Homotopy and fundamental groups

9/28/20 11:23 AM

Recall homotopy between $f: \overset{S^1}{X} \to \overset{\Sigma(g,r)}{Y}$ & $g: X \to Y$

$H: X \times [0,1] \to Y$ cont.

$H(X,0) = f$. $H(X,1) = g$.

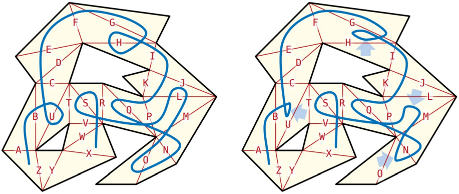
We say f & g are homotopic. $f \simeq g$.

How to test two curves on surface are homotopic?

Cut surface into polygonal schema.
each face is a disk. and no extra vertices! "frugal triangulation"

Keep track of how f & g crosses the edges.

"Reduce" the crossing sequence. (w/ signs!)



AB~~UU~~CDEFGGH~~HH~~IJLPQQPMN~~OO~~NM~~LL~~PQRSSSVW → ABCDEFGGHIJLPQQPMNNMPQRSSSVW

Lemma. Every crossing sequence can be reduced uniquely.

Pf. $\omega \underset{\to v \to}{\overset{\to u \to}{\ }} \hat{\omega}$ two different pairs cancelled.

(1) $\omega = w_1 x^{+} x^{-} w_2 y^{-} y^{+} w_3$, thus $\hat{\omega} = w_1 w_2 w_3$

(2) $\omega = w_1 x^{+} x^{-} x^{+} w_2$, then $\hat{\omega} = w_1 x^{+} w_2$

Claim:

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd}$$

Turns to

$$\begin{tikzcd} u \arrow[dr] & & v \arrow[dl] \\ & \hat{w} & \end{tikzcd}$$

Now say $\omega \to w_1^*. w_2^*$ both reduced.

If $w_1^* \leftrightarrow y_1 \leftrightarrow y_2 \leftrightarrow \cdots \leftrightarrow y_k \leftrightarrow w_2^*$.

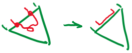
and $\exists\, y_{i-1} \leftarrow y_i \rightarrow y_{i+1} \Rightarrow y_{i-1} \rightarrow z_i \leftarrow y_{i+1}$ by Claim.

thus $w_1^* \to \cdots \to y_i \leftarrow \cdots \leftarrow w_2^*$, at most one $w_1^*. w_2^*$ is reduced.

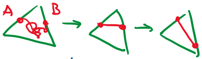
∎.

Thm. Any two curves are homotopic iff they share the same reduced word.

Pf. ⇐ carry out cancelling pairs using homotopy.



two curves in same face are homotopic.



⇒: every homotopy/vertex move can be carried out by elem. move.

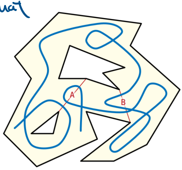
now the same curve has the same reduced crossing seq. ☑

Thm. Homotopy testing between two k-edge polygonal paths in polygonal domain. takes $O(n\log n + nk)$ time.

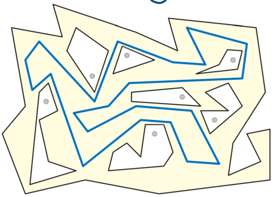
Pf. length of crossing sequence is $O(nk)$.

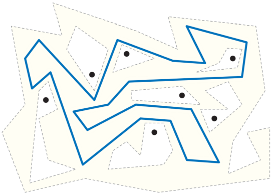
Observation. (1) minimal cutting into disks is enough.
we never use the fact that
each face is △.

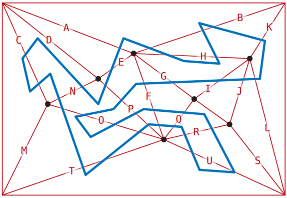
⇒ system of loops.



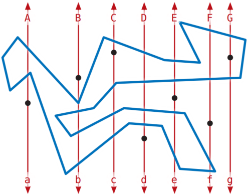
(2) since triangulation doesn't matter, replace w/ punctures.







A triangulation of $R \setminus S$
and a loop with crossing sequence
CNPEABHHBBKJIGFPOOPFQRUURQFPOTTNC.



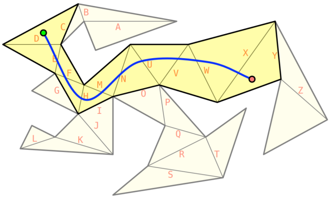
A partition of $\mathbb{R}^2 \setminus S$ into vertical slabs
and a loop with crossing sequence
AbcDeffeDcbbcDEFgGFEDCbA.

⇒ $O(\underbrace{n}_{\text{complexity of domain}} + \underbrace{X \log X}_{\text{sort punctures}} + \underbrace{Xk}_{\text{length of crossing seq.}})$-time algorithm.

