- Starting from Homework 1 you are allowed to work in of group up to 3 people (for undergrads there is no limit to the group size). Please write the names of *all* group members on the first page of your submission. *One, and only one* person from each group is responsible in submitting the solution to Canvas.
- *Each member* of the group is required to cite all the people and resources you used when trying to solve the problems; but you don't have to cite other members from your group. The standard citation rules applies (see the course webpage).

Let γ be a *generic* planar curve where all self-intersection points are transverse double-crossings. Any planar curve can be made generic by slightly perturbing the curve. In this homework we will be playing with generic planar curves.

1. ***Gauss code.*** A ***Gauss code*** is a cyclic string of $2n$ symbols where each symbol occurs exactly two times; it is ***signed*** if in addition each symbol x is attached with a plus/minus sign +/-, one for each occurence of x. A Gauss code is ***planar*** if it encodes the sequence of crossings we see as we traverse an n-vertex planar curve γ; the signing of the Gauss code correspond to the Gauss signs of the crossings of γ.

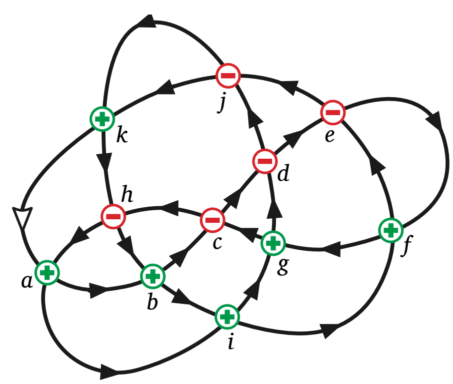

Figure 1. A planar curve with Gauss code [abcdefgchaigdjkhbifejk] and signing [++---+++--+-+-++---++-].

   (a) Describe and analyze an algorithm whether a given *signed* Gauss code is planar.

   (b) Construct an (unsigned) non-planar Gauss code. Identify the reason why your code cannot be realized by a planar curve (as a condition of the code).

   (c) Construct another Gauss code that is not planar, for a *different* reason than the one you stated above.

   ⋆(d) Try to prove that any Gauss code excluding the above conditions must be planar. If the proof fails, identify more reasons to why a code can be non-planar, and try again. Generate a list of necessary conditions that becomes sufficient. *[Hint: Lemma* ≥ 2 *and* ≤ 2 *for the Jordan polygon theorem.]*

   ★(e) Can you provide such characterization for Gauss codes encoding curves on the torus?[1]

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[1] The *really* big star here indicates that the problem, as far as I know, is open.

2. ***Simplifying planar curves.*** Any homotopy between two generic planar curves can be decomposed into a finite number of local operations:

   - **1**→**0**: removing an empty *loop*;
   - **2**→**0**: removing an empty *bigon*;
   - **3**→**3**: flipping a *triangle*, or equivalently, moving a strand across another crossing.

   We refer to these operations, *together with their inverses*, as the ***homotopy moves***.

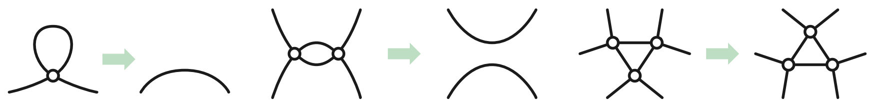

Figure 2. Homotopy moves 1→0, 2→0, and 3→3.

During the lectures we showed that any planar curve can be reduced to some canonical curve with the same rotation number using only *regular* homotopy moves (where only 0→2, 2→0 and 3→3 are allowed), by iteratively emptying the loops. From there one can always simplify the canonical curves (that is, remove all crossings) using 1→0 moves.

The goal of this problem is to simply a planar curve *without ever creating new self-intersections*. In other words, any planar curve can be simplified using only 1→0, 2→0, and 3→3 moves. Such moves are referred as ***monotonic***.

(a) As a first step, we show that at least one of the monotonic homotopy moves can be applied at any given time. In other words, prove that there is always a face of degree at most 3 in any planar curve.

The main difficulty to adapt the loop-removal approach here is that emptying a loop in general requires 0→2 moves, which is not monotonic. However, we can search for a (possibly non-empty) *bigon* instead. A ***bigon*** is the region bounded by two simple interiorly-disjoint subpaths of γ that shares the same two endpoints. (See Figure 3.)

(b) Prove that after all empty loops are removed, there must be a bigon in γ (or γ is already simple).

(c) A bigon is *(inclusion-wise) minimal* if no other bigon lies inside. Prove that, after all empty loops are removed, any non-empty minimal bigon in γ must have a triangle on the bounding curves (and therefore can be removed using a single 3→3 move).

(d) Prove our main statement that any planar curve can be simplified monotonically. How many moves does your algorithm take?

As an alternative approach, one can prove termination by introducing a *potential*. Consider the breadth-first search tree on the dual graph of γ, starting at the external face (called the *root*). Each face is labelled with its distance to the root on the BFS tree, called the ***depth***. Define the ***depth potential*** to be the sum of depths on all faces of γ.

★(e) Prove that there is always a homotopy move that decreases the depth potential.[2]

[2]The only proof I know conducts an ugly case-by-case-by-case analysis, and is not done using the curve language. Is there a way that the combinatorial Gauss-Bonnet theorem can help?

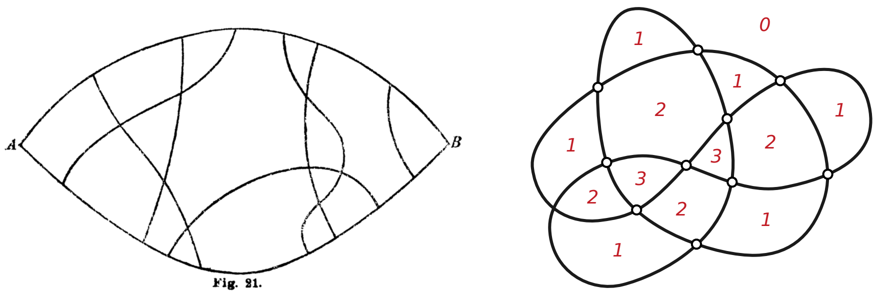

Figure 3. Left: A (non-minimal) bigon. Right: Face depths.

Since the depth potential of any n-vertex planar curve is bounded by $O(n^2)$, the algorithm will terminate after at most $O(n^2)$ moves.

★(f) Can you simplify an arbitrary n-vertex planar curve monotonically in $o(n^2)$ moves?