Winding number and its friends

9/16/20 10:19 AM

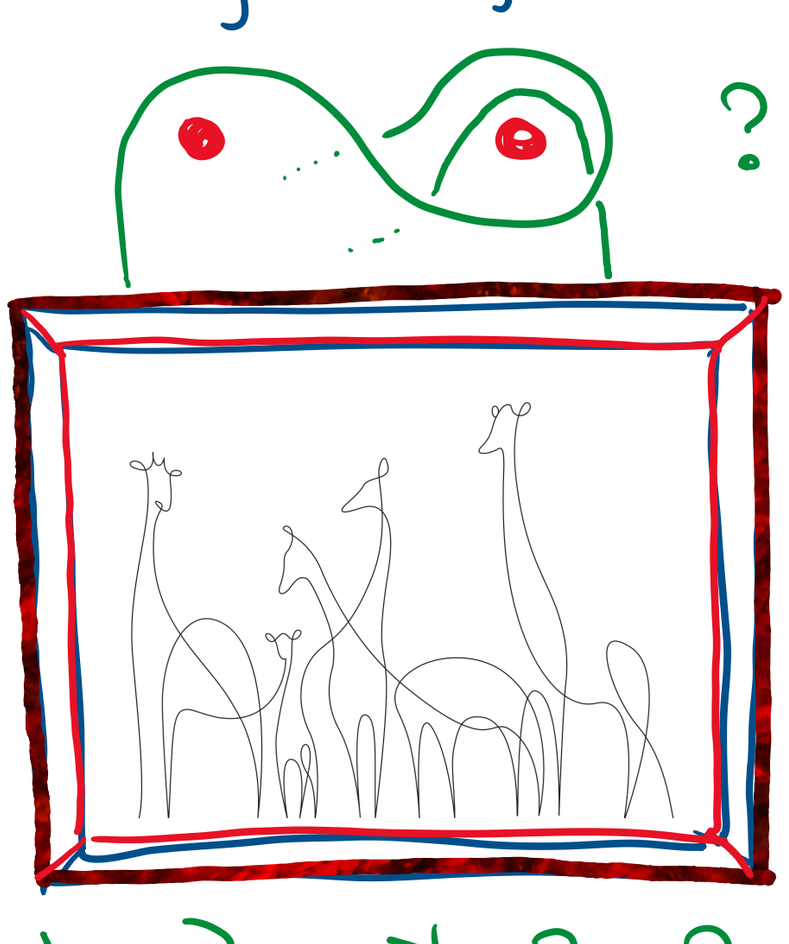
But first, administrivia.

- HW0 is due this Friday (9/18). will give you my assessment afterwards.
- Office hours: Fri 4p-5p, Mon 9a-10a. Come & talk to everyone!
- Slack: Q&A, discussions

Now onto our main topic today! Continue our tour in 18/19th-century topology.

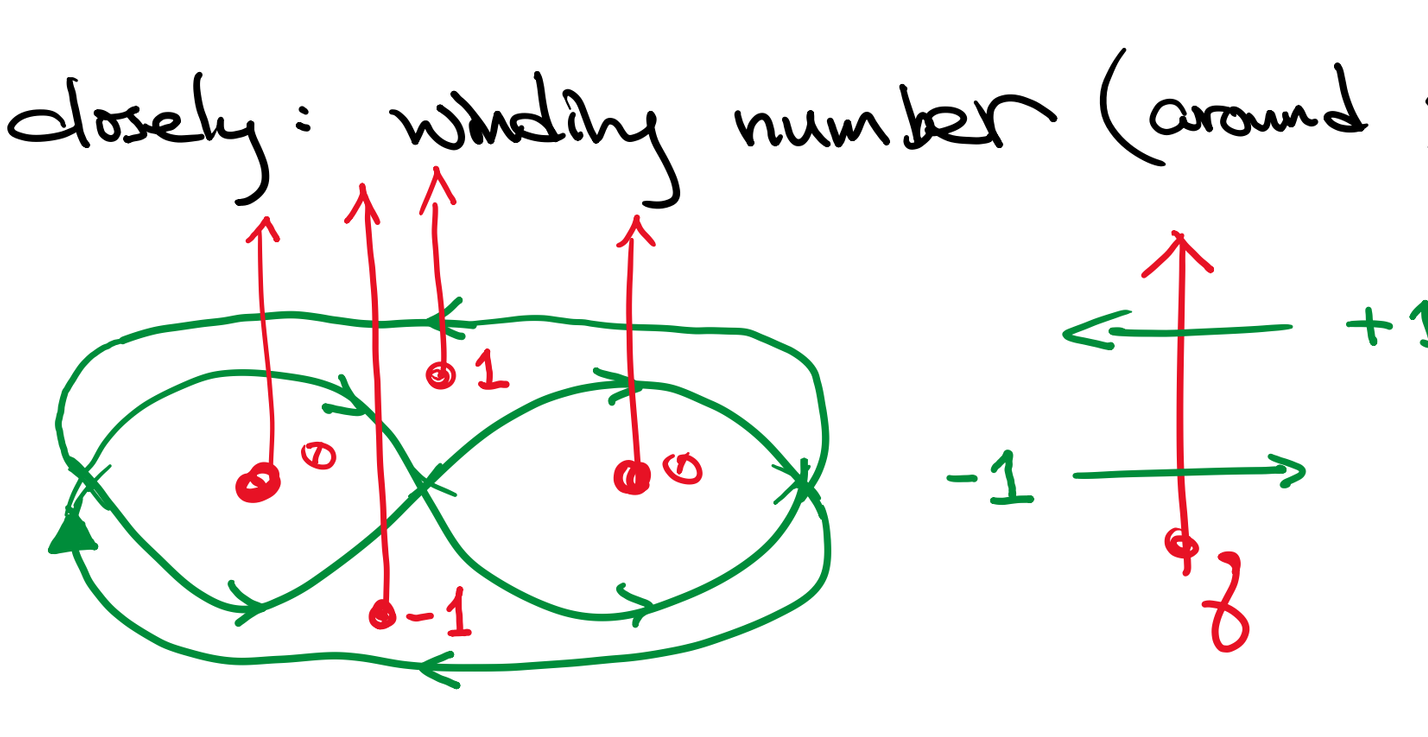
Picture hanging puzzle:

"Can you hang a picture w/ 2 nails, such that the picture falls if either nail is pulled?"



Bonus: How about 3 nails? 2-out-of-3? Some arbitrary Boolean fcn?

