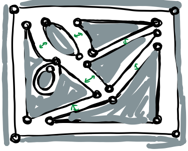
Euler characteristic

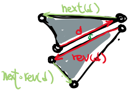
9/23/20 9:50 AM

Euler characteristic $\chi(\Sigma)$

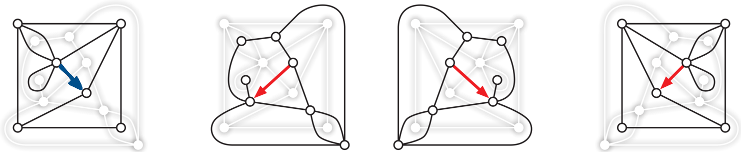
Setting: orientable surface w/o boundary Σ
actually: plane

Remember: polygonal schema IS the (dual) surface graph.





(if the faces are disks; cellular embedding)



primal G	dual G^*	primal G	dual G^*
vertex v	face v^*	empty loop	spur
dart d	dart d^*	loop	bridge
edge e	edge e^*	cycle	bond
face f	vertex f^*	even subgraph	edge cut
$tail(d)$	$left(d^*)$	spanning tree	complement of spanning tree
$head(d)$	$right(d^*)$	$G \setminus e$	G^* / e^*
$succ$	$rev \circ succ$	G / e	$G^* \setminus e^*$
clockwise	counterclockwise	minor $G \setminus X / Y$	minor $G^* \setminus Y^* / X^*$

Correspondences between features of primal and dual planar maps

Tree-cotree decomposition. The edges of plane graph G can be decomposed into two sets:

$$E = \underbrace{T}_{\text{spanning tree of } G} \sqcup \underbrace{C}_{\text{spanning tree of } G^*}$$

Pf.

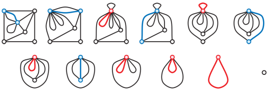
SpanningTree(G):

```
for any edge e in G:
    if e is a loop:
        L delete e
    if e is a bridge:
        L contract e
    o.w.
        L delete or contract e
return all contracted edges T
```

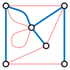
SpanningTree(G^*):

```
for any edge e* in G*:
    if e* is a ~~loop~~ bridge:
        L ~~delete~~ contract e*
    if e is a ~~bridge~~ loop:
        L ~~contract~~ delete e*
    o.w.
        L ~~delete~~ contract or ~~contract~~ delete e*
return all contracted edges C
```

did I change anything ...?



Computing a spanning tree of a graph.



☒

Euler's formula. $E = (V-1) + (F-1)$.

$$V - E + F = 2$$ Euler characteristic $\chi(S^2)$

A property of the surface graph / polygonal schema / surface.

(Careful, not a property of the abstract graph)

What happen if we run spanning tree algorithm on surface graph?

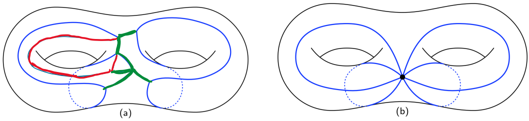
Key. Preserve cellular embedding

SpanningTree(G):

```
for any edge e in G:
    if e ~~is a loop~~ has diff. incident face:
        L delete e.
    if e ~~is a bridge~~ has diff. endpt.:
        L contract e
    o.w.
        L delete or contract e
return all contracted edges T & deleted edges C & leftovers L
```

Leftover edges L defines a system of loops

surface graph w/ 1 vertex & 1 face

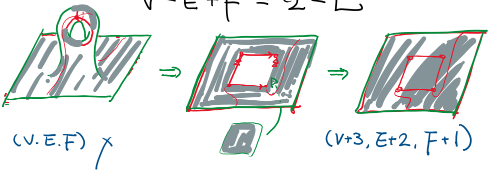


(a) (b)

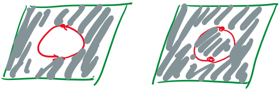
$$E = \underbrace{T}_{\text{tree}} \cup \underbrace{C}_{\text{cotree}} \cup \underbrace{L}_{\text{leftover}}$$

