1. ***Weighted Delaunay graph.*** Let P be a collection of points in the plane and $\pi : P \to \mathbb{R}$ be real weights on the points in V. The ***parabolic lift*** of (P, π) lift each point $p = (x, y)$ in P to the paraboloid $z = x^2 + y^2$, then move the point downwards by $\pi(p)$. In other words, each point p corresponds to the point

$$p^+ := (x, y, x^2 + y^2 - \pi(p)).$$

Now if we take the convex hull of the lifted points $P^+ := \{p^+ : p \in P\}$ and project the convex hull down into the plane, we arrive at a plane graph called the ***weighted Delaunay graph***. (Notice that if the weights $\pi(\cdot)$ are all zero and the point locations are generic, then the weighted Delaunay graph is just the regular Delaunay triangulation).

(a) Assume that the weighted Delaunay graph is a triangulation (where all faces have degree 3). An edge xy in the triangulation is ***locally Delaunay*** if (1) xy is on the boundary, or (2) the two triangles that has xy as one of the sides, say xyz and xyw, satisfy the property that z lies outside the circumcircle of xyw (and thus, w lies outside the circumcircle of xyz as well).

Prove the weighted version of the ***Delaunay lemma***, that if every edge of a triangulation G is locally Delaunay, then G is a weighted Delaunay graph.

(b) Establish the last step in the Cremona-Maxwell-Delaunay equivalance: Every convex polyhedral lift (G, P, z) is in one-to-one correspondence with a weighted Delaunay graph (G, P, π). *[Hint: The backward direction is straightforward. How to ensure every point in P is indeed in the weighted Delaunay graph for the forward direction?]*

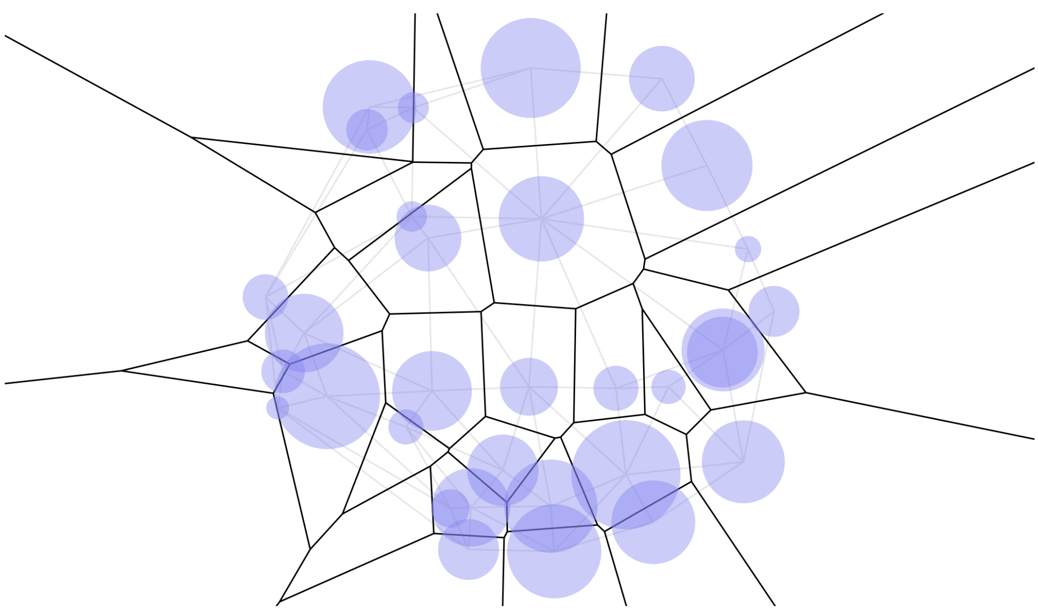

Figure 1. Power diagram and its associated weighted Delaunay triangulation. Figure generated using marmakoide.

2. ***Power diagram***, also known as the *Laguerre-Voronoi diagram*, of points P with weights $\pi : P \to \mathbb{R}$ is a Voronoi diagram with respect to the ***power distance*** $\boldsymbol{d_\pi}$:

$$d_\pi(x, p) := \|x - p\|^2 - \pi(p).$$

Put it differently, the power diagram consists of a collection of cells at points in P, where each cell at q is defined as

$$\boldsymbol{cell_P(q)} := \left\{ x \in \mathbb{R}^2 : d_\pi(x, q) \le d_\pi(x, p) \text{ for every other point } p \in P \right\}.$$

(a) Prove that the power diagram of points P with weights π is equivalant to the reciprocol diagram of the associated weighted Delaunay triangulation (G, P, π).

(b) Explain intuitively why the circle packing of plane graph is a special instance of a power diagram. *[Hint: Watch out! Why can all the radii constraints be satisfied simultaneously?]*