Look at it closely: winding number (around q)

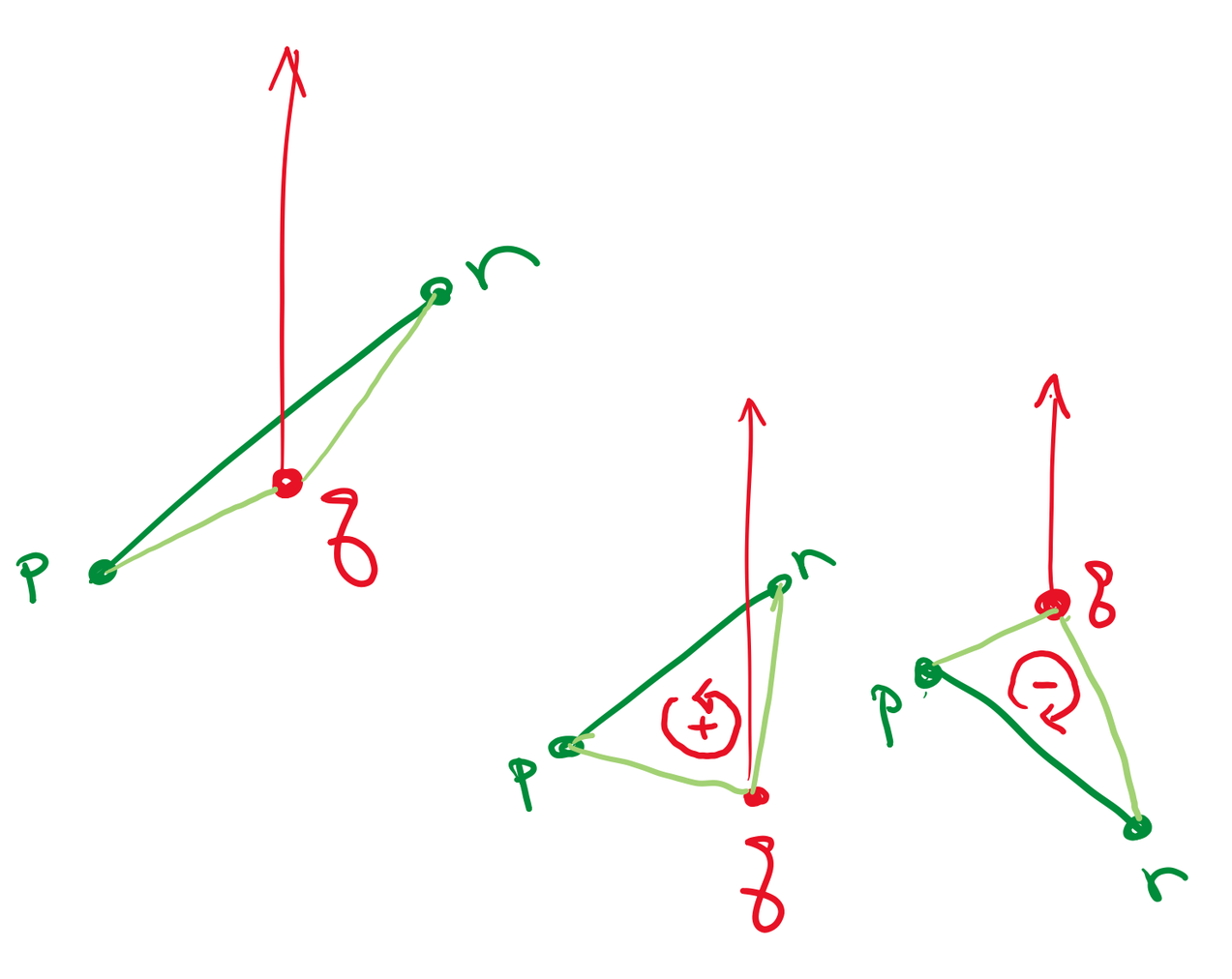


Resembles inside/outside labeling

Compute winding numbers

```
WindingNumber (P, q):
  foreach segment pr:
    Δ ← Δ(p, q, r)
    if p.x < q.x < r.x:
      wind--
    if p.x > q.x > r.x:
      wind++
  return wind
```

(crossed out: InsidePolygon?; sign ← −Δ·sign; sign ← Δ·sign; return sign)

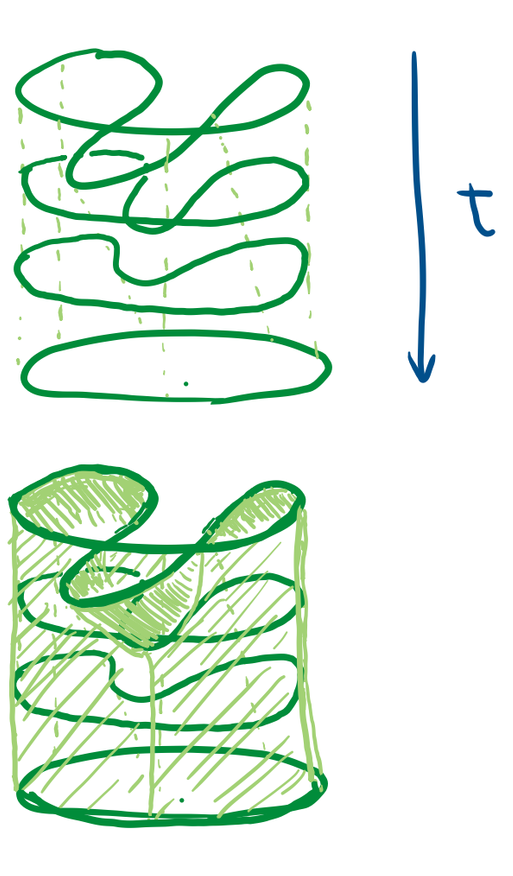


Winding number is invariant under change:

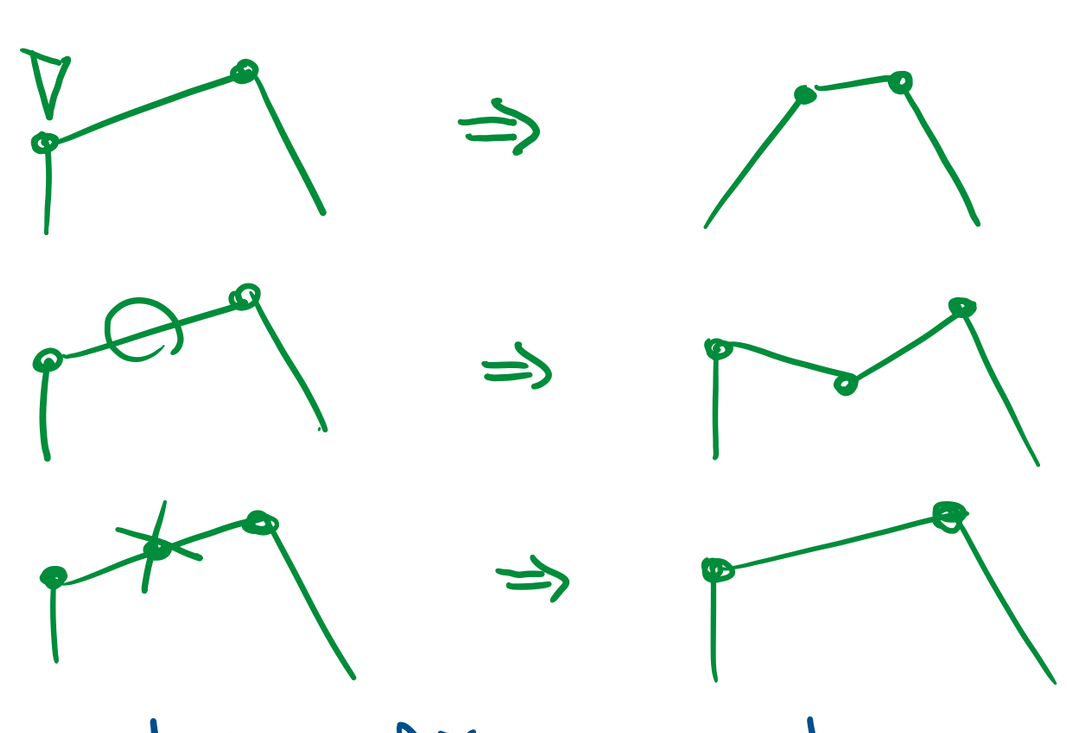
= homotopy: continuous change from γ to γ'

