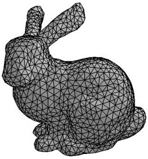
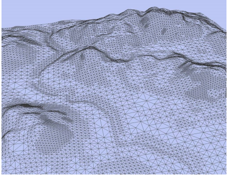
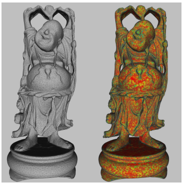
Surfaces

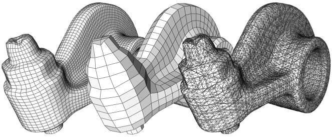
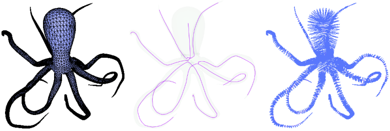
9/21/20 10:55 AM

Admin stuff:

- HW⓪ deadline has passed. Still, submit to introduce yourself!
- HW1 out some time this week.
- Scribbles will be uploaded.

Surface (2-manifolds w/ boundaries)

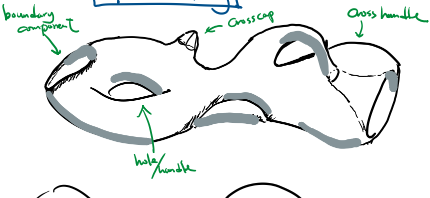




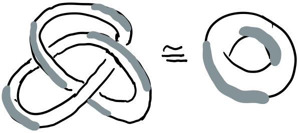



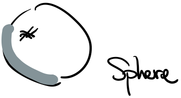
No geometry. What's left?

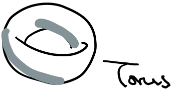
Formally: A $\boxed{\text{Surface}}$ is a Hausdorff 2nd-countable Topological space that is locally homeomorphic to the plane.

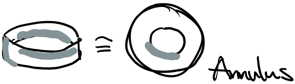
$\boxed{\text{w/ boundary}}$: " or the half-plane.

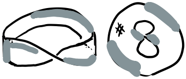




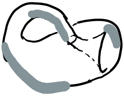




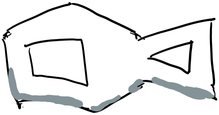




Möbius band



Klein bottle.



Polygonal domain.

Representation: Triangulated surface.

[Kerékjártó – Radó 1925]

Triangulation Thm. Any surface can be

pf. JCT + Schönflies Thm, yadda yadda

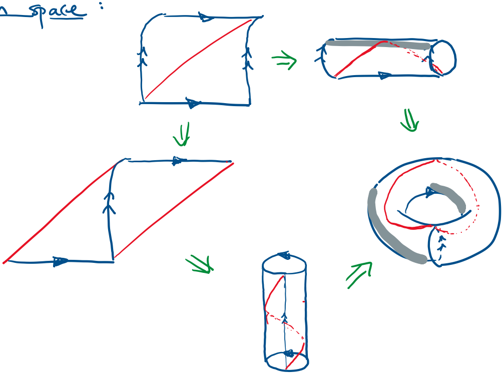
[Moise '77]

Refinement Thm. Any two Triangulations have a common refinement.

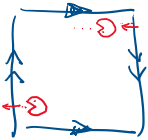
Both are false in ≥4-dimensions. [Freedman '82, RCS+97]

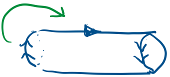
Polygonal Schemata.

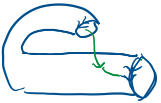
Pacman space:



Klein bottle:

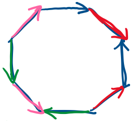






Exercise: what is this surface?

Moral: It doesn't really matter how the surface is cut.



[double Torus]

Question. Can we obtain all possible surfaces using only cut & paste?

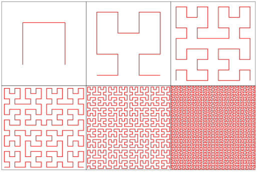
Homeomorphism: $f: X \to Y$ cont. bijection, f^{-1} also cont.

Be careful.

$f: X \to Y$ cont. bijection

$g: Y \to X$ cont. bijection

not enough!



$[0,1] \times [0,1]^2$ not homeomorphic.

Space-filling curve.

