1. ***Gauss code.*** A ***Gauss code*** is a cyclic string of $2n$ symbols where each symbol occurs exactly two times; it is ***signed*** if in addition each symbol x is attached with a plus/minus sign +/-, one for each occurrence of x. A Gauss code is ***planar*** if it encodes the sequence of crossings we see as we traverse an n-vertex planar curve γ; the signing of the Gauss code correspond to the Gauss signs of the crossings of γ.

   Describe and analyze an algorithm whether a given signed Gauss code is planar.

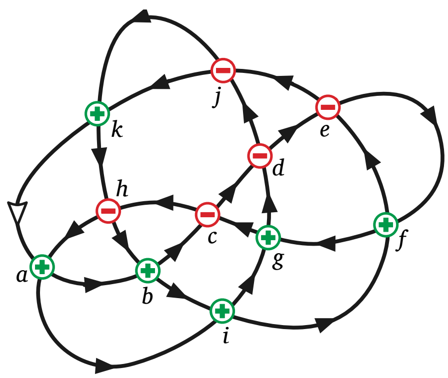

Figure 1. A planar curve with Gauss code `[abcdefgchaigdjkhbifejk]` and signing `[++---+++--+-+-++---++-]`.

2. ***Spanning trees as α-orientations.*** Let G be a plane graph and G^* be its dual, drawn in the plane in such a way that every crossing correspond to exactly one primal-dual edge pair from (G, G^*). Consider the ***overlay graph*** G^+:

   - Add all vertices in G and G^*, and all the crossings in the drawing as vertices of G^+;
   - Subdivide each edge (u, v) in G and G^* at the crossing point x, and add the two edges (u, x) and (x, v) as edges of G^+.

   (Alternatively, one can construct the overlay graph by performing the radial construction twice on the primal graph G: $G^+ := G^{\diamond\diamond}$.[1])

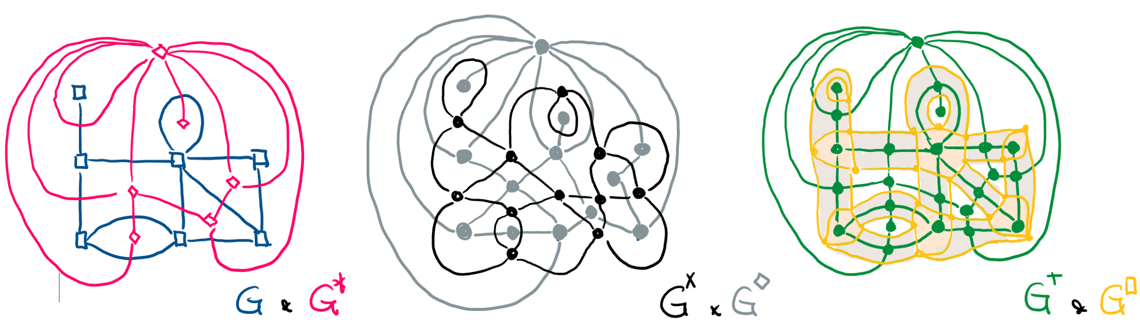

Figure 2. (a) Plane graph G and its dual G^*. (b) Medial graph $G^\times$ and radial graph $G^\diamond$. (c) Overlay graph G^+ and its dual $G^\square$.

[1]The overlay graph G^+, obtained by performing the radial construction twice, is a subgraph of the barycentric subdivision of G. The dual graph of G^+, conveniently denoted as $G^\square$, can be obatined by performing the medial construction twice ($G^\square := G^{\times\times}$), and is a *minor* of the band decomposition/ribbon graph of G.

(a) Prove that there is a feasible function α defined on the overlay graph G^+, such that a tree-cotree pair in the primal-dual plane graph (G, G^*) is in bijection with α-orientations of G^+, after fixing one primal "root" and one dual "root" from the vertices of G^+.

(b) Prove that the essential cycles for the above collection of α-orientations are exactly the faces of G^+ (which are exactly the corners of G) not incident to the two roots.

*3. ***Improving presentation.*** In class we showed that given any ±-labeling on the edges of a planar graph with vertex set V, we have

$$\sum_{v \in V} alt(v) < 4|V|$$

where $alt(v)$ is the number of sign alternation around the vertex v.

Provide a new proof to the result using discrete Gauss-Bonnet Theorem.