Simple planar curves

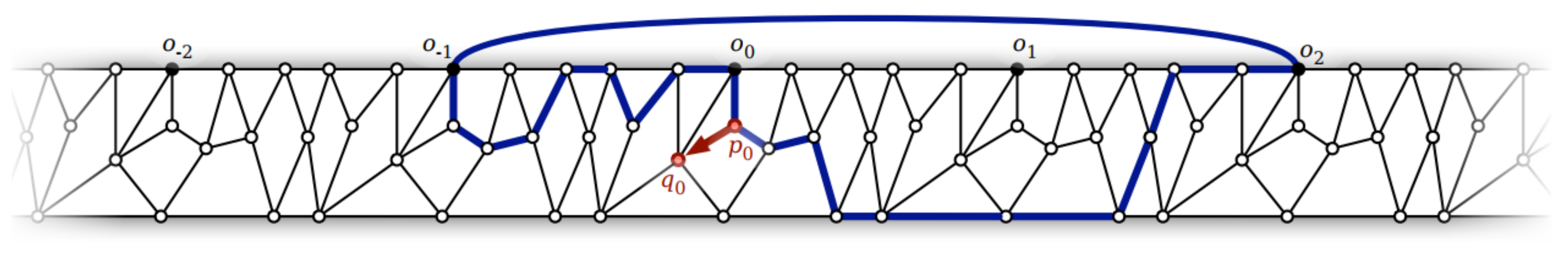
9/14/20 11:57 AM

What is a curve?

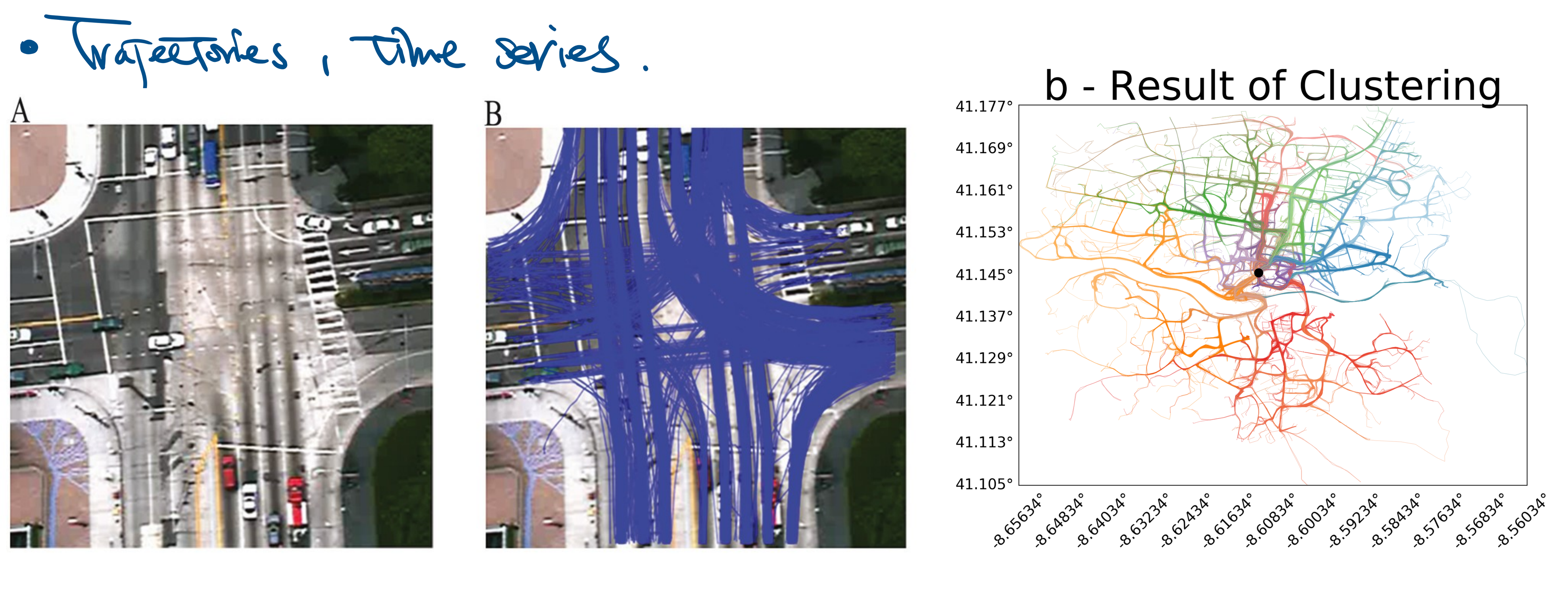
mapping $\gamma : S^1 \to X$ $X = \mathbb{R}^2$: planar curve

Why curves? I like doodles. Building blocks of topology

- Graph walks & traversals

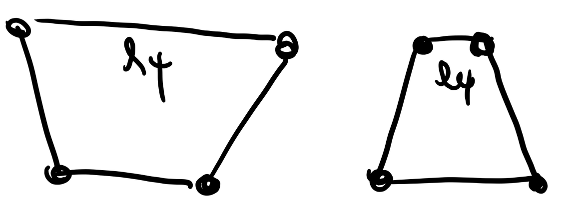


- Trajectories, time series.