Classification of surfaces. Every connected surface is homeomorphic to one of the following schemata + boundary:

1. a sphere w/ handles $\Sigma(g,0)$ orientable
2. a sphere w/ crosscaps $\Sigma(0,r)$ non-orientable: containing a Möbius strip.

[Conway's ZIP proof]

Pf. Claim 1. Every surface is homeomorphic to one w/ finitely many $\underset{T}{\text{handles}}$, $\underset{K}{\text{crosshandles}}$, $\underset{P}{\text{crosscaps}}$, & boundary components

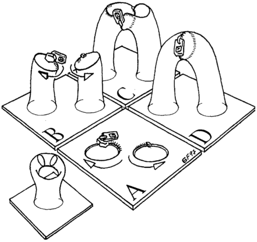


Figure 1. Handle

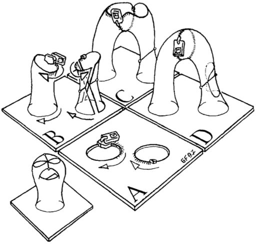


Figure 2. Crosshandle

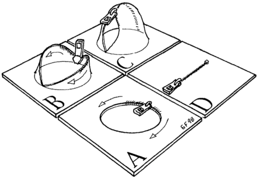


Figure 3. Cap

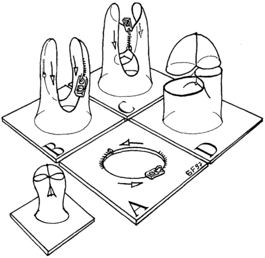
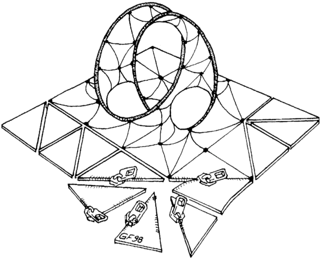
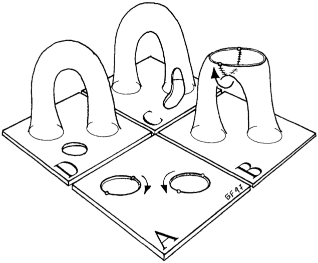
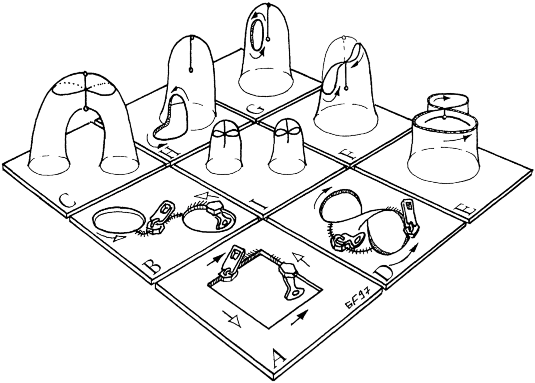


Figure 4. Crosscap



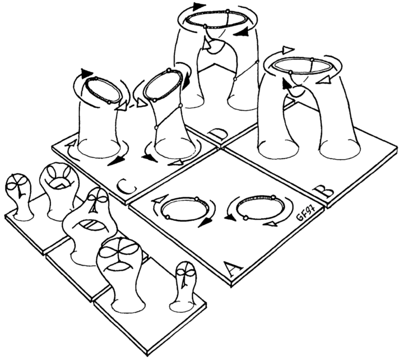


Claim 2. One crosshandle = two crosscaps $(K = P \# P)$



[Dyck 1888]

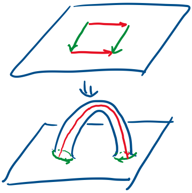
Claim 3. Handle + crosscap = crosshandle + crosscap. $(T\#P = K\#P)$



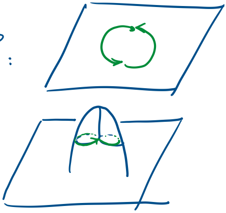
$\exists$ crosshandle or crosscap : turn all handles into crosscaps. $\Sigma(0,r)$

$\nexists$ " : all handles. $\Sigma(g,0)$ ☒

Toroidal hole :



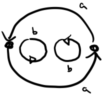
Crosscap hole :

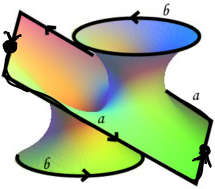


Every surface belongs to one of the following schemata :

1. ε (sphere)
2. $a_1 b_1 \bar{a}_1 \bar{b}_1 \cdots a_g b_g \bar{a}_g \bar{b}_g$ (g-handle Torus)
3. $a_1 a_1 \cdots a_r a_r$ (r-crosscaps)

Exercise : what is this surface?





