- The homework is due on May 25, 23:59pm. Please submit your solutions to Gradescope.
- Starting from Homework 1, all homework sets allow *group submissions* up to 2 people. Please write down the names of the members *very clearly* on the first page of your solutions.
- Answer the questions in a way that is clear, correct, convincing, and concise. The level of details to aim for is that your peers in this class should be convinced by your solutions.
- You are expected to spend a reasonable amount of time (measured in hours) working on these problems. Remember you are allowed to utilize any resources. Make sure to cite all the people/webpages/source of infomation that helped.

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1. ***Diameter-treewidth property.*** The ***diameter*** of an unweighted graph G is the length of the longest shortest path in G, measured in number of edges.

   (a) Prove the *diameter-treewidth property* for planar graphs: Any planar graph G with diameter Δ has treewidth at most $f(\Delta)$ for some function Δ. (How small your function $f(\cdot)$ can get?)

   (b) Explain why the diameter-treewidth property does not hold for general minor-free graphs by providing an example.

2. ***Spectral cuts for cycles.*** Recall the Cheeger inequality we learn in class:

   Cheeger's inequality. Let G be any simple undirected graph and L be the (unweighted) Laplacian matrix of G, with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Denote $\phi(G)$ the conductance of G. Then

   $$\frac{\lambda_2}{2} \le \phi(G) \le \sqrt{2\lambda_2}.$$

   Workout the example when G is a cycle of length n, denoted by C_n.

   (a) Show that the 2nd Laplacian eigenvalue of C_n is $\lambda_2 = 1 - \cos(2\pi/n) = O(1/n^2)$.

   (b) Prove that the conductance of C_n is $\Theta(1/n)$.

   (This demonstrates that Cheeger inequality is tight on the upper bound side.)

3. ***Drawings of K_5.*** A consequene of the (strong) Hanani-Tutte theorem says: *Any drawing of K_5 contains two indepedent edges that intersect.* What is wrong with the following proof?

   > Let D be a given drawing of K_5. If there is a pair of independent edges in D intersect then we are done; otherwise, since K_5 is not planar, a pair of edges that share an endpoint at vertex x much intersect each other. Resolve the crossing by the "smoothing" operation:
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   > Take a small neighborhood of the crossing point so that the two edges intersects the boundary of the neighborhood at four points, denoted p_1 to p_4 in cyclic order, where the first edge starting from x enters from p_1 and leaves at p_3, and the second edge from p_2 to p_4. Now erase the drawing inside the neighborhood, and reconnect p_1 with p_4 and p_2 with p_3 using two disjoint curves.
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   > Repeat the process until all crossings are removed, in which case we get a planar drawing of K_5, a contradiction.