$H: S^1 \times [0,1]$ cont.

$H(\cdot, 0) = \gamma$, $H(\cdot, 1) = \gamma'$

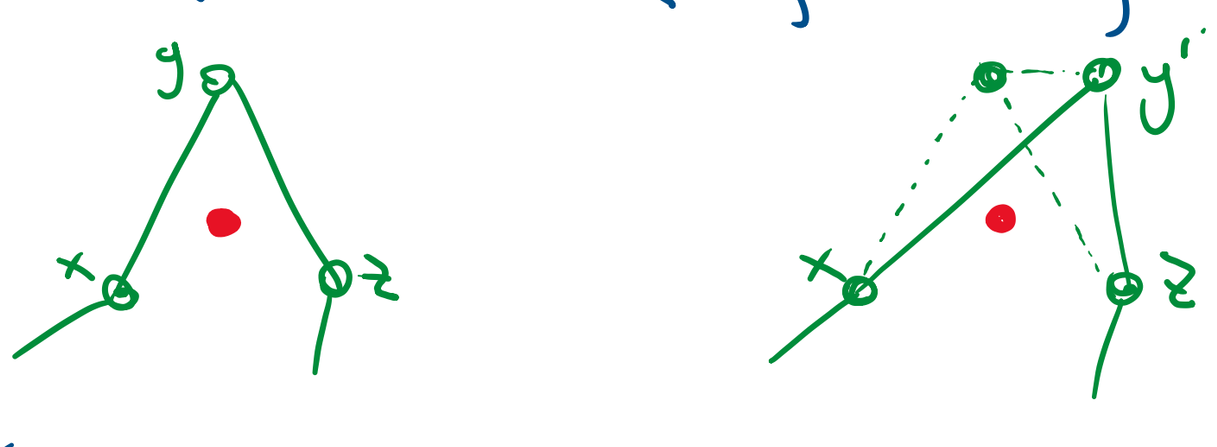


= vertex move:



Always turn a polygon into a polygon.

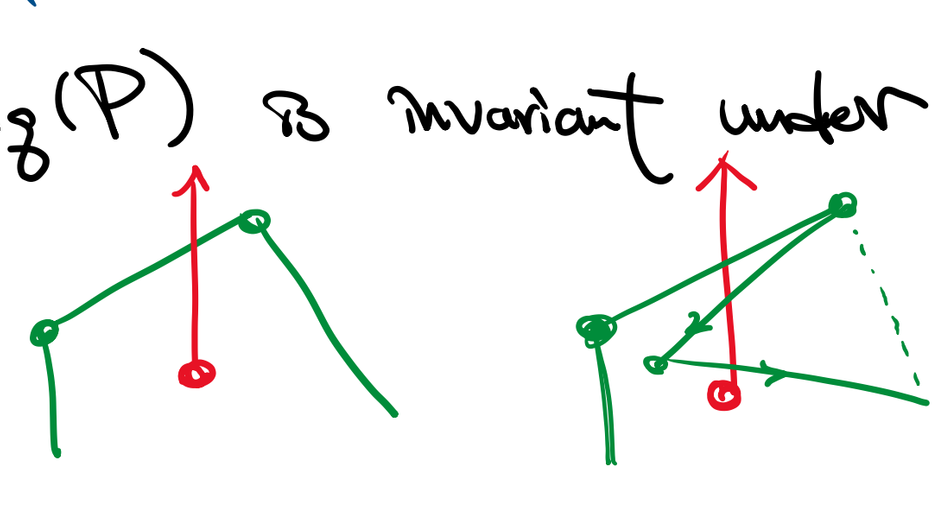
= safe vertex move: (defined only for P in $\mathbb{R}^2 \setminus q$)



= $\left(\{\text{Polygons in } \mathbb{R}^2\setminus q\}, \{\text{safe vertex moves}\}\right) \simeq \left(\{\text{Curves in } \mathbb{R}^2\setminus q\}, \{\text{homotopy}\}\right)$

Thm. $\text{wind}_q(P)$ is invariant under safe vertex moves.

pf.