[Leiserson-Maley] [Hershberger-Snoeyink]

Shortest homotopic path: Funnel algorithm

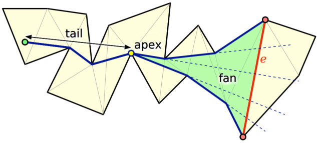
¡Geometry!



Sleeves.

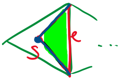
Compute reduced word.

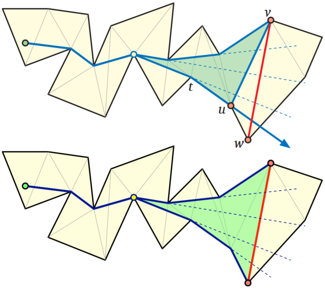
Gluing △s touched.



Funnels.

starting funnel:



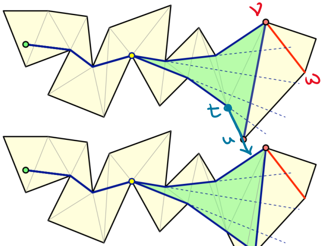


Extending the funnel.

test if $\vec{tu} \cap \overline{vw} \neq \emptyset$:

insert w as fan vertex.

O(1) time by orientation test.

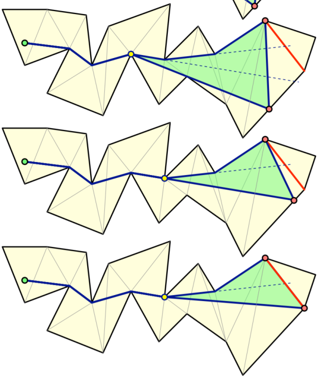


Contracting the funnel.

If $\vec{tu} \cap \overline{vw} = \emptyset$:

move u along $\overline{uw}$ to w, removing fan vertices as we go.

charge deletion to insertion, O(1) amortized time



Overall, running time propositional to length of reduced sequence.

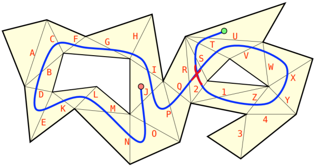
O(nk) time.

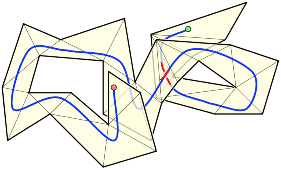
Observation. The exact same algorithm works for polygon w/ holes!

Why? The algorithm behaves locally.

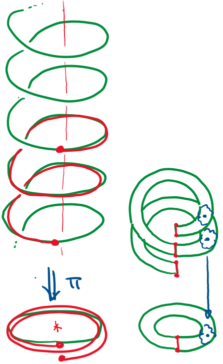
Does not see intersects between two copies of the same △.

⇒ Hint to Covering spaces!





Fundamental groups & Covering spaces



Remember winding number?

"Lift" a curve to its covering space.

projection $\pi : \mathbb{R} \to S^1$
$s \mapsto (\cos 2\pi s, \sin 2\pi s)$.

lift of $\overset{\text{canonical}}{\text{curve}}$ $w_n : [0,1] \to S^1$ of wind. num. n
$s \mapsto (\cos 2\pi n s, \sin 2\pi n s)$

$\tilde{w}_n : [0,1] \to \mathbb{R}$ s.t. $\pi \circ \tilde{w}_n = w_n$.
$s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$

For any pt x_0 on S^1, fix $\tilde{x}_0$ in $\pi^{-1}(x_0)$

Lemma. Two closed curves γ, γ' are homotopic iff $\tilde{\gamma}, \tilde{\gamma}'$ end at same pt.

Pf. Technical but obvious.

Fundamental group. $\pi_1(X, x_0)$

path-connected, S^1

Define $[\gamma]$ to be the class of curves homotopic to γ in X.

$\boxed{\pi_1(X, x_0)} := \{ [\gamma] :$ closed curve γ in S^1 s.t. $\gamma(0) = \gamma(1) = x_0 \}$.

Lemma. $\pi_1(X, x_0)$ is a group.

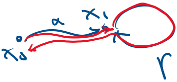
Pf. define $\gamma \circ \gamma'$ to be concatenation at x_0.

check $[\gamma] \circ [\gamma'] := [\gamma \circ \gamma']$ is well-def.

Identity: $[\,\bullet\,]$ trivial loop inverses: $[\bar{\gamma}]$, $\bar{\gamma}(t) := \gamma(1-t)$. ▨

Lemma. $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ as groups.

Pf.



$[\alpha^{-1} \gamma \alpha] \in \pi_1(X, x_0)$
if $[\gamma] \in \pi_1(X, x_1)$. ▨
"conjugation"

Thm. $\pi_1(S^1) \cong \mathbb{Z}$.

Pf. $\Phi : \mathbb{Z} \to \pi_1(S^1)$ is a group isomorphism.
$n \mapsto [w_n]$ ▨

Trivial? Amazing implications.

