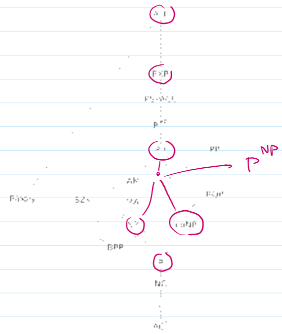
# NP-hardness reductions

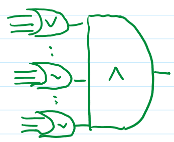
Wednesday, May 7, 2025 12:42 PM

To prove Problem A is NP-hard, reduce known NP-hard problem to A




## 3SAT

input: 3-CNF formula
output: Is the formula satisfiable?



$\phi: (a \vee b \vee c \vee d) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b})$ Clauses CNF

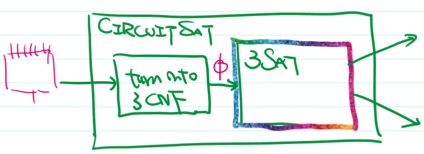
OR AND

NP-hard. NP-hard!

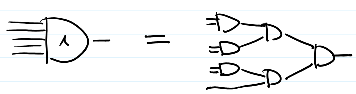
CIRCUITSAT $\leq$ 3SAT :

turn ✓1. circuit into 3-CNF formula ✓4. in poly time

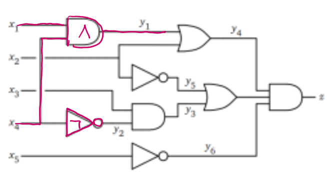
s.t. ✓2. yes-inst. $\Rightarrow$ yes-inst.
3. no- no-




mapping reduction.



Pf.




$y_1 = x_1 \wedge x_4$

$y_2 = \neg x_4$

$(y_1 = x_1 \wedge x_4) \wedge (y_2 = \overline{x_4}) \wedge (y_3 = x_3 \wedge y_2) \wedge (y_4 = y_1 \vee x_2) \wedge$
$(y_5 = \overline{x_2}) \wedge (y_6 = \overline{x_5}) \wedge (y_7 = y_3 \vee y_5) \wedge (z = y_4 \wedge y_7 \wedge y_6) \wedge z$

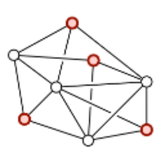
$a = b \wedge c \longmapsto (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee b) \wedge (\bar{a} \vee c)$
$a = b \vee c \longmapsto (\bar{a} \vee b \vee c) \wedge (a \vee \bar{b}) \wedge (a \vee \bar{c})$
$a = \bar{b} \longmapsto (a \vee b) \wedge (\bar{a} \vee \bar{b})$

| a | b | c | a=b∧c | ( )∧( )∧( ) |
|---|---|---|---|---|
| 0 | 0 | 0 | | |
| 0 | 0 | 1 | | |
| | ⋮ | | | |
| 1 | 1 | 1 | | |

$(a \vee b) \mapsto (a \vee b \vee x) \wedge (a \vee b \vee \bar{x})$

## MaxIndSet

input: graph G, int k.
output: largest ind. set in G $\geq k$?



NP-hard. NP-hard.

3SAT $\leq$ MaxIndSet

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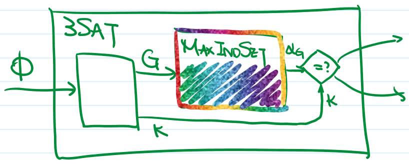
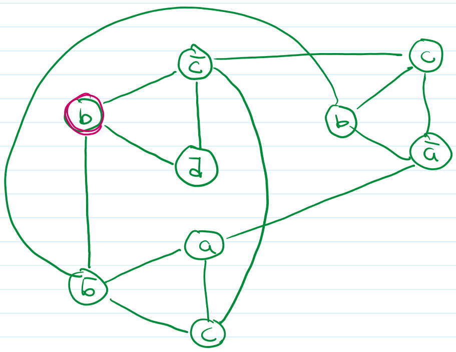
3CNF $\Phi$ $\longmapsto$ $(G, k)$




pf sketch. $\Phi \Rightarrow (G, k)$



Build G: clause gadget:

$(b^{T} \vee \bar{c}^{T} \vee \bar{d}^{T})$

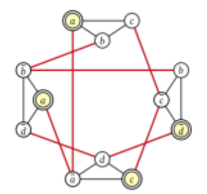
$\wedge (a \vee \bar{b}^{F} \vee c)$

conflict gadget:

Set k: #clauses.