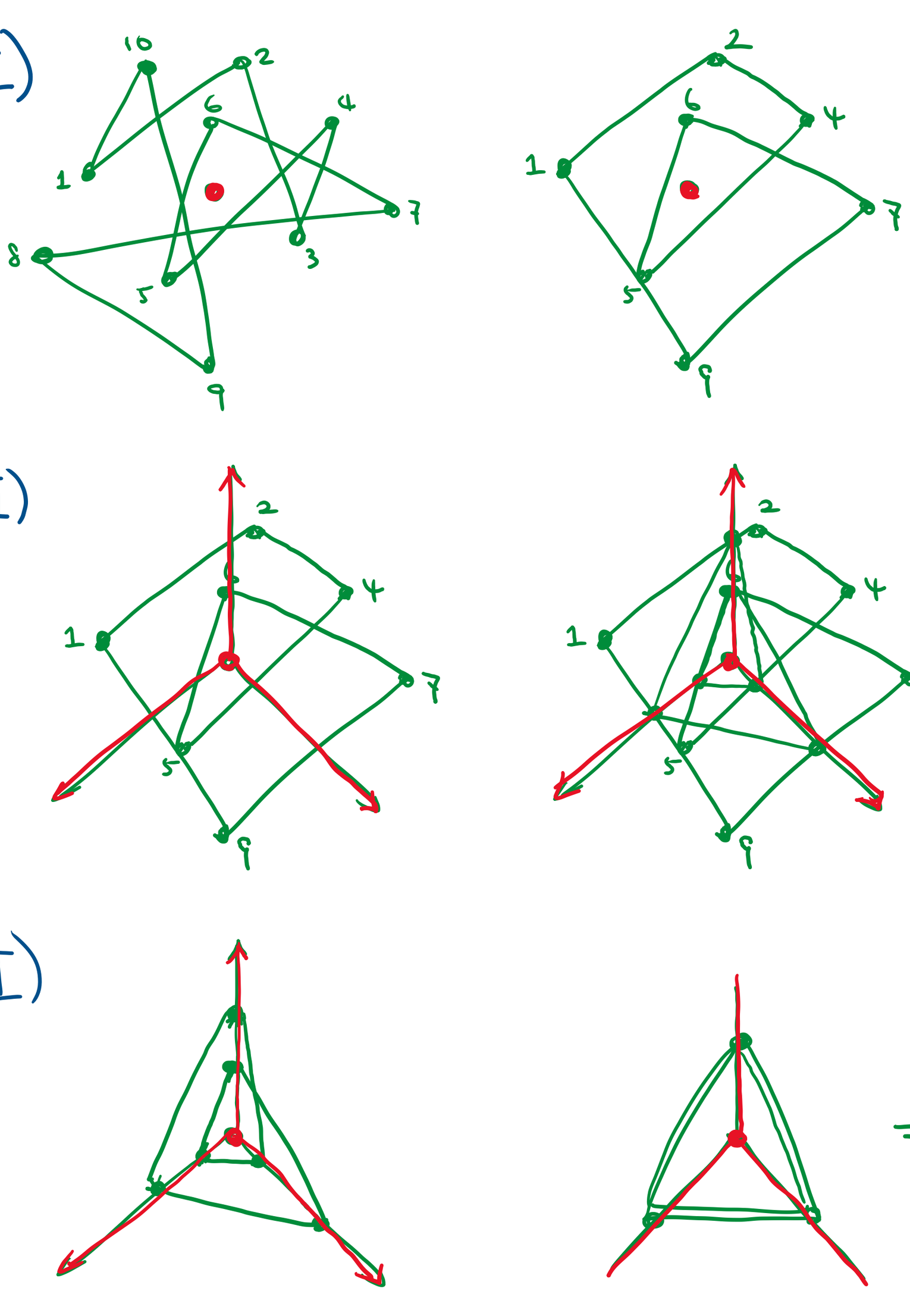
any safe Δ intersects the ray once in each direction. (or not at all) ▨

[Hopf 1935]

Thm. Two polygons P & Q are homotopic in $\mathbb{R}^2\setminus q$ if and only if they have the same $\text{wind}_q(\cdot)$

Idea. Turn both P & Q into $\Delta^{\text{wind}(P)} = \Delta^{\text{wind}(Q)}$

pf. (I) (II) (III)



```
3-coin
(p, q, r) ← (1, 2, 3)
while r ≠ 2:
  if Δ(p, q, r) is safe:
    (p, q, r) ← (p, r, succ(r))
  else (if Δ(p, q, r) contains σ):
    (p, q, r) ← (q, r, succ(r))
```

$= \Delta^{\text{wind}(\gamma)}$ ▨

Wind_q is a complete homotopy invariant!

A property of the space $\mathbb{R}^2\setminus q$.

Consider all closed curves on $\mathbb{R}^2\setminus q$, identifying homotopic ones. then there's only Δ^k for $k \in \mathbb{Z}$.

Foretaste of topological object: (1st) Homotopy group $\pi_1(\mathbb{R}^2\setminus q) \cong \mathbb{Z}$