- Configuration spaces / motion planning

  Properties unchanged under length tweaking



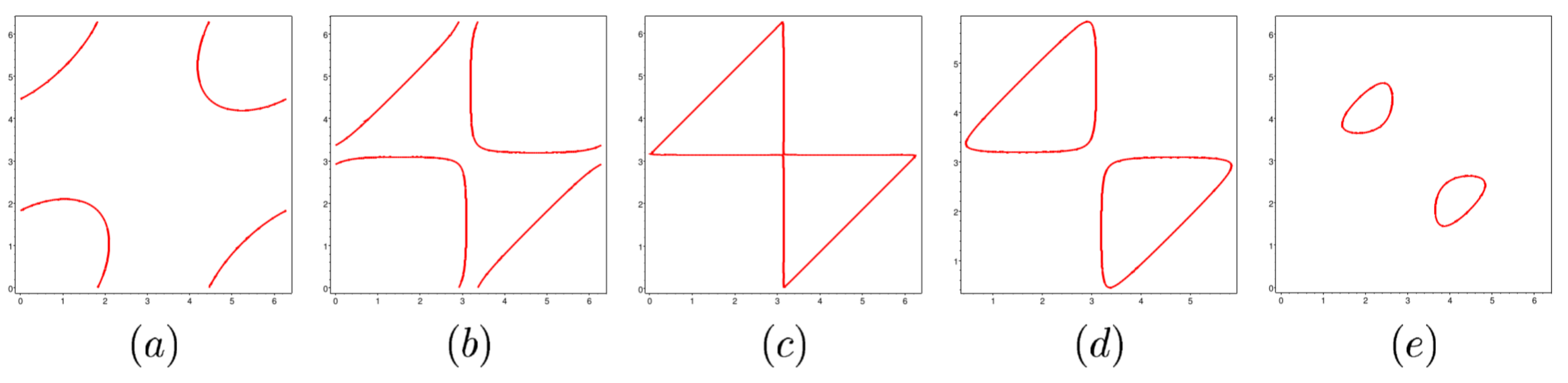
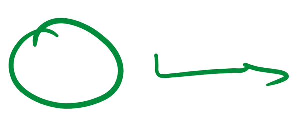
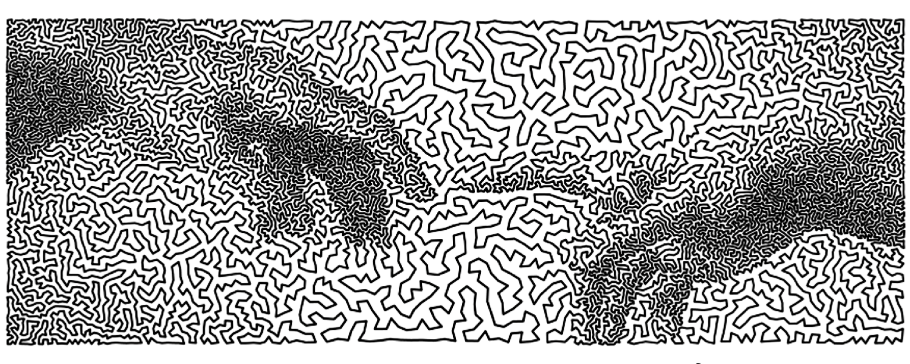


Figure 4: 4-gonal linkage $\mathcal{P}(1, 1, 1, l_4)$ with $l_4 \in \{1.5, 1.1, 1, 0.9, 0.5\}$

Topology of simple planar curves

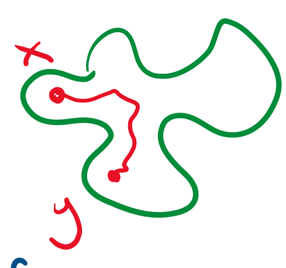
$\gamma : S^1 \xhookrightarrow[\text{embedding}]{} \mathbb{R}^2$





