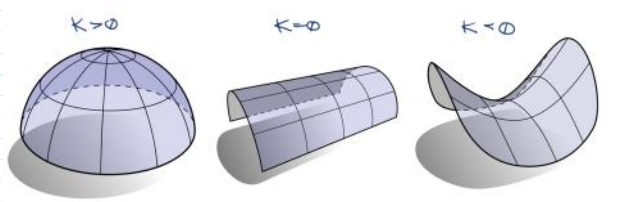
Administrivia.

- Talk w/ me about the presentation schedule!
- HW 0. HW 1 out.

$\boxed{\text{Planar graphs have ... Gauss-Bonnet Theorem}}$

Q. How do you know there are only 5 Platonic Solids?



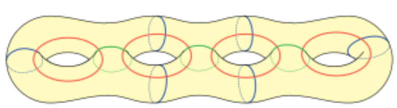
Uniformization Thm [Poincaré 1907] [Koebe 1907]

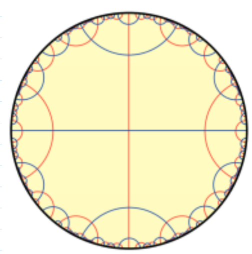
Any simply-connected surface is conformally equivalent to

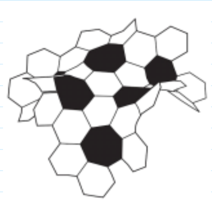
(a) (open) unit-disk, $\chi<0$

(b) complex plane, $\chi=0$

(c) sphere. $\chi>0$



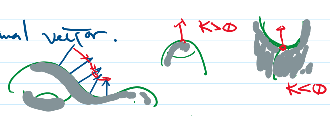




Geometry!

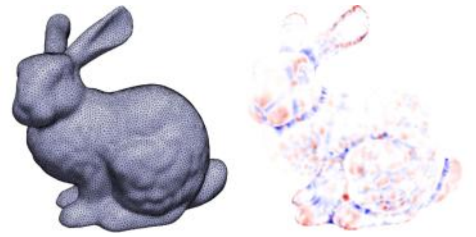
Gauss map $\gamma: \Sigma \to S^2$ unit-normal vector.

Gauss curvature $K = \det(D\gamma)$



Gauss-Bonnet Thm. Σ orientable surface w/o boundary, then

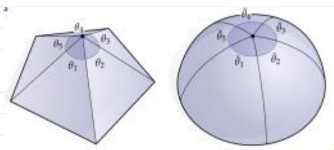
$$\int_\Sigma K \, dA = 2\pi \chi(\Sigma)$$

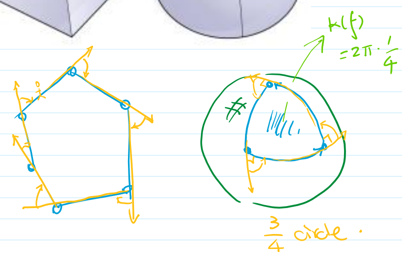


Combinatorial GB : Assign arbitrary real interior angle $\angle C$ to each corner of the faces.

$$\underline{K(v)} := 2\pi \cdot \Big(1 - \sum_{\angle C \in v} \boxed{\angle C}\Big) \rightarrow \text{angle } C$$

$$K(f) := 2\pi \cdot \Big(1 - \boxed{\sum_{\angle C \in f} \big(\tfrac{1}{2} - \angle C\big)}\Big)$$





Then. $\sum_v K(v) + \sum_f K(f) = 2\pi \chi(S)$

Pf. $\sum_v K(v) = 2\pi \cdot \big(V - \sum_{\angle C} \angle C\big)$

$+$

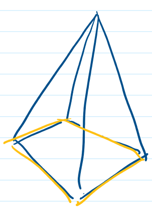
$\sum_f K(f) = 2\pi \cdot \big(F - E + \sum_{\angle C} \angle C\big)$

$\|$

$\sum_v K(v) + \sum_f K(f) = 2\pi \cdot \chi(S)$

Q. What makes the polyhedron solid?

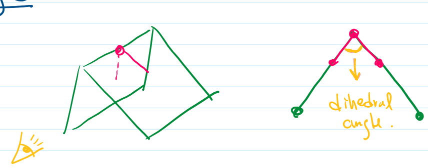
- fix all edge lengths.
- convexity
- bijection between faces.
  the face pairs are congruent.