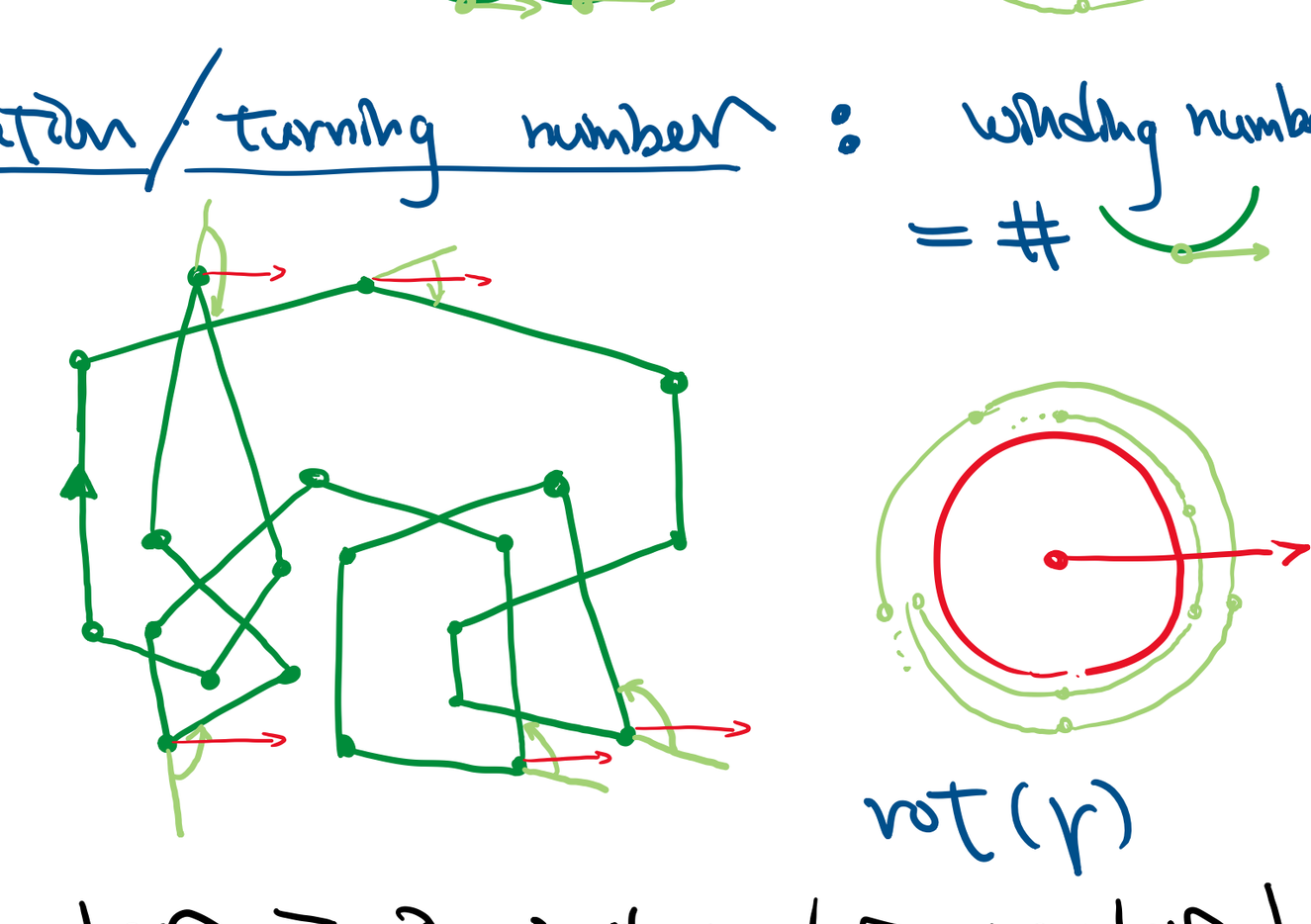
Regular homotopy

Intuition: No corners & cusps during homotopy

γ has a cont. derivative $\gamma': S^1 \to \mathbb{R}^2 \setminus (0,0)$

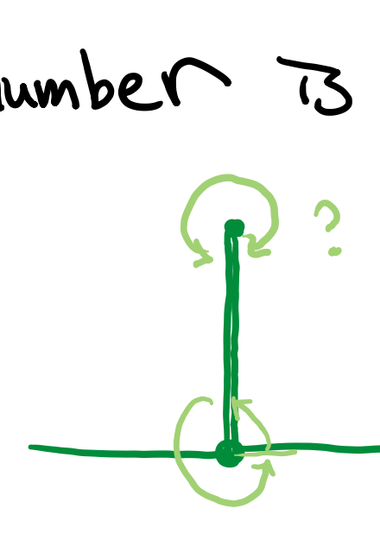
all $\gamma'(t)$ are well-def & nonzero

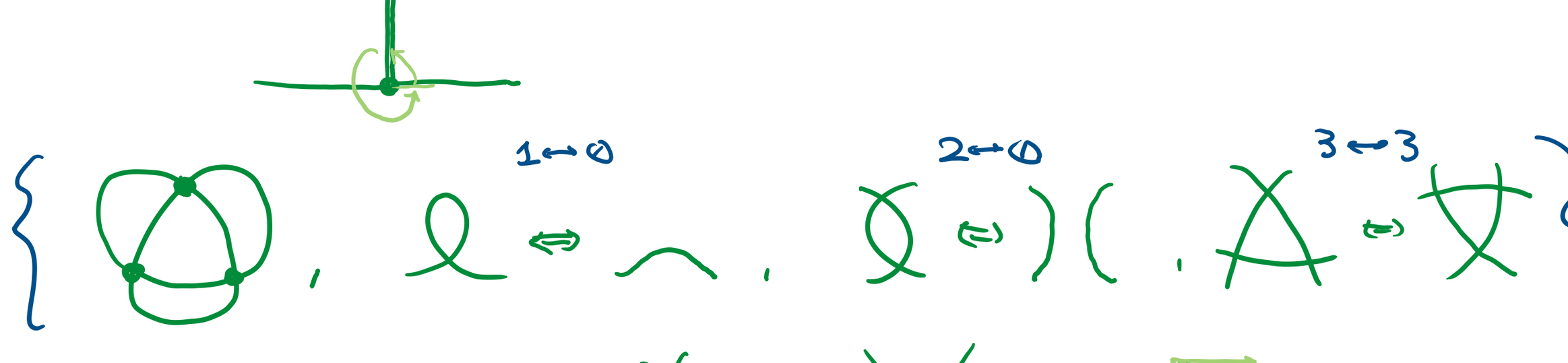
rotation/turning number: winding number of γ' = # ◡ − # ◠



$\text{rot}(\gamma)$

Rotation number is invariant under regular homotopy.





pf.

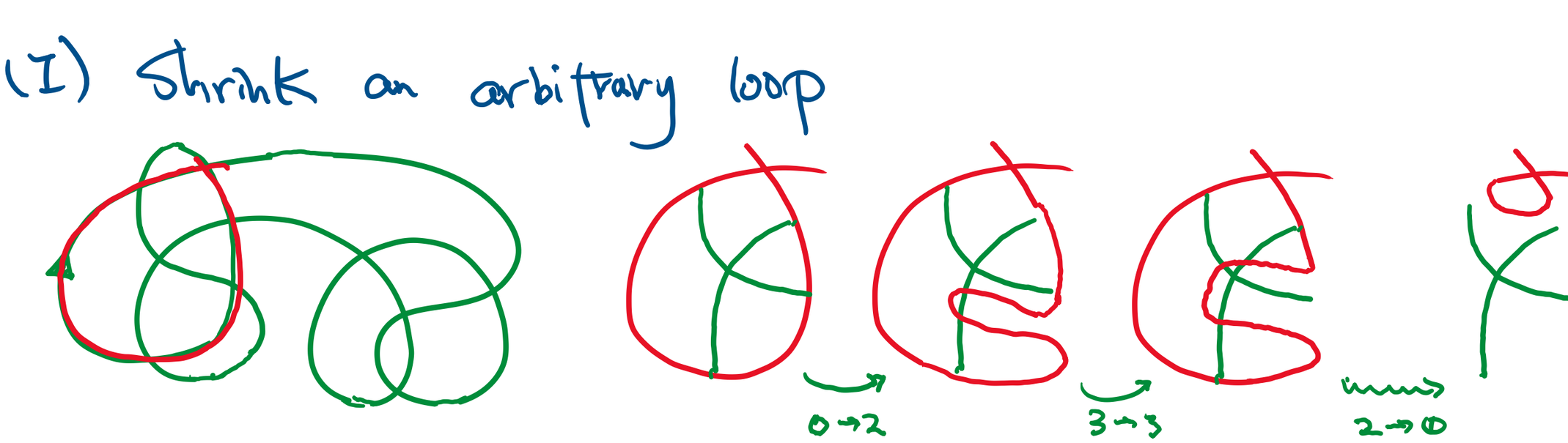
regular homotopy: 2↔0 & 3↔3 moves only.

[Whitney 1937] [Boy 1901] [Meister 1770]