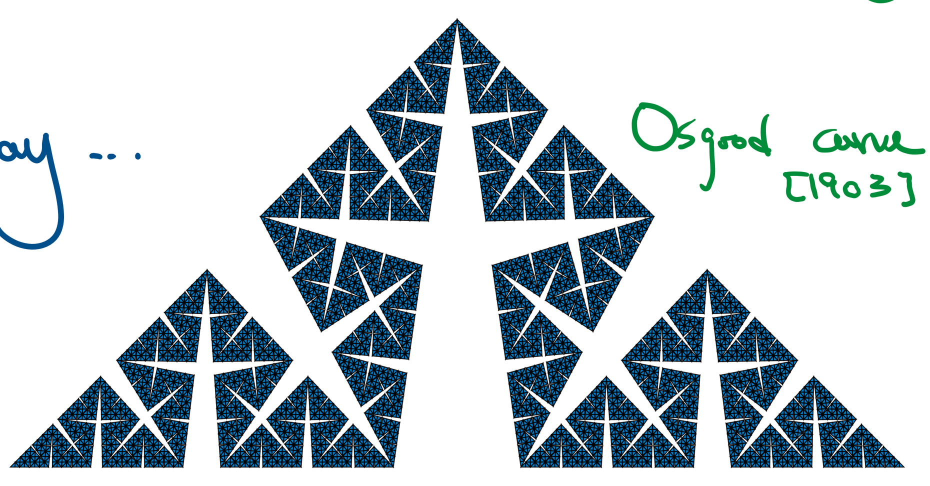
First non-trivial topological fact (Jordan Curve Theorem) [Jordan 1887]

Every simple closed planar curve separates $\mathbb{R}^2$ into exactly 2 components.



component: $\forall x, y \in R$. $\exists$ path in R between x, y

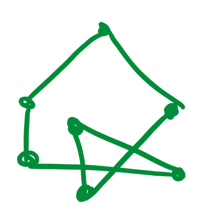
Nontrivial! In a subtle way...



Osgood curve [1903]

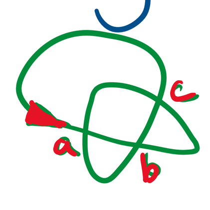
Representation of curves.

- Polygons: cyclic seq. of points $[p_0 \ldots p_{n-1}, p_n = p_0]$



- Generic curve as 4-reg. plane (multi-) graph

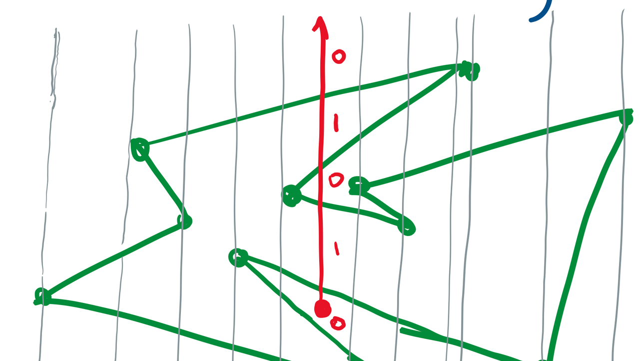
- Gauss code:



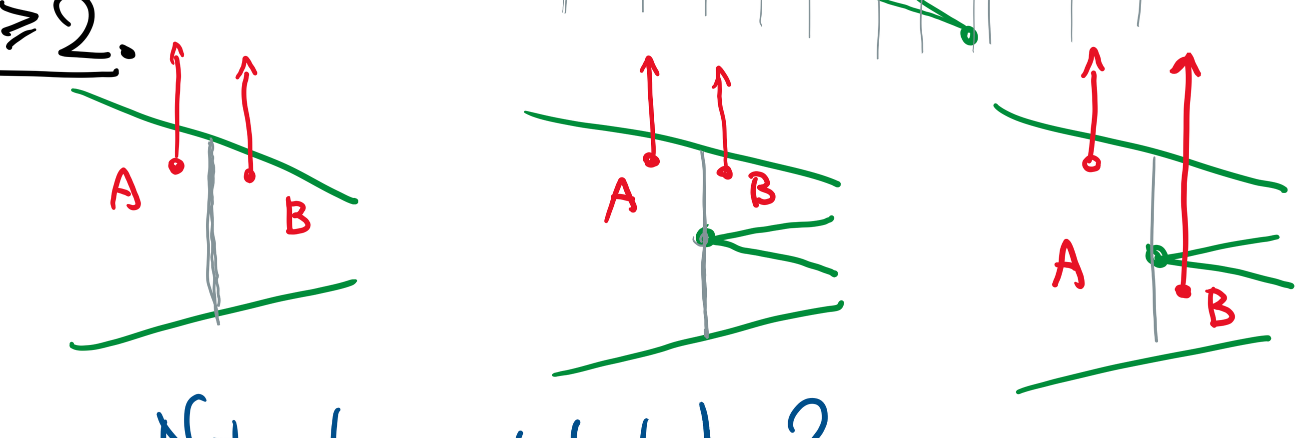
[abcabc]

Jordan Polygon Thm. Any simple polygon P separates $\mathbb{R}^2 \setminus P$ into exactly 2 components.