[Dyck's surface]

I made a mistake during the lecture;
It should be P # K = P # P # P

How to tell they are different?

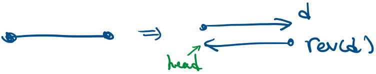
* * *

Graphs on Surfaces.

We can Treat the cutting lines as a graph.

Abstract graphs. $G = (V, D, \text{rev}, \text{head})$:

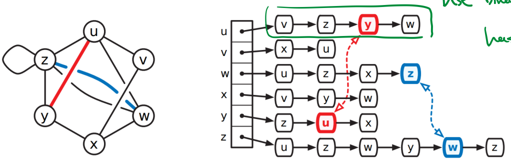
- V : vertices.
- D : darts
- rev : reversal map. $\text{rev}(\text{rev}(d)) = d \neq \text{rev}(d)$
- head : head vertex map from D to V



Convention :

- Tail : $\text{head}(\text{rev}(d))$
- edge : $\{d, \text{rev}(d)\}$

Data structure : incidence list



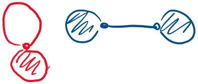
Use binary search tree or hash table

An incidence list representation of a graph, with the dart records for two edges emphasized. For clarity, most reversal pointers are omitted.

Query time: O(1) insert/delete, O(log n) adjacency
O(1)

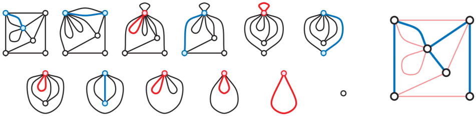
Spanning tree.

loops and bridges.



SpanningTree(G):

```
for any edge e in G:
    if e is a loop:
        delete e.
    if e is a bridge:
        contract e
    o.w.
        delete or contract e
return all contracted edges
```



Computing a spanning tree of a graph.

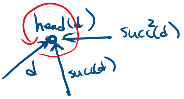
Whatever-first search works.

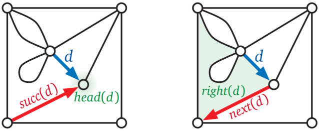
```
WHATEVERFIRSTSEARCH(s):
    put (∅, s) in bag
    while the bag is not empty
        take (p, v) from the bag                 (⋆)
        if v is unmarked
            mark v
            parent(v) ← p
            for each edge vw                     (†)
                put (v, w) into the bag          (⋆⋆)
```

Surface graphs: abstract graph w/ rotation system

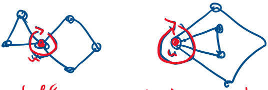
We assume orientability here. Otherwise one needs an extra fcn.

- succ: successor of dart d around head(d)
- next: next dart around right face incident to d: rev(succ(d)) (in clockwise order)
- faces: orbits of next(·) = rev ∘ succ(·)





The successor and dual successor of a dart in an embedded planar graph.



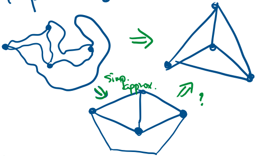
Careful! Same abstract graph can have different rotation system.

Thm. Every topological graph on surface has a rotation system.

Pf. J-Sch Thm.

Thm. Every plane graph has a straight-line embedding.

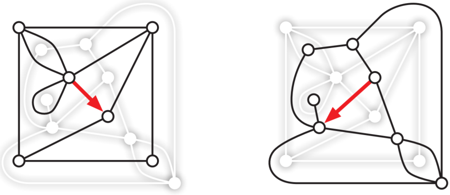
Pf. Non-trivial, even assuming polygonal paths!



Dual surface graph. Again, assume orientability.

Formal dual of a rotation system:

$$(\mathrm{succ}, \mathrm{rev}) \longleftrightarrow (\overset{\mathrm{rev}\circ\mathrm{succ}}{\overset{\|}{\mathrm{next}}}, \mathrm{rev})$$



- dual graph G^*: $G^* = (V^*. D^*)$

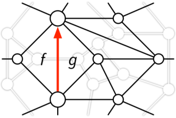
V^* : orbits of next = rev ∘ succ

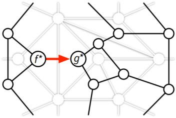
D^* : Same as D but w/ (next. rev)

F^* : orbits of succ.