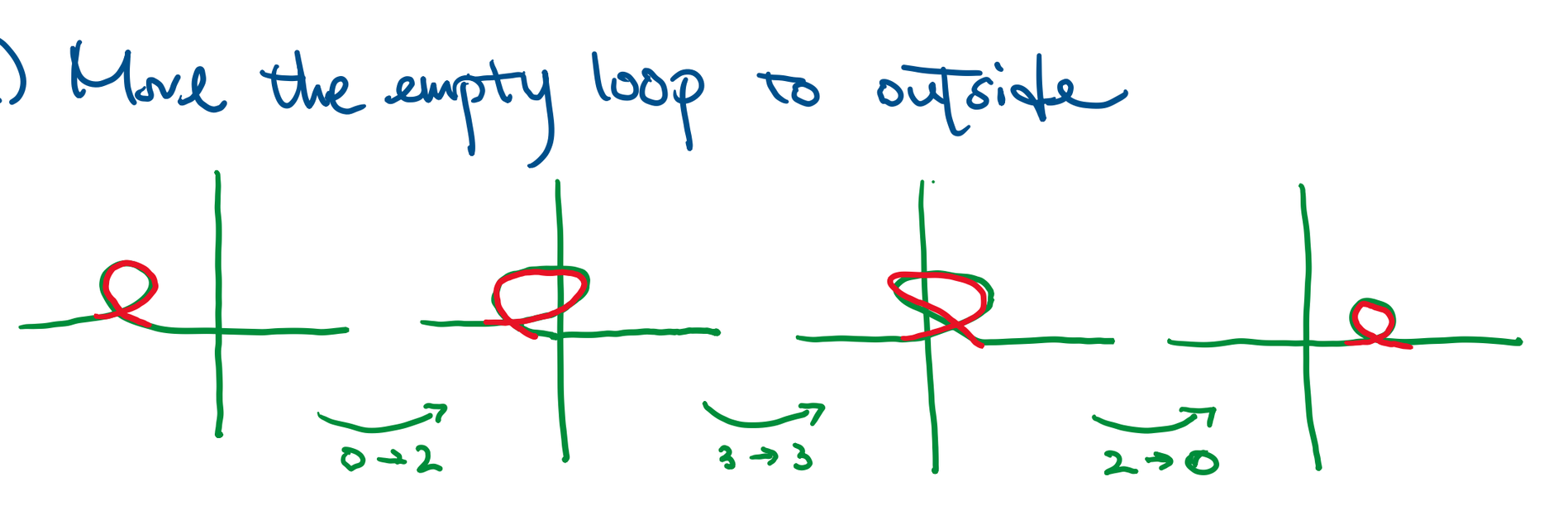
Whitney–Graustein Thm. Any two regular curves are regularly homotopic iff their rotation numbers are the same.

pf. Turn a regular curve γ into canonical curves $O^{\text{rot}(\gamma)}$

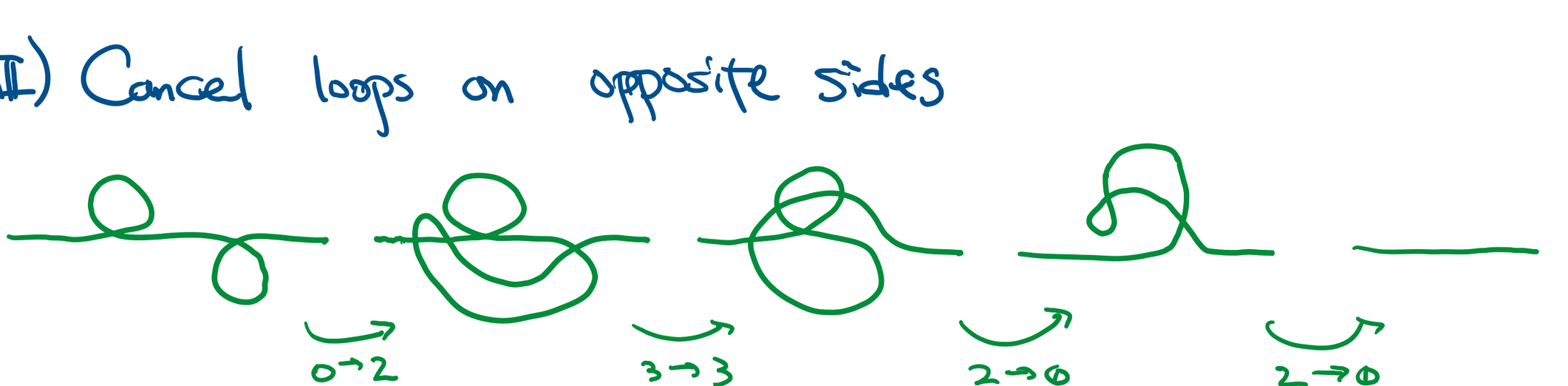
(I) Shrink an arbitrary loop



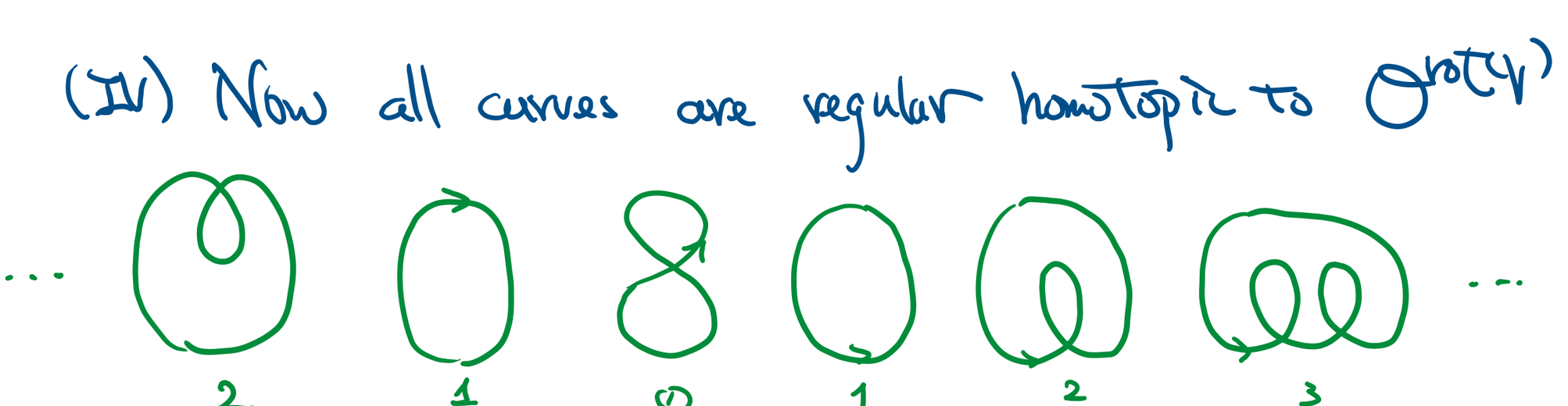
(II) Move the empty loop to outside



(III) Cancel loops on opposite sides



(IV) Now all curves are regular homotopic to $O^{\text{rot}(\gamma)}$:



$O(n^2)$ moves in total! ▨

Rotation number is a complete reg. homotopy invariant for planar curves

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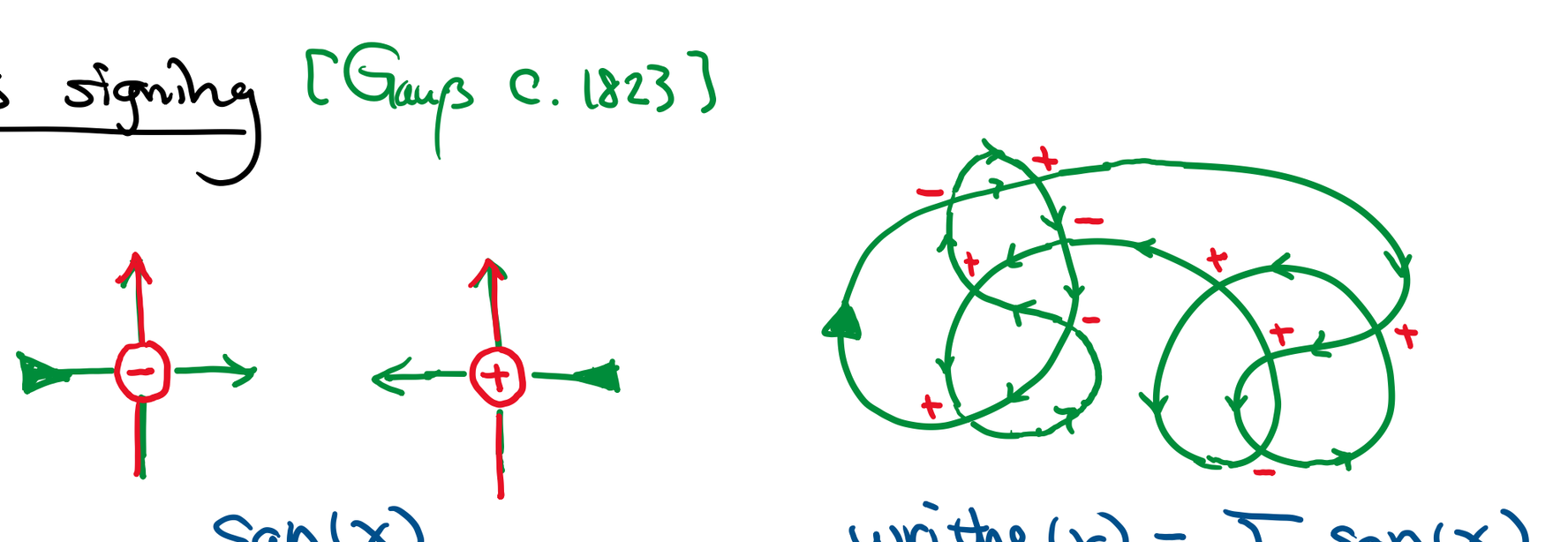
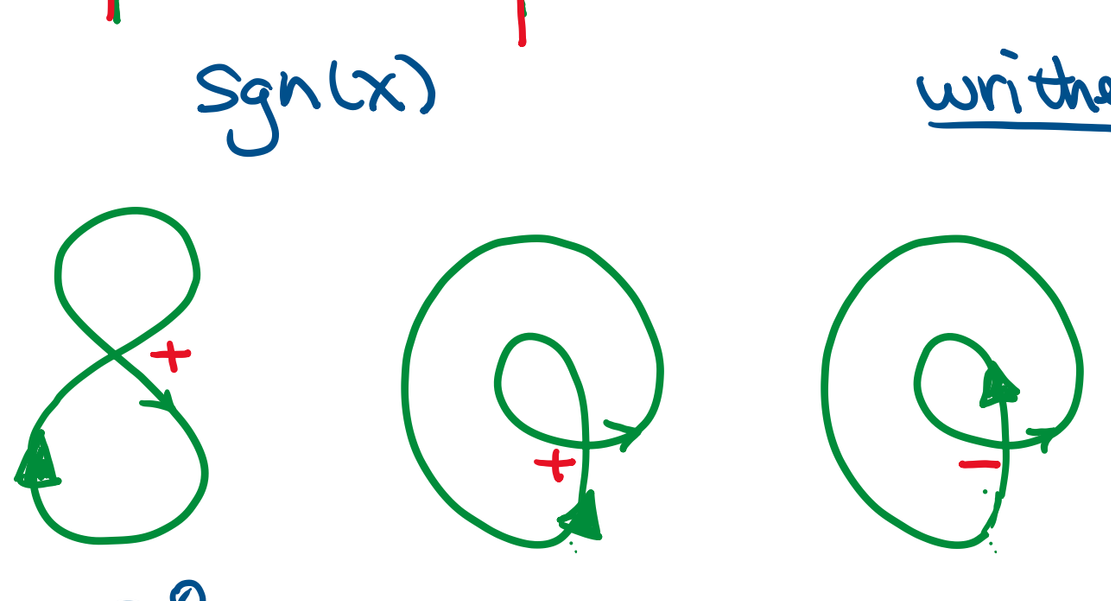
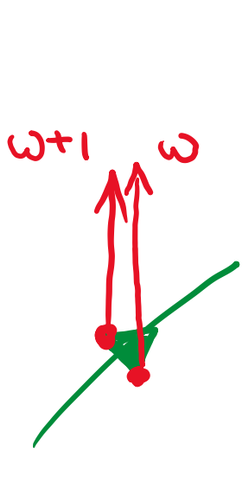
All Roads Lead to Rome

Gauss signing [Gauss c. 1823]