Idea. Parity argument



Lemma ≥ 2.

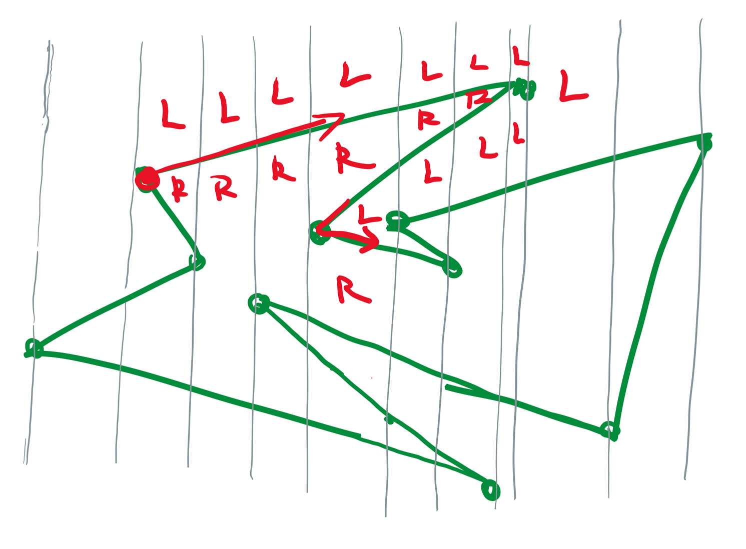


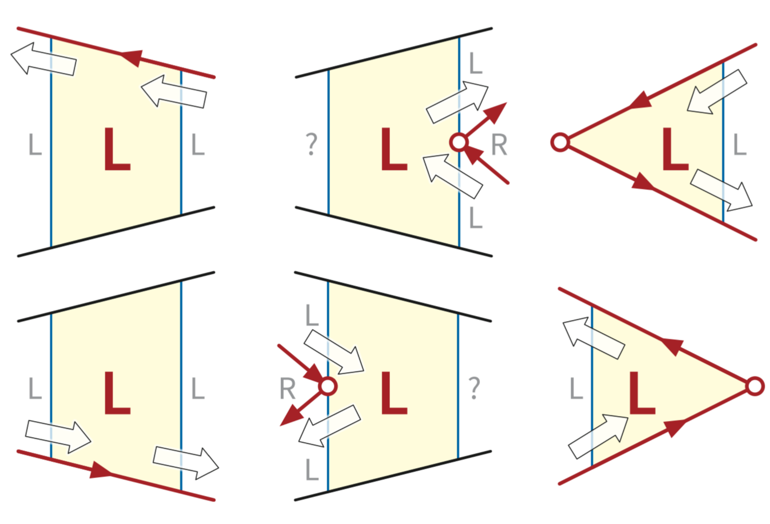
Not done yet! Why?

Lemma ≤ 2.

≥ 2 is a property of space $\mathbb{R}^2$

≤ 2 is a property of curve simple

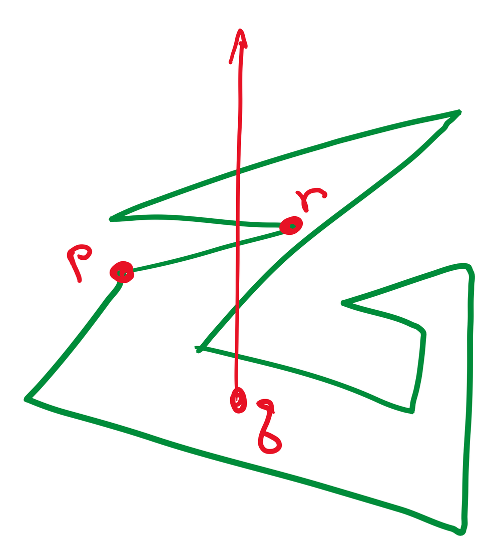




Inside polygon testing.

InsidePolygon?(P, q):

```
for each segment pr:
    Δ ← Δ(p, q, r)
    if p.x ≤ q.x < r.x:
        sign ← -Δ · sign
    if r.x ≤ q.x < p.x:
        sign ← Δ · sign
return sign
```



$O(n)$ time.

Data structure

Build Trapezoid decomposition in $O(n \log n)$ time using sweep-line algorithm.

Label all trapezoids in $O(n)$ time

Query the trapezoid containing q in $O(\log n)$ time.

Check in/out label in $O(1)$ time.

... J – Schönflies Thm. [1906]

... and each component of ~~$\mathbb{R}^2 \setminus \gamma$~~ $\mathbb{R}^2 \setminus P$ can be morphed into ~~a disk.~~ a convex polygon

[Dehn 1899]