$\Phi$ sat. $\Rightarrow$ (G, k) has ind. set of size k ✓

$\Phi$ un-sat. $\Rightarrow$ (G, k) doesn't have ind. set of size k



$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

a = T  c = T  d = F  b = F

MaxIndSet $(G, S)$ MaxClique $(\bar{G}, S)$ MinVertexCover $(G, \bar{S})$

3Color

| |
|---|
| input: graph G<br>output: Is G 3-colorable? |

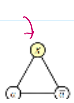
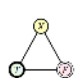
NP-hard. NP-hard.

3SAT $\leq$ 3Color

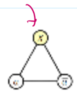
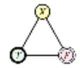
$\Phi$ G

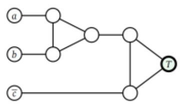
pf sketch. $\Phi \Rightarrow G$.

Build G:

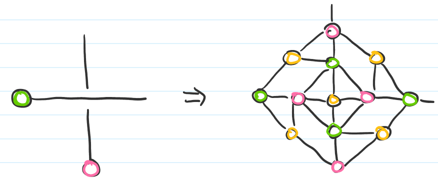
- variable gadget.  • Truth gadget. 

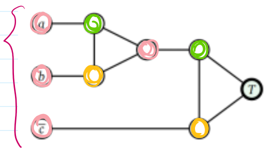
Build G :

- variable gadget.  • Truth gadget. 

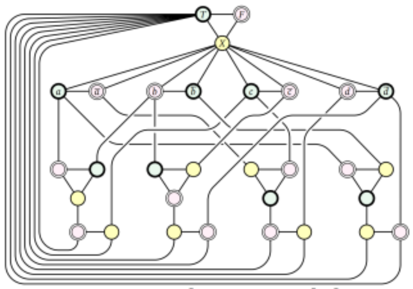
- clause gadget. 
  $(a \vee b \vee \bar{c})$

algorithm design! in 3COLOR language.





- $\phi$ sat. $\Rightarrow$ G 3-colorable
- $\phi$ sat. $\Leftarrow$ G 3-colorable



$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

**You may assume the following problems are NP-hard:**

**CircuitSat:** Given a boolean circuit, are there any input values that make the circuit output True?

**3Sat:** Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment? ] gadgets. problems w/ 3.

**MaxIndependentSet:** Given an undirected graph $G$, what is the size of the largest subset of vertices in $G$ that have no edges among them?

**MaxClique:** Given an undirected graph $G$, what is the size of the largest complete subgraph of $G$? ] largest object

**MinVertexCover:** Given an undirected graph $G$, what is the size of the smallest subset of vertices that touch every edge in $G$? ] smallest object

**3Color:** Given an undirected graph $G$, can its vertices be colored with three colors, so that every edge touches vertices with two different colors? ] partition into subsets.

**HamiltonianPath:** Given an undirected graph $G$, is there a path in $G$ that visits every vertex exactly once?

**HamiltonianCycle:** Given an undirected graph $G$, is there a cycle in $G$ that visits every vertex exactly once?

**DirectedHamiltonianCycle:** Given an directed graph $G$, is there a directed cycle in $G$ that visits every vertex exactly once?

**TravelingSalesman:** Given a graph $G$ (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in $G$? ] ordering stuff.

**Draughts:** Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

**Super Mario:** Given an $n \times n$ level for Super Mario Brothers, can Mario reach the castle? ] don't.