Primal : succ counterclockwise

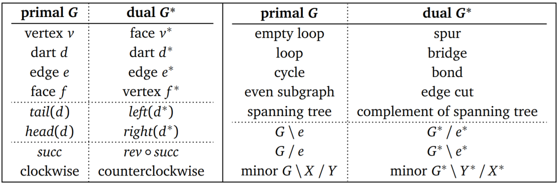
Dual : next clockwise





Duality is an involution (because rev is).

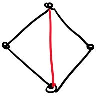
Plane graph.

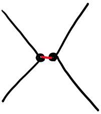


primal G	dual G^*	primal G	dual G^*
vertex v	face v^*	empty loop	spur
dart d	dart d^*	loop	bridge
edge e	edge e^*	cycle	bond
face f	vertex f^*	even subgraph	edge cut
$tail(d)$	$left(d^*)$	spanning tree	complement of spanning tree
$head(d)$	$right(d^*)$	$G \setminus e$	G^* / e^*
$succ$	$rev \circ succ$	G / e	$G^* \setminus e^*$
clockwise	counterclockwise	minor $G \setminus X / Y$	minor $G^* \setminus Y^* / X^*$

Correspondences between features of primal and dual planar maps

Exercise : deletion $\xleftrightarrow[\text{dual}]{}$ contraction

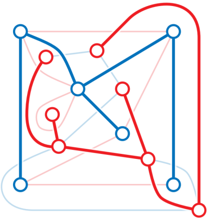




Tree-cotree decomposition.

Lemma. Let G be a plane graph.
Then T is a spanning tree of G
iff $G^* \setminus T^*$ is a spanning tree of G^*

Pf. cycle ⟷ bond



[Euler 1750]

Euler's formula for plane graph. Any connected plane graph G satisfies

$$E=(V-1)+(F-1) \Rightarrow \boxed{V-E+F=2}$$

What if $V-E+F \neq 2$? rotation system still gives an embedding. but not in the plane.

polygonal schema $\overset{}{\underset{\text{dual}}{\longleftrightarrow}}$ rotation system

polygons/faces — orbits of next(·)

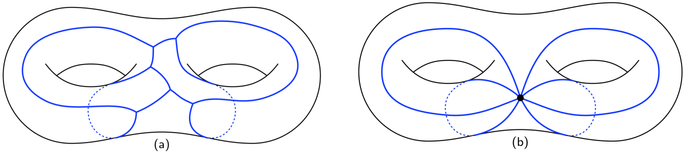
gluing — identify d w/ rev(d).

SpanningTree(G):

```
for any edge e in G:
    if e ~~is a loop~~ has same endpt.:
        delete e.
    if e ~~is a bridge~~ has same incident face:
        contract e
    o.w.
        delete or contract e
return all contracted edges & deleted edges
```

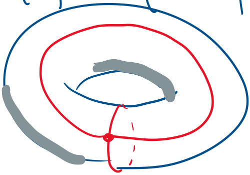
$$E = \underbrace{T}_{\text{tree}} \cup \underbrace{C}_{\text{cotree}} \cup \underbrace{L}_{\text{leftover}}$$

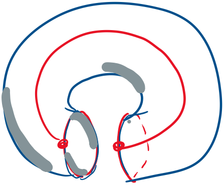
Leftover edges L defines a system of loops
surface graph w/ 1 vertex & 1 face



(a) (b)

What if we cut the loops further?





Euler's formula

$(V+1,\ E+1,\ F+2)$

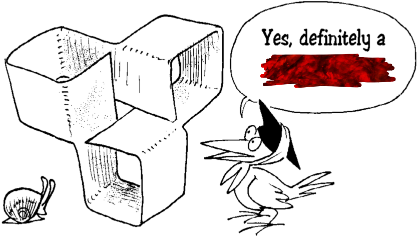
Euler char. $\chi(M)$

$$V - E + F = 2 - L = \underline{2 - 2g - r - b}$$

$$[g, r] = [g-1, r+2]$$

if $r > 0$

Therefore, (Euler characteristic, orientability, # components) is a complete invariant of surfaces w/ boundary under homeomorphism.

Exercise: What is this surface?