1. ***Regular or not?*** *Prove or disprove* that each of the languages below is regular (or not). Let Σ^+ denote the set of all *nonempty* strings over alphabet Σ; in other words, $\Sigma^+ = \Sigma \cdot \Sigma^*$. Denote $n(w)$ the integer corresponding to the binary string w.

   (a) $\{\mathtt{3}x\mathtt{=}y :\ x, y \in \{\mathtt{0}, \mathtt{1}\}^*,\ n(y) = 3n(x)\}$

   Solution: Denote the language in the problem 1(a) as L_a. We prove that L_a is not regular by constructing a fooling set for L_a of infinite size.

   Let $F = \{\mathtt{310}^i : i \geq 0\}$. For two distinct prefixes $x = \mathtt{310}^i$ and $y = \mathtt{310}^j$ in F, let z be $\mathtt{=110}^i$.

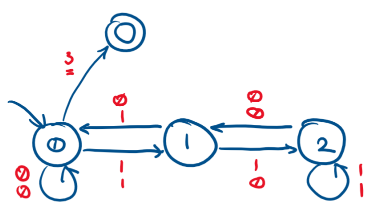
   - $xz = \mathtt{310}^i\mathtt{=110}^i$; because $n(\mathtt{110}^i) = 3n(\mathtt{10}^i)$, we have xz in F.
   - $yz = \mathtt{310}^j\mathtt{=110}^i$; because $n(\mathtt{110}^i) \neq 3n(\mathtt{10}^j)$ if $i \neq j$, we have yz not in F.

   This implies that F is a fooling set of infinite size, and thus L_a is not regular. ■

   (b) $\left\{ \begin{smallmatrix}\mathtt{3x}\\ \mathtt{=y}\end{smallmatrix} :\ \begin{smallmatrix}\mathtt{x}\\ \mathtt{y}\end{smallmatrix} \in \left\{ \begin{smallmatrix}\mathtt{0}\\ \mathtt{0}\end{smallmatrix}, \begin{smallmatrix}\mathtt{0}\\ \mathtt{1}\end{smallmatrix}, \begin{smallmatrix}\mathtt{1}\\ \mathtt{0}\end{smallmatrix}, \begin{smallmatrix}\mathtt{1}\\ \mathtt{1}\end{smallmatrix} \right\}^*,\ n(y) = 3n(x) \right\}$

   Solution: Denote the language in the problem 1(b) as L_b. We prove that L_b is regular by constructing an NFA recognizing L_b.

   We construct NFA recognizing the reverse of the language, L_b^R; by the exercise problems, L_b is regular if and only if L_b^R is regular.



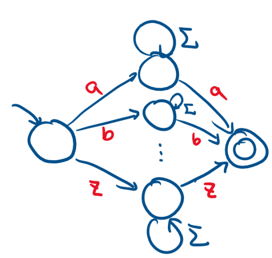
   The NFA reads the input from the least significant bits of x and y, and records the amount of carry at any moment. The transitions are implemented so that the machine only continues if the current digit of y equals to (the least significant bit of) three times the corresponding digit in the x plus the carry. After reading the full strings x and y, if there is any carry left then we reject; otherwise the NFA finishes off by reading the leading $\begin{smallmatrix}\mathtt{3}\\ \mathtt{=}\end{smallmatrix}$ and accepts. ■

   (c) $\{wxw^R : w, x \in \Sigma^+\}$

   Solution: Denote the language in the problem 1(c) as L_c. We prove that L_c is regular by constructing an NFA recognizing L_c, which is equivalent to the following language:

$$L_c' := \{\sigma x' \sigma : x' \in \Sigma^+, \sigma \in \Sigma\}.$$

   For $L_c' \subseteq L_c$, take $w = \sigma$ and $x = x'$; for $L_c \subseteq L_c'$, take σ to be the first symbol in w and x' to be whatever is left.



   The constructed NFA reads the first and the last symbol, and accepts if they match; therefore the NFA correctly recognizes language L_c'. More formally, create one state q_σ for each symbol

$\sigma \in \Sigma$; and add two extra states s and t. Let s be the only starting state and t be the only accepting state. For each symbol σ, add transitions s to q_σ and q_σ to t on reading σ, and self-loop transition at q_σ on reading all symbols.

