1. ***Drawings of $K_5$.*** A consequene of the (strong) Hanani-Tutte theorem says: *Any drawing of $K_5$ contains two indepedent edges that intersect.* What is wrong with the following proof?

   Let $D$ be a given drawing of $K_5$. If there is a pair of independent edges in $D$ intersect then we are done; otherwise, since $K_5$ is not planar, a pair of edges that share an endpoint at vertex $x$ much intersect each other. Resolve the crossing by the "smoothing" operation:

   Take a small neighborhood of the crossing point so that the two edges intersects the boundary of the neighborhood at four points, denoted $p_1$ to $p_4$ in cyclic order, where the first edge starting from $x$ enters from $p_1$ and leaves at $p_3$, and the second edge from $p_2$ to $p_4$. Now erase the drawing inside the neighborhood, and reconnect $p_1$ with $p_4$ and $p_2$ with $p_3$ using two disjoint curves.

   Repeat the process until all crossings are removed, in which case we get a planar drawing of $K_5$, a contradiction.

2. ***Diameter-treewidth property.*** The ***diameter*** of an unweighted graph $G$ is the length of the longest shortest path in $G$, measured in number of edges.

   (a) Prove the ***diameter-treewidth property*** for planar graphs: Any planar graph $G$ with diameter $\Delta$ has treewidth at most $f(\Delta)$ for some function $\Delta$. (How small your function $f(\cdot)$ can get?)

   (b) Explain why the diameter-treewidth property does not hold for general minor-free graphs by providing an example.

3. ***Neighborhood balls in planar graph.*** Let $G$ be an unweighted planar graph, and let $d_G(x, y)$ denote the shortest-path distance between nodes $x$ and $y$. Consider $r$-neighborhood balls in $G$, one for each node $v$:

$$N_r[v] := \{x \in V(G) : d_G(v, x) \le r\}.$$

   (a) Provide an example that the $r$-neighborhood ball system $\{N_r[v] : v \in V(G)\}$ is not a pseudo-disk system. *[Hint: How should "pseudo-disks" be defined on graphs?]*

   (b) Prove that the $r$-neighborhood ball system is nevertheless "non-piercing". (Two regions $D$ and $D'$ are ***non-piercing*** is both $D \setminus D'$ and $D' \setminus D$ are connected.)