- The homework is due on May 04, 23:59pm. Please submit your solutions to Gradescope.
- Starting from Homework 1, all homework sets allow *group submissions* up to 2 people. Please write down the names of the members *very clearly* on the first page of your solutions.
- Answer the questions in a way that is clear, correct, convincing, and concise. The level of details to aim for is that your peers in this class should be convinced by your solutions.
- You are expected to spend a reasonable amount of time (measured in hours) working on these problems. Remember you are allowed to utilize any resources. Make sure to cite all the people/webpages/source of infomation that helped.
- Some problems are marked with a *star*; these are more challenging (and fun) extra credit problems. They are optional and do not count toward raw grades.

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1. ***Sources, sinks, and saddles.*** Let G be an arbitrary embedded directed planar graph. Put it differently, assume a rotation system of the underlying undirected planar graph is given, and every edge is orientation in one of the two possible ways. (Assum the underlying undirected graph is connected.)

   A ***source*** in G is a vertex with only outgoing edges. A ***sink*** in G is a vertex with only outgoing edges. We call a vertex v in G ***normal*** if its cyclic list of incident edges consists of a single interval of incoming edges followed by a single interval of outgoing edges. We call v a ***saddle*** if four of the edges incident to v are directed into v, out of v, into v, and out of v in cyclic order around v. (There might be more incident edges or alternations.)

   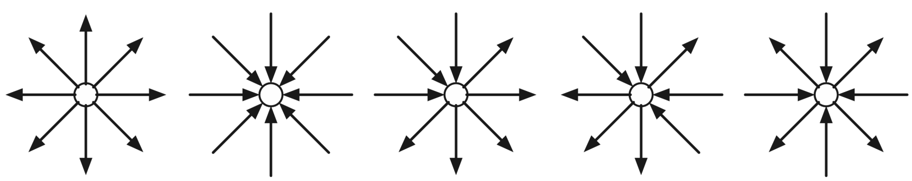

   Figure 1. From left to right: A source, a sink, a normal vertex, a saddle, and another (monkey) saddle.

   Prove that if G is an embedded directed *acyclic* planar graph with k sources and ℓ sinks, G contains at most $k + \ell - 2$ saddle vertices, using discrete Gauss-Bonnet theorem.

   (In particular, if G has a unique source s and a unique sink t, then every other vertex except must be normal.)

2. ***Path separators.*** Let G be a planar graph with n vertices. Prove that for any given parameter $p \geq 1$, we can compute a separator S that partition V_G into subsets V_1 and V_2:

   - the total size of V_i is at most $3n/4$ for each $i \in \{1, 2\}$;
   - S is consists of two part P and Q. P is the union of at most two paths of length $O(p)$ in G, and Q has at most $O(n/p)$ many vertices.

   (In one extreme when $p = n$, we get a *path separator* when S is the union of two paths; in the other extreme when $p = \sqrt{n}$, we have our regular separator.)