Induced homomorphism.

$\varphi: X \to Y$. $\varphi_*: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ push loops forward.

$[\gamma] \mapsto [\varphi\circ\gamma]$

$\varphi\circ\gamma : [0,1] \xrightarrow{\gamma} X \xrightarrow{\varphi} Y$

Lemma. φ_* is a group homomorphism.

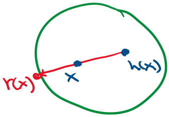
Application. Brouwer fixed-point Thm:

Every map $h: D^2 \to D^2$ has a fixed pt. ($h(x)=x$).

Pf. Suppose not. Define $r: D^2 \to S^1$:

r is a retraction from D^2 to S^1:

$r: D^2 \to D^2$ s.t. $r|_{S^1}$ identity.



Lemma. If X $\overset{\text{deformation-}}{\text{retracts}}$ to A, the induced homomorphism

$i_*: \pi_1(A) \to \pi_1(X)$

of inclusion $i: A \to X$ is $\overset{\text{isomorphism}}{\text{injective}}$.

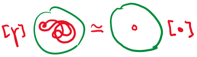
Pf: $r\circ i = \mathbb{1}$. thus $r_*\circ i_* = \mathbb{1}$. thus i_* injective. ▨

$r(\cdot, t)$ deformation retract

then $r_t\circ\gamma$ is a homotopy from γ to loop in A.

i_* surjective. ▨

Now, $i_*: \underset{\mathbb{Z}}{\pi_1(S^1)} \to \underset{\emptyset}{\pi_1(D^2)}$ is injective. ⨯ ▨

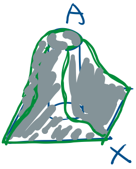


deformation retract: $H: X\times[0,1] \to X$ s.t.

$H_0(\cdot) = \mathbb{1}$

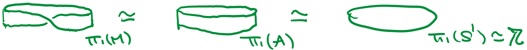
$H_t(\cdot)|_A = \mathbb{1}_A$.

$H_1(X) = A$



A type of homotopy equivalence. $X \simeq Y$ if:
$\exists\, f: X \to Y,\ \exists\, g: Y \to X$ s.t. $f \circ g \simeq g \circ f \simeq \mathbb{1}$

example. Möbius band is homotopic to normal band/annulus.



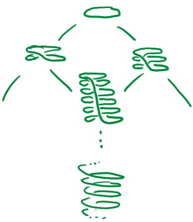
Lemma. Homotopy equiv. induces group isomorphism on π_1

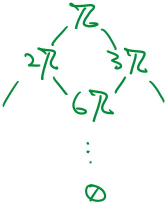
Moral. $\pi_1(X)$ is a homotopy invariant of X.

example. What is $\pi_1(\underset{\substack{\cong\\ \text{Torus}}}{\Sigma(1.0)})$? $\pi_1(\underset{\substack{\cong\\ \text{Proj. plane}}}{\Sigma(0.1)})$?

⓪

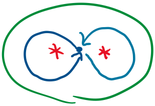
Covering spaces: Galois correspondence between
$\{\text{connected covering spaces of } X\} \longleftrightarrow \{\text{subgroups of } \pi_1(X)\}$

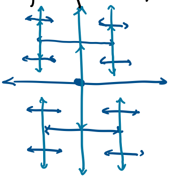




$\{\tilde{X} \xrightarrow{p} X\} \longrightarrow \{\text{subgroups of } \pi_1(X)\}$
$\tilde{X} \longmapsto p_*(\pi_1(\tilde{X})) \leq \pi_1(X).$

Thm. $\exists$ simply-connected universal cover of $\tilde{X}$
of any good enough space X.



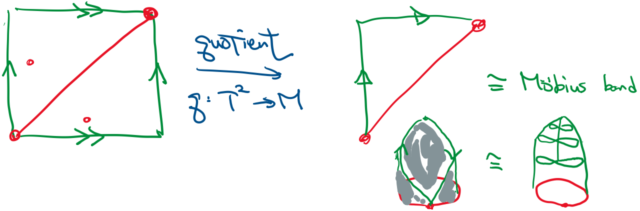


End with a puzzle: mixing colonies

Two yeast colonies each developed in cycles/phases

Say we mix them together, arriving at phase θ

$$f: S^1 \times S^1 \to S^1 \qquad (1)\ f(\theta,\theta)=\theta.$$
$$(\alpha, \beta) \mapsto \theta \qquad (2)\ f(\alpha,\beta)=f(\beta,\alpha).$$



So $T^2 \xrightarrow{q} M \xrightarrow{\hat{f}} S^1$. $\hat{f}$ is a retract.

$\hat{f}_*: \pi_1(M) \to \pi_1(S^1)$ injective;

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(M) \xrightarrow{\hat{f}_*} \pi_1(S^1)$$

α α^2 β^2 $\nrightarrow$!?

loop around once