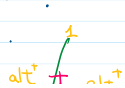
[Cauchy 1813, Steinitz-Rademacher 1934]

Cauchy Rigidity Thm. Two convex polyhedra w/ identical skeletons & with rigid edges & faces must be congruent.

dihedral angle :



Pf. Let G be the skeleton, which is (3-conn) planar.
Mark edges w/ signs in dihedral angle changes.



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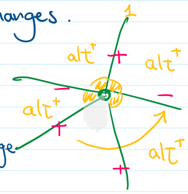
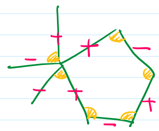


$\text{alt}(v) :=$ # $\pm$-labeling changes around v.

$\Sigma := \sum_v \text{alt}(v)$, $\Sigma^+ := \Sigma$ but change $\emptyset$ to $+$.

Claim. $\Sigma \le \Sigma^+ < 4V$

Pf. Double counting the alternations!



- Claim. $\sum_v \text{alt}(v) = \sum_f \text{alt}(f)$
- How large can $\text{alt}(f)$ be?
  - If $|f|$ even, $\text{alt}(f) \le |f|$.
  - If $|f|$ odd, $\text{alt}(f) \le |f| - 1$

  $\Big\} \le 2|f| - 4 \quad \forall |f| \ge 3$

Thus $\Sigma^+ = \sum_v \text{alt}(v) \le \sum_f (2|f| - 4)$

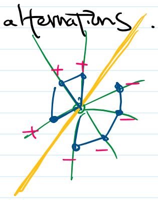
$= 4E - 4F = 4V - 8 < 4V.$ ☒

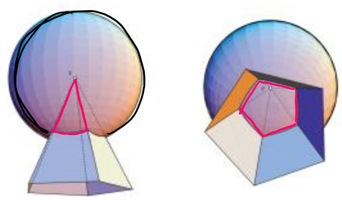
Intuition. Count only alt as real $(\frac{1}{4})$ angles.
We want $\overset{\text{avg.}}{\text{positive curvature}}$ on vertices. $\sum_v \text{alt}(v) < 4V$.

- $\angle$ corner on face gives $\frac{1}{2}$ to angle.
- $\angle$ 〃 $\frac{1}{4}$

So either $2 \times \angle$ or $\angle + \angle + \angle$ or $4 \times \angle$ makes a face non-positive, which is always true. ☒.

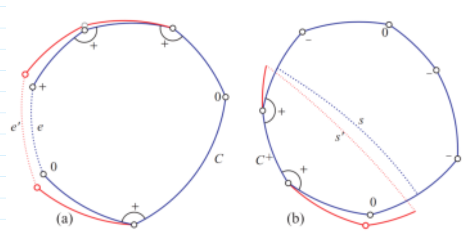
Some vertex has $\underline{0}$ or 2 alternations.





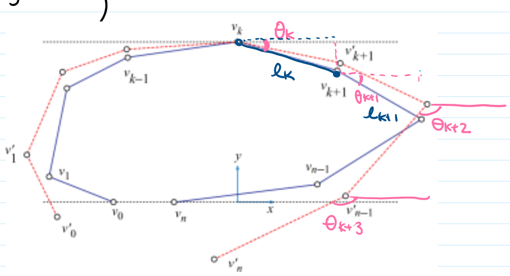
Cauchy-Steinitz Lemma. Any $\pm$-labeling on the corners of rigid convex polygon must have ≥ 4 alternations.

Pf.



Cauchy Arm Lemma. Opening up a convex $\overset{\text{rigid}}{\text{chain}}$ by increasing internal angles only increases the distance between two endpoints.

[Zaremba 1967]



$v_n . x = l_k \cdot \cos\theta_k + l_{k+1} \cdot \cos\theta_{k+1} + \cdots + l_{n-1} \cdot \cos\theta_{n-1}$

$\theta_i \in [-\pi, 0]$ cosine increases as θ_i increases.

$v'_n . x \ge v_n . x$

Cauchy-Steinitz implies $\Sigma \ge 4V$, contradicting to $\Sigma^+ < 4V$ ☒.

Next time: Schnyder wood. straight-line embeddings