# Regular languages are automatic: NFAs

Wednesday, April 9, 2025 12:55 PM

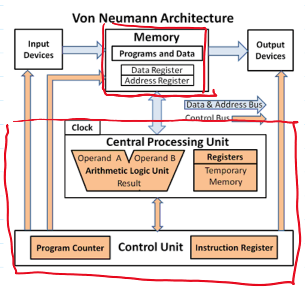
Administrivia:

- HW1 due on Wednesday (4/16)
- Quiz next Monday (4/14)

Modeling/Philo.

Question. Does DFA really captures CPU...?

CPU




Regular: representable by reg. expressions.

Automatic: accepting by DFAs.

Let's try to prove our first nontrivial result:

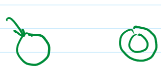
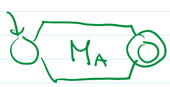
All regular languages are automatic.

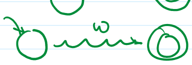
$\Rightarrow$ Construct DFA for a R.E.

Reg. expression:

| | | | |
|---|---|---|---|
| $\phi$ | $\phi$ | | construct DFAs. |
| $\omega$ | $\{\omega\}$ | $\forall \omega \in \Sigma^*$ | |
| $A+B$ | $L(A) \cup L(B)$ | $\forall$ reg. $A, B$ | combine DFAs. |
| $AB$ | $L(A) \cdot L(B)$ | " | |
| $A^*$ | $L(A)^*$ | " | |

Proof (attempt)

$\phi$:  

$\omega$: 

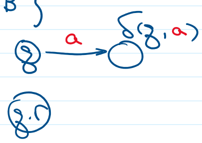
$A+B$: $\exists$ DFAs $M_A, M_B$

"Product construction"

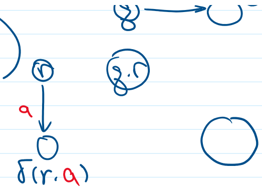
$Q := Q_A \times Q_B \quad s := (s_A, s_B) \quad \Sigma := \Sigma_A \cup \Sigma_B$

$A := \{(q, q') \in Q : q \in A_A \text{ or } q' \in A_B\}$

$\delta((q,r), a) := (\delta(q,a), \delta(r,a))$