$\text{sgn}(x)$ $\qquad$ $\text{writhe}(\gamma) = \sum_x \text{sgn}(x)$



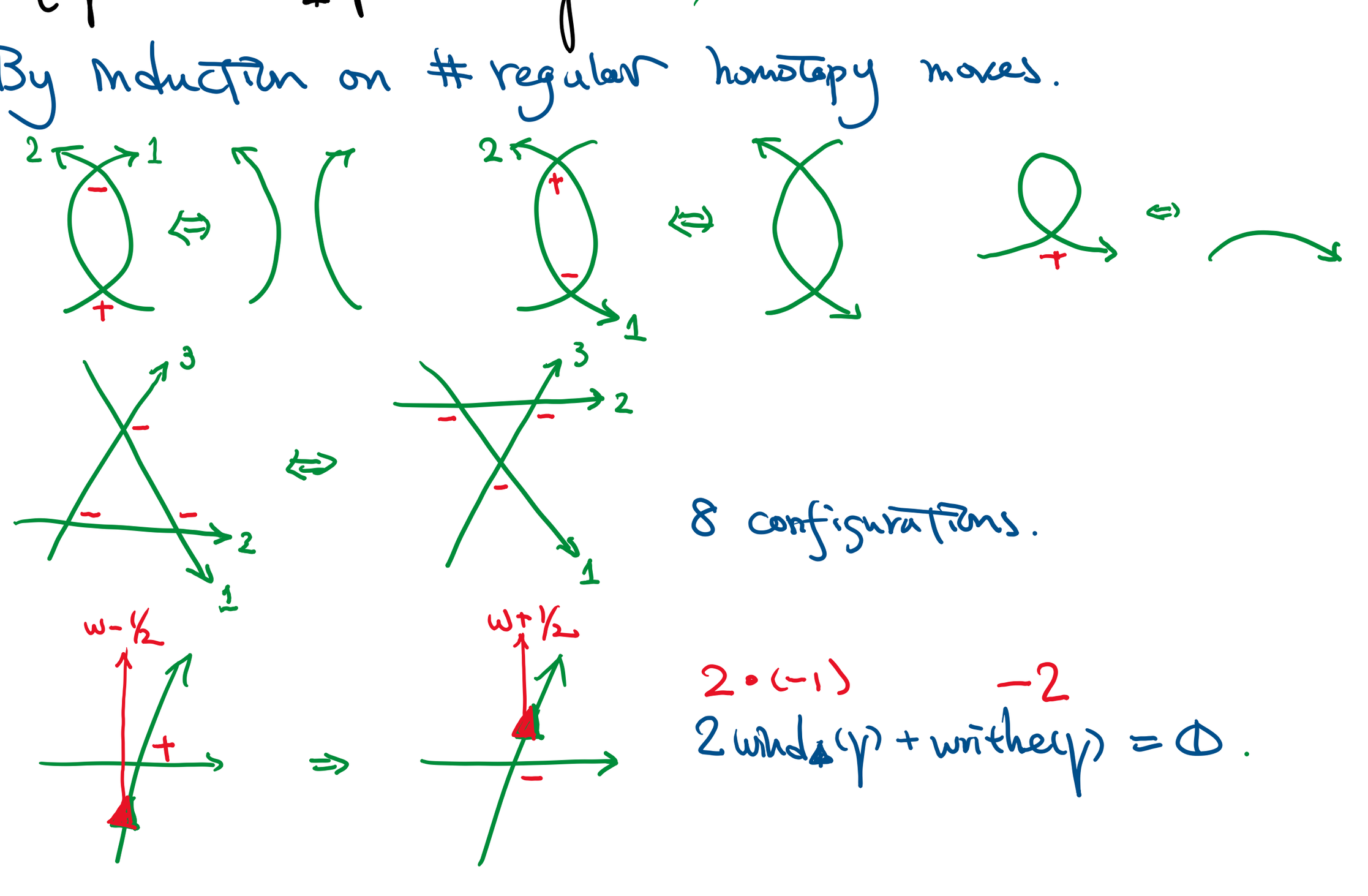


$\text{wind}_\blacktriangle(\gamma) = w + 1/2$

[Titus 1960] [Whitney 1937] $\text{wind}_\blacktriangle(\gamma) + \text{wind}_\blacktriangle(\gamma)$

Thm. $\text{rot}(\gamma) = 2\,\text{wind}_\blacktriangle(\gamma) + \text{writhe}(\gamma)$

pf. By induction on # regular homotopy moves.

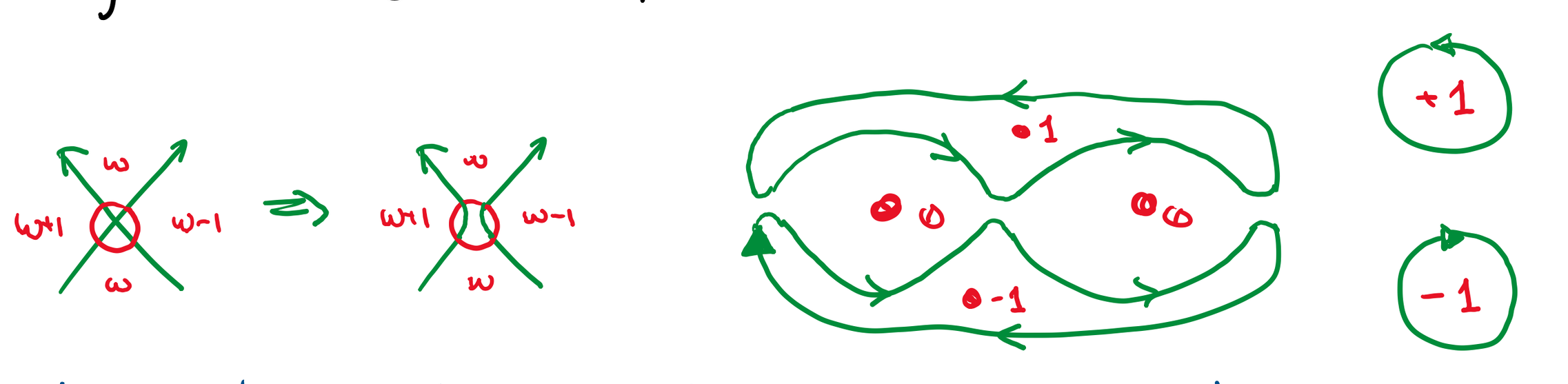


8 configurations.

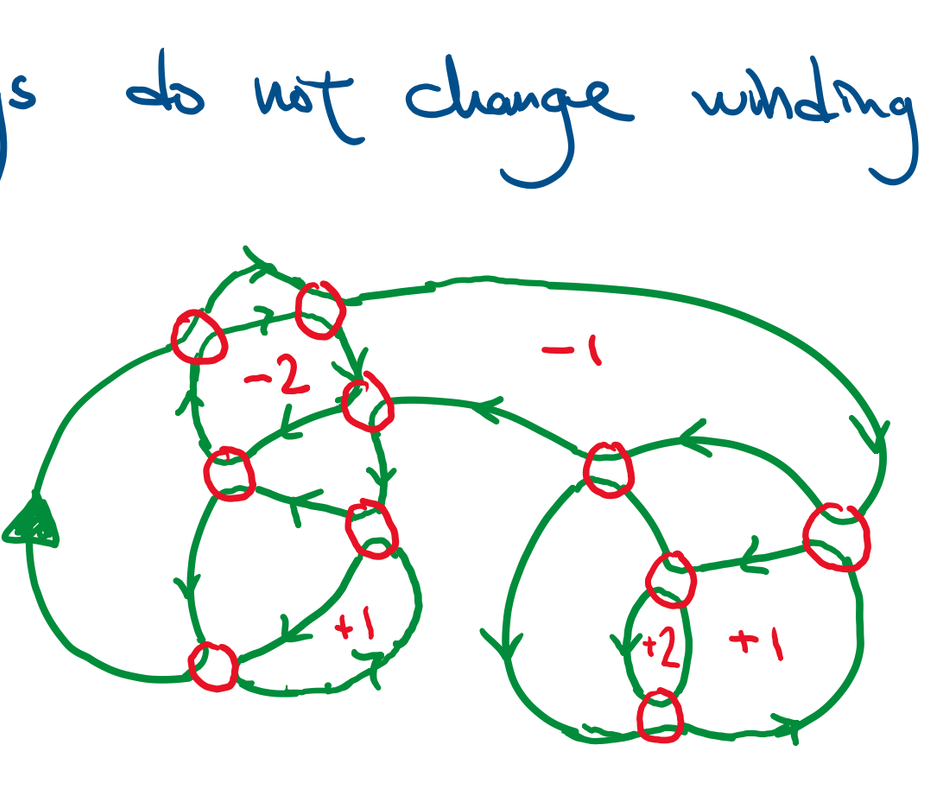
$2\cdot(-1)$ $\quad -2$

$2\,\text{wind}_\blacktriangle(\gamma) + \text{writhe}(\gamma) = 0$.

Smoothings & Seifert decomposition [Seifert 1931]



Smoothings do not change winding & rotation numbers.



Closing Question: What are the canonical curves in Poincaré space?