Exponential-time hypotheses

Friday, May 9, 2025 1:23 PM

Assuming $P \neq NP$. (CNFSAT $\notin$ P) all NP-hard problems can't be solved in poly-time.

No subexp. time algorithms in practice!

COLORING. INDSET. DOMSET. HITTINGSET. HAMPATH ...

n^c poly-time. $2^{\sqrt{n}}$ subexp. $c^{n^{o(1)}}$ exp. time.

CNFSAT: n variables. m clauses. $m = O(n)$ by sparsification lemma

k-SAT : k literals per clause.

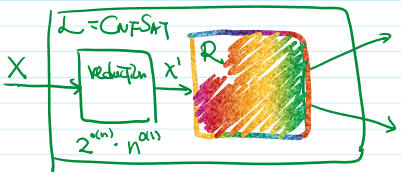
ETH: CNFSAT can't be solved in $2^{o(n)}$ time.

k-SAT " $2^{s_k \cdot n}$ for some s_k. [IPZ'01]

↳ $s_k \sim (1 - \frac{1}{\Theta(k)})$

Subexp-time reduction.

- subexp. time ($2^{o(n)} \cdot \text{poly}$ $\forall \varepsilon > 0$) during reduction.
- ⊙ near-linear blowup to input size. $N \sim \tilde{O}(n)$



$L \leq R$

size X : n

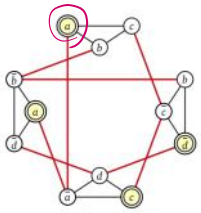
size X' : $n^2 =: N$

We solve R in $2^{o(\sqrt{N})}$ time

$\Rightarrow$ We solve L in $2^{o(n)}$ time

example. MAXINDSET

G:



ϕ: $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

m = # clauses

3SAT (ϕ : every variable occurs 3 times.) $\leq$ MAXINDSET

$\phi \longmapsto (G, m)$ = # clauses in ϕ.

size of G: $O(n+m)$

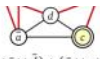
$|V|$: $3m$

$|E|$: $O(1) \cdot$ #variables

ϕ' : n' variables, m' clauses

n' : $3m$.

m' : $m + 3m = O(m)$



ϕ: $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

$|E| : O(1) \cdot \#\text{variables} \quad (n)$ $\qquad m' : m + 3m = O(m)$

m : #clauses.

Cor. MaxIndSet not in $2^{o(n)}$ time assuming ETH.

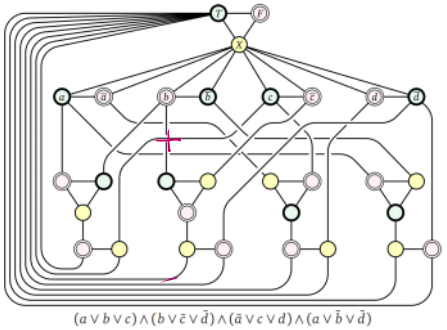
Claim. We can modify ϕ s.t. every variable occurs 3 times.

$(x_1 \vee a \vee b) \wedge \cdots \wedge (x_2 \vee c \vee d) \wedge (\bar{x}_3) \wedge (\bar{x}_4)$

$(x_1 \Rightarrow x_2) \wedge (x_2 \Rightarrow x_3) \wedge (x_3 \Rightarrow x_4) \wedge (x_4 \Rightarrow x_1)$

$(\neg x_1 \vee x_2)$

example. 3Color



$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

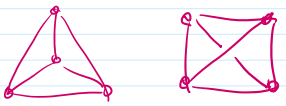
3Sat $\leq$ 3Color.

$\phi \mapsto G$

size of G: $O(n+m)$

$|V| : 3 + 2n + 5m$.

$|E| : O(1) + O(n) + O(m) = O(n+m)$



Cor. 3Color can't be solved in $2^{o(n)}$ time assuming ETH.

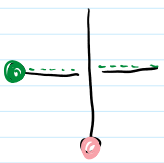
Planar3Color

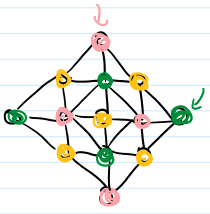
Input: planar graph G

output: Is G 3-colorable?

3Sat $\leq$ Planar3Color

Construct $\bar{G}$:



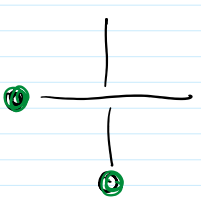


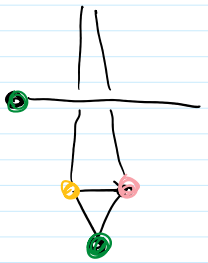
size of $\bar{G}$:

m edges in G

$\Rightarrow O(m^2)$ crossings.

$\Rightarrow O(m^2)$ edges in $\bar{G}$





Cor. Planar3Color can't be solved in $2^{o(\sqrt{n})}$ time.

HAMCycle

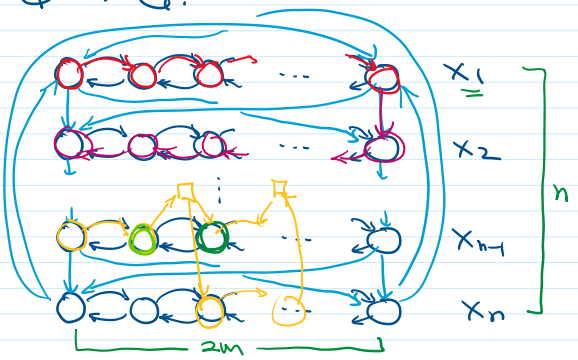
input: directed graph G.

output: ∃ cycle through every vertex in G?

$3SAT \leq_{\phi} HAMCycle_{\vec{G}}$

pf sketch. $\phi \Rightarrow \vec{G}$.

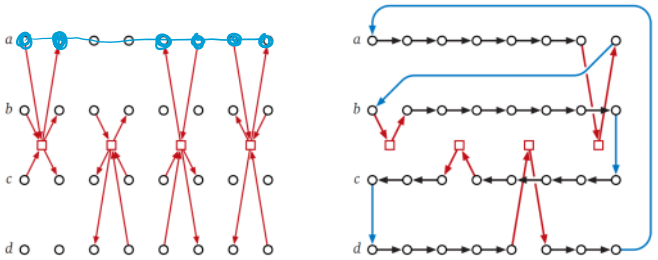
variable:



size of $\vec{G}$:

$V = n \cdot 2m$

clause:



size $\vec{G}'$:

$|V| = 6 \cdot n$

Cor. HAMPATH can't be solved in $2^{o(n)}$ time

Question. How to make $\vec{G}$ linear in size?