■

(d) $\{ww^R x : w, x \in \Sigma^+\}$

Solution: Denote the language in the problem 1(d) as L_d. We prove that L_d is not regular by constructing a fooling set for L_d of infinite size. Without loss of generality we assume that `0` and `1` are in Σ.

Let $F = \{\mathtt{01}^i\mathtt{0} : i \text{ is an odd integer}\}$. For two distinct prefixes $u = \mathtt{01}^i\mathtt{0}$ and $v = \mathtt{01}^j\mathtt{0}$ in F (without loss of generality assuming $i < j$), consider the suffix $z = \mathtt{01}^i\mathtt{00}$.

- $uz = \mathtt{01}^i\mathtt{001}^i\mathtt{00}$; by taking $w = \mathtt{01}^i\mathtt{0}$ and $x = \mathtt{0}$, this shows that uz is in F.
- $vz = \mathtt{01}^j\mathtt{001}^i\mathtt{00}$. Because j is odd, ww^R cannot be of the form $\mathtt{01}^j\mathtt{0}$; which means the first run of `1`s must lie in w completely. But then there are not enough `1`s in the rest of the word to form w^R. Therefore, not matter what x is, word vz cannot be of the form $ww^R x$. This shows that vz is not in F.

This implies that F is a fooling set of infinite size, and thus L_d is not regular. ■

Rubric: Standard 5-point grading scale (plus deadly-sins and sudden-death rules) *for each subproblem*, scaled to 2.5 points. (Thus 10 points in total for problem 1.) Maximum 0.5 points if one tries to prove a regular language to be non-regular, or vice versa. Maximum 0.5 points if the fooling set is in fact not fooling.

Full credit for subproblem (d) if one correctly proves the language to be regular *when Σ is unary*. This was an oversight.

2. ***Telling DFAs apart.***

Let M_1 and M_2 be two DFAs, each with exactly n states. Assume that the languages associated with the two machines are different (that is, $L(M_1) \neq L(M_2)$), there is always some string in the symmetric difference of the two languages.

Prove that there is a string w of length polynomial in n in the symmetric difference of $L(M_1)$ and $L(M_2)$. What is the best upper bound you can get on the length of w?

Solution: First we construct a DFA M', described by $(Q', s', A', \Sigma', \delta')$, that recognizes the symmetric difference of the two languages $L(M_1)$ and $L(M_2)$, using the product construction. Denote M_i by the tuple $(Q_i, s_i, A_i, \Sigma_i, \delta_i)$ for $i \in \{1, 2\}$.

- States Q': $Q_1 \times Q_2$ — pairs of states, one from each M_i
- Starting state s': (s_1, s_2)
- Accepting states A': $\{(r_1, r_2) \in Q' : \text{either } r_1 \in A_1 \text{ and } r_2 \notin A_2, \text{ or } r_1 \notin A_1 \text{ and } r_2 \in A_2\}$
- Alphabet Σ': $\Sigma_1 \cup \Sigma_2$
- Transition function δ': $\delta_1 \times \delta_2$, mapping $\delta'((q_1, q_2), \mathtt{a})$ to $(\delta_1(q_1, \mathtt{a}), \delta(q_2, \mathtt{a}))$ on reading any symbol $\mathtt{a} \in \Sigma'$

DFA M' recognizes the symmetric difference of the two languages $L(M_1)$ and $L(M_2)$ and has n^2 states. Now by problem statement M' accepts at least one word. Now any walk from the starting state s' to an accepting state in M' can be turned into a simple path between the same two endpoints, without ever visiting the same state twice. This shows that there is a word of length at most n^2 that is accepted by M', and thus in the symmetric difference of $L(M_1)$ and $L(M_2)$. ■

Rubric: Any complete solution with a (justified) polynomial bound receives full credit. Any *subquadratic* bound receives extra credit.

Standard 5-point grading scale plus deadly-sins and sudden-death rules.