Euler's formula. $E = (V-1) + (F-1) + L$

$$V - E + F = 2 - L$$

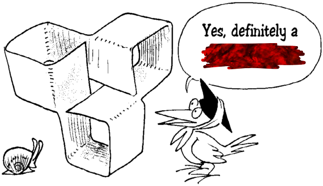


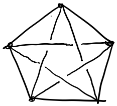
(V, E, F) χ $(V+3, E+2, F+1)$ $\chi+2$



$(V+1, E+1, F+1)$ $\chi+1$

$$V-E+F = 2-L = 2-2g-r-b$$

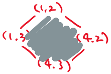




Configuration space of distinct vertex (ordered) pairs:

$(1,2) \;—\; \left\{ (1,3)\ (1,4)\ (1,5)\ (3,2)\ (4,2)\ (5,2) \right\}$

faces in $\mathrm{Conf}^2(K_5)$:



$\mathrm{Conf}^2(K_5)$ is a surface! what is it?

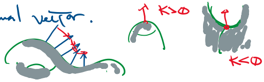
$$V-E+F = 20-60+30 = -10 = 2-2g$$

$\boxed{g=6}$ $\Sigma(6,0)$

__Geometry!__

Gauss map $\gamma: \Sigma \to S^2$ unit-normal vector.

Gauss curvature $K = \det(D\gamma)$

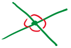


__Gauss-Bonnet Thm.__ Σ orientable surface w/o boundary, then

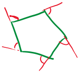
$$\int_\Sigma K\, dA = 2\pi\chi(\Sigma)$$

Combinatorial GB: Assign arbitrary real interior angle $\angle C$ to each corner of the faces.

$$K(v) := 2\pi - 2\pi \sum_{c \in v} \angle C$$



$$K(f) := 2\pi - 2\pi \sum_{c \in f} \left(\tfrac{1}{2} - \angle C\right)$$

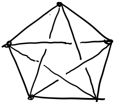


Then. $\sum_v K(v) + \sum_f K(f) = 2\pi\chi(\Sigma)$

pf. $\sum_v K(v) = 2\pi \left(V - \sum_c \angle C \right)$

$\sum_f K(f) = 2\pi \left(F - \tfrac{1}{2}\cdot 2E + \sum_c \angle C \right)$

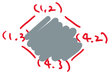
$2\pi(V-E+F)$. ▨



Configuration space of distinct vertex (ordered) pairs:

$(1,2) - \{(1,3)\ (1,4)\ (1,5)\ (3,2)\ (4,2)\ (5,2)\}$

faces in $\mathrm{Conf}^2(K_5)$:



$\mathrm{Conf}^2(K_5)$ is a surface!

- assign each face corner with $\angle C = 1/4$. $\Rightarrow K(f) = 0$.

$$K(v) = 2\pi\Big(1 - \sum_{C \in v} \angle C\Big) = 2\pi(1 - 6 \cdot 1/4) = -\pi.$$

Thus $2\pi\chi(\Sigma) = 20 \cdot (-\pi)$. $\chi = -10$.

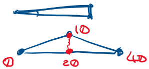
Moral: Curvatures can be pushed around, but not removed.

⓪

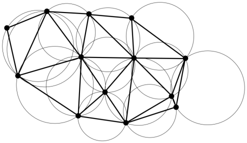
Mesh generation.

Goal. "Good" mesh.

- no sharp angles.
- no flat angles
- confine to constraints.



Delaunay Triangulation.



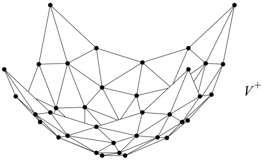
Assume:
no 4pts on a circle.

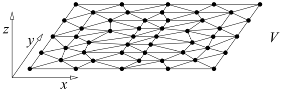
- Δ is **Delaunay** if the circumcircle is empty.
- A triangulation is **Delaunay** if all its Δs are.

parabolic lift. $\mathrm{lift}^+ : V \longrightarrow \{z = x^2 + y^2\}$

$$p = (x, y) \longmapsto p^+ = (x, y, x^2 + y^2)$$

Consider convex hull of $V^+ := \{p^+ : p \in V\}$

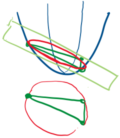




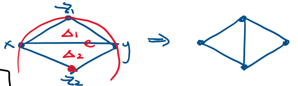
[Seidel 1982]

Thm. Δ is Delaunay iff Δ^+ is a face of $\operatorname{conv}(V^+)$.

pf. Δ Delaunay. then $\operatorname{circ}(\Delta) \cap V = \emptyset$.
Consider hyperplane H containing $\operatorname{circ}(\Delta)^+$.
no pt in V^+ is below H.
thus Δ^+ is a face of $\operatorname{conv}(V^+)$. ☑



Flip algorithm.



