)^i\cdot\sigma[0\dots\hat{i}\dots n]$$

example.



$\partial_1 e = v-w$



$$\partial_2 f = [1,2]-[0,2]+[0,1] = e_0+e_1+e_2$$

Lemma. $\partial_{n-1}\circ\partial_n = 0$ [Fundamental Lemma of Homology]

Pf. $$\partial_{n-1}\cdot\partial_n(\sigma) = \sum_{j<i}(-1)^i(-1)^j\sigma[0\dots\hat{j}\dots\hat{i}\dots n] + \sum_{j>i}(-1)^i(-1)^j\sigma[0\dots\hat{i}\dots\hat{j}\dots n] = 0$$ ☒

Chain complex $(C_\bullet, \partial_\bullet)$

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

example. $$\underset{\wr}{C_2} \xrightarrow{\partial_2} \underset{\wr}{C_1} \xrightarrow{\partial_1} \underset{\wr}{C_0} \xrightarrow{\partial_0} 0$$

$\langle f\rangle \quad \langle a.b.c.d\rangle \quad \langle v\rangle$

$\partial_2(f) = a+b-a-b+c+d-c-d = 0.$ $\quad H_2^\Delta = \mathbb{Z}$

$\partial_1(a) = \cdots = \partial_1(d) = v-v = 0.$ $\quad H_1^\Delta = \mathbb{Z}^4$

$\partial_0(v) = 0.$ $\quad H_0^\Delta = \mathbb{Z}$

Cycles. chain σ s.t. $\partial_n\sigma = 0$; $\ker \partial_n$.

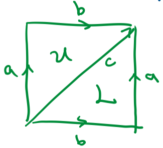
Boundaries. chain σ of the form $\partial_{n+1}\tau$; $\operatorname{im} \partial_{n+1}$.

Looking for cycles that are not boundaries.

Simplicial Homology Groups $H_\bullet^\Delta(X)$.

$$H_n^\Delta(X) := \ker\partial_n \big/ \operatorname{im}\partial_{n+1} = n\text{-Cycles}\big/ n\text{-Boundaries}$$

examples.



$C_2 = \langle U.L\rangle$

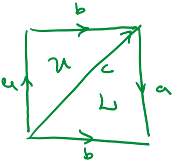
$C_1 = \langle a.b.c\rangle$

$C_0 = \langle v\rangle$

$\partial_1 = 0.$

$\partial_2 U = a+b-c = \partial_2 L$

$H_1^\Delta(T) \simeq \mathbb{Z}^2.$

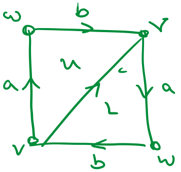


$\partial_2 U = a+b-c$

$\partial_2 L = a-b+c$

$H_1^\Delta(K) = \mathbb{Z}\oplus\mathbb{Z}_2$

$H_0^\Delta(K) = \mathbb{Z}.$



$\partial_2 U = a+b-c$

$\partial_2 L = a+b+c$

$H_1^\Delta(\mathbb{P}) = \dfrac{\langle a.a+b-c.c\rangle}{\langle a+b-c, 2c\rangle}$

$\simeq \mathbb{Z}/2\mathbb{Z}$

$H_0^\Delta(\mathbb{P}) = \langle v.w\rangle \big/ \langle w-v\rangle$

Great! What's missing? **Induced homomorphism.** f_*.

To get f_* we need to work hard.

Goal. $f: X\to Y \Rightarrow f_*: H_n(X)\to H_n(Y)\ \forall n\in\mathbb{N}.$

Euler characteristic redux.

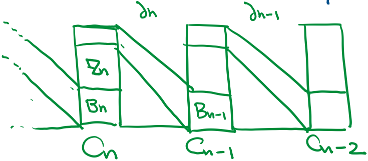
Betti numbers : $\dim H_n(X)$.

Euler characteristic : $V-E+F = \sum_n (-1)^n \cdot \dim C_n(X) = \chi(X)$

Euler-Poincaré Thm. $\chi(X) = \sum_n (-1)^n \cdot \dim H_n(X)$

pf. telescope sum : $\dim C_n = \dim \overset{Z_n}{\text{Cycles}_n} + \dim \overset{B_{n-1}}{\text{Boundaries}_{n-1}}$

$\dim \overset{Z_n}{\text{Cycles}_n} = \dim \overset{B_n}{\text{Boundaries}_n} + \dim H_n$.



$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

$\operatorname{Im} i = \ker \partial_n$

Puzzle time! SO^3 (all rotations in $\mathbb{R}^3$) think about unit ball gluing antipodal pts.

what is $H_1(SO^3)$?