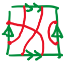
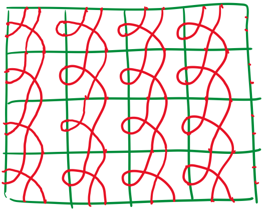
Combinatorial group theory and optimization

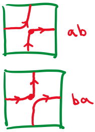
9/30/20 9:56 AM

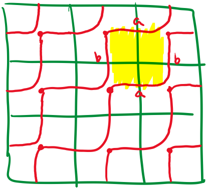
Motivating Question. What is $\pi_1(\Sigma(g,r))$?

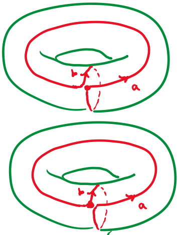
Example: $\pi_1(\mathbb{T}) = \pi_1(\Sigma(1,0))$:











$\pi_1(\mathbb{T}) = \langle a,b \mid ab=ba \rangle$

$\langle a,b \rangle = \left(\{\text{words over } \{a,b\}\}, \text{concatenation} \right)$

"free group"

Prop. $\pi_1(\mathbb{T}) \simeq \mathbb{Z}^2$

Pf. $ab = ba \Rightarrow \bar{b}\bar{a} = \bar{a}\bar{b}$, $\bar{b}a^k b = a^k$, $a b^k \bar{a} = b^k$.

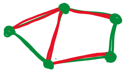
$\bar{b}a = a\bar{b}$ $\qquad$ $b\bar{a} = \bar{a}b$

Move all $b, \bar{b}$ after $a, \bar{a}$. $\Rightarrow a^k b^\ell$. $\boxtimes$.

Consider graph G w/ spanning tree T & edges $\bar{T}$ not in T.

$\pi_1(G)$ = free group on $\bar{T}$ $= \langle e : e \in \bar{T} \rangle$

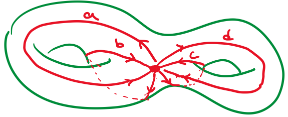
(rank of free group = # generators = $|\bar{T}|$)



Now consider a surface Σ and system of loops L

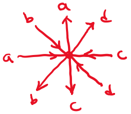
the single face has a facial walk, as a word over L.

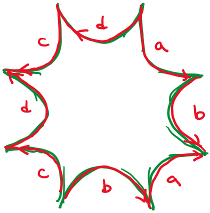
The facial word has to be identity.
(it's a face!)



$\pi_1(\Sigma(2,0))$ $|L| = 2g = 4$

$= \langle a,b,c,d \mid ab\bar{a}\bar{b}cd\bar{c}\bar{d} = 1 \rangle$





<u>Thm.</u> $\pi_1(\Sigma) = \langle \bar{T} \mid F \rangle$, where
$\bar{T} := C \cup L$ F: faces.
cotree leftovers

<u>Lemma.</u> $\langle C \mid F \rangle$ is indep. to the choice of G.

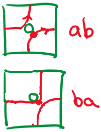
Pf. Check edge contractions $^{\text{in } T}$ & deletions $^{\text{in } C}$
(~~that~~ preserves cellular embeddability)
does not change $\langle C \mid F \rangle$ as a group. ☒

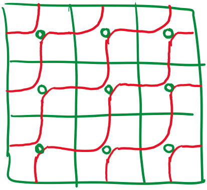
<u>Thm.</u> $\pi_1(S^2)$ trivial.

$\pi_1(\Sigma(g,0)) \simeq \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \bar{a}_1 \bar{b}_1 \ldots a_g b_g \bar{a}_g \bar{b}_g = 1 \rangle$

$\pi_1(\Sigma(0,r)) \simeq \langle a_1 \ldots a_r \mid a_1 a_1 \cdots a_r a_r = 1 \rangle$

What if surface Σ has boundary?





no longer $ab = ba$!

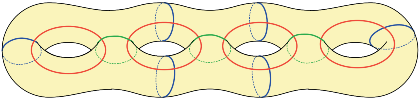
in general, for punctured Σ,
$\pi_1(\Sigma)$ is <u>free</u>.
(because universal cover has to be simply-connected.)

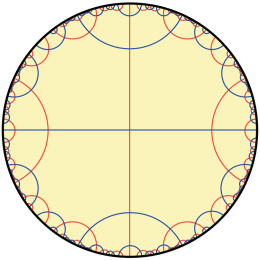
- I said in class that # generators doesn't change; that is <u>wrong</u>. Each puncture after the first one adds a new generator.

<u>Uniformization Thm</u> [Poincaré 1907] [Koebe 1907]

Any simply-connected surface is <u>conformally equivalent</u> to
(a) (open) unit-disk, $\chi < 0$
(b) complex plane, $\chi = 0$

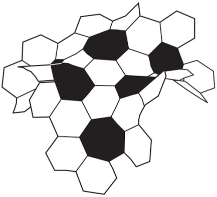
(c) sphere. $\chi > 0$

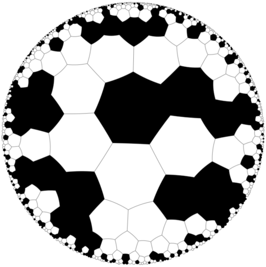




¡Hyperbolic Geometry!

Puzzle:





Observation from Gauss-Bonnet:

Assign each $\angle c = 1/3$. $\kappa(v) = 0$.

$\kappa(f) = 2\pi \cdot \deg f \cdot \left(\frac{1}{2} - \frac{1}{3}\right)$. $\chi = \frac{1}{6} \deg f$.

If $\deg f < 6$: $\chi > 0$. spherical

$= 6$: $\chi = 0$. planar

> 6 : $\chi < 0$. hyperbolic

Isoperimetric Inequality.

Consider a union of polygons/tiles $\mathcal{U}$ in hyperbolic tiling $\hat{\Sigma}$

Area := # tiles. Perimeter := #vertices on $\partial \mathcal{U}$

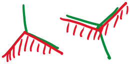
Then. Area $\leq C \cdot$ Perimeter. (compare w/ $A = \pi \cdot r^2$!)

Pf: Consider cutting Σ with deg-3 graph G.

$V = 4g-2$, $E = 6g-3$, $F = 1$ $\Rightarrow$ $(12g-6)$-gon

Assign each corner $\angle c = 1/3$. $\kappa(v) = 0$. $\kappa(f) = 2\pi(2-2g)$

each convex vertex on ∂U has $\kappa(v) = 2\pi(\underline{1/2} - 1/3) = 2\pi/6$



concave .. $\kappa(v) = 2\pi(\underline{1/2} - 2/3) = -2\pi/6$

$\Rightarrow$ #convex $-$ #concave $= 6 \cdot (2g-2)$ Area $+ 6$

Thus, some face is incident to more than $12g-12$ convex vert.

$\Rightarrow$ at most 4 interior edges.

Remove a face drops Area by 1 & Perimeter by $\geq 12g-14$

$$\text{Area} \leq \text{Perimeter} / (12g-14)$$

(linear if $g > 1$). $\square$

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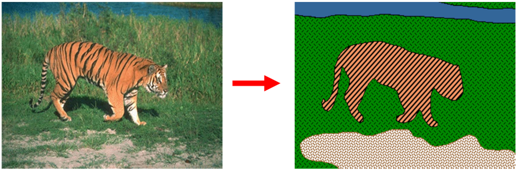
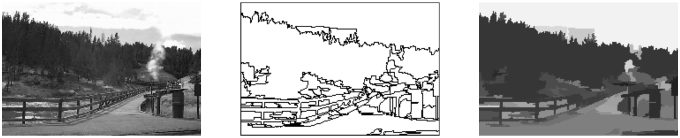
<u>Optimization</u>. Minimum (s,t)-cut in planar graph

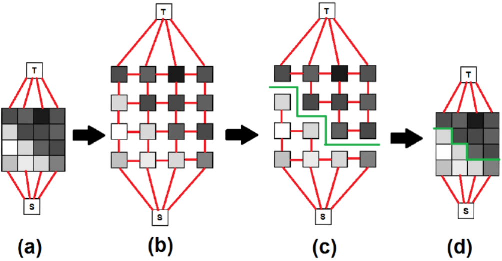
Application: Image segmentation.





[Wang-Siskind 2001]

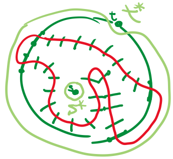
[Falcão-Miranda-Rocha 2006]



Minimum (s,t)-cut: Given undirected graph G & two vertices s & t, find a min-weight edge cut separating s & t.

$\{\text{edge cuts}\} \xleftrightarrow{\text{dual}} \{\text{union of cycles}\}$

minimum (s,t)-cut ⟷ minimum cycle separates s^* & t^*.



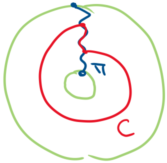
Minimum nontrivial cycle in annulus.

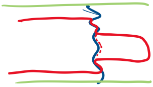
Find a min-cycle C homotopic to s^*/t^* !

Observations: (1) Shortest cycle C must pass through any path π from s^* to t^*.

(2) C intersects π at one segment if π is shortest.

$\Rightarrow \mathrm{wind}(C) = \pm 1$





MinCut(G,s,t):

```
Find shortest path $\pi$ from $s^* \rightsquigarrow t^*$
Cut open $G^*$ along $\pi$.
for each vertex $x$ on $\pi$ :
    find shortest path $x^+ \rightsquigarrow x^-$
Return length of min$\{x^+ \rightsquigarrow x^-\}$
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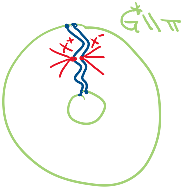

Analysis : Shortest paths by Dijkstra $O(|\pi| \cdot n\log n)$

[Henzinger et al. 1997] $O(|\pi| \cdot n)$

JCT to the rescue!

[Reif 1983]

MinCut(G,s,t):

```
Find shortest path $\pi$ from $s^* \rightsquigarrow t^*$
Cut open $G^*$ along $\pi$.
for ~~each~~ middle vertex $x$ on $\pi$ :
    find shortest path $\pi_x : x^+ \rightsquigarrow x^-$
    MinCut$(G^* \backslash\backslash \pi_x, s^*, \pi_x^*)$
    MinCut$(G^* \backslash\backslash \pi_x, \pi_x^*, t^*)$
Return length of min$\{x^+ \rightsquigarrow x^-\}$
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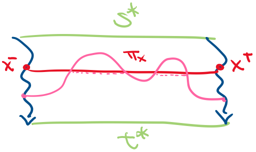


Divide & Conquer. $O(n\cancel{\log n} \cdot \log|\pi|) = O(n\log^{\cancel{2}} n)$

It was the end of story. until [Italiano et al. 2011]

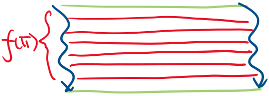
$O(n\log\log n)$.

Intuition is clear: Cut into many strips simultaneously.

Say, by magic:

$O(n\log\log n)$ preprocessing.

$O(n)$ $f(\pi)$ parallel shortest paths.



$$T(n,\pi) \le O(n) + \sum_{i=1}^{f(\pi)} T\left(n_i, \frac{\pi}{f(\pi)}\right) \qquad F(\pi) = \frac{\pi}{f(\pi)}$$

$$T(n,\pi) = O(n\cdot F^*(n)) \quad \text{where } F^*(n) = \#\text{times } F^{(\alpha)}(n) \le C.$$

- $F(n) = n-2$, $F^*(n) \le n/2$
- $F(n) = n/2$, $F^*(n) \le \log n$.
- $F(n) = \log n$, $F^*(n) := \log^*(n)$.
- $F(n) = \sqrt{n}$, $F^*(n) \le \log\log n$.

Tools to get there:

- (Cycle) separation decomposition / r-division. $O(n)$
- Monge heap / dense distance graph. $O(\frac{n}{r}\cdot r\log r)$
- FR-Dijkstra $O(\log n\cdot \frac{n}{\sqrt{r}} \log n \log r)$
- Reif. $O(\frac{n}{r}\cdot r\log r)$

Take $r = \text{polylog}\, n$. done. $f(\pi) = \frac{\pi}{\text{polylog}\,\pi}$!

Open Question. Can we improve this?

⑧

Where to go from here?

- Optimization algorithms, data structures.
- Homology structures, obstructions, some applied topology.