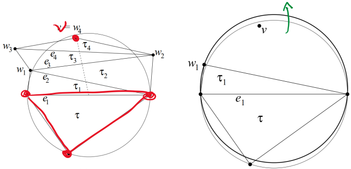
FLIP(Σ):

```
while ∃ bad edge e:
    flip e
    update set of bad edges
return final triangulation.
```

bad edge: $\operatorname{circ}(\Delta_1) \ni z_2$.
(i.e. $\operatorname{circ}(\Delta_2) \ni z_1$).

Lemma. All edges in triangulation Σ is good iff Σ Delaunay.

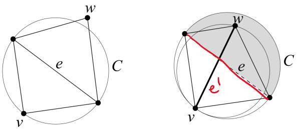
pf. $\Leftarrow$ easy. $\Rightarrow$:



(a) (b)

☑

Lemma. If e is not good, flipping e make it good.



(a) (b)

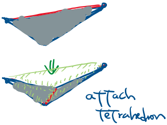
Thm. Flip algorithm terminates in $O(n^2)$ steps.

Pf. Look at the parabolic lift Σ^+

the volume above Σ^+ grows monotonically.

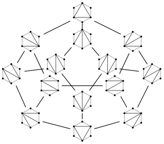
a disappearing edge won't appear again.

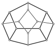
$\Rightarrow \binom{n}{2}$ steps.



Cor. Every pt set in $\mathbb{R}^2$ has a Delaunay triangulation.

Cor. The ~~flip graph~~ is connected. actually $\lceil \frac{n}{2} - 2 \rceil$-connected [WW2020]





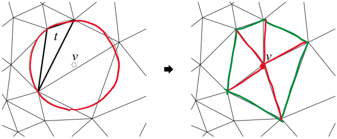
[Lawson 1977]

Cor. Delaunay triangulation maximizes min-angle
minimizes max-circumcircle.

* * *

Practical algorithms / Delaunay refinements.

Idea. Insert vertex near "bad" triangles.



[Lawson]

Bowyer-Watson(T, v):

Delete all Δs whose $\text{circ}(\Delta) \ni v$.
Subdivide cavity polygon w/ v.

{ find one Δ using pt-location.
find the rest using DFS.

Lemma. All Δs whose $\text{circ}(\Delta)$ contains v is connected.

Pf. faces$^+$ visible to v^+ must be connected.

Running time: If pts are inserted randomly, then $O(n)$ Δs are generated in expectation.

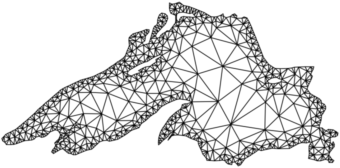
$O(n \log n)$ time using pt-location data structure

Main Question: Where to insert v?

look at the ratio between circumradius & shortest side.

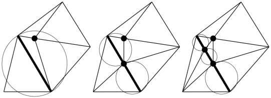
Ruppert's refinement: threshold $B \leq \sqrt{2} \Rightarrow 20.7° \sim 138.6°$
[Ruppert 1992]

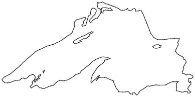
size-optimal: $O(1)$-approx to #Δs.



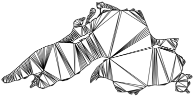
Ruppert's refinement:

1. Split edges w/ non-empty diametral circle in half.
2. Try to insert center of circ(Δ) for bad Δ;
   if center lies in some diametral circle, split the edge of the circle.

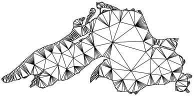




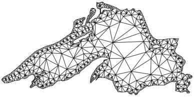
Lake Superior PSLG.



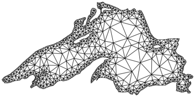
Triangulated with no minimum angle.



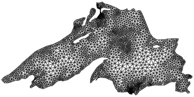
Triangulated with 5° minimum angle.



Triangulated with 15° minimum angle.



Triangulated with 25° minimum angle.



Triangulated with 34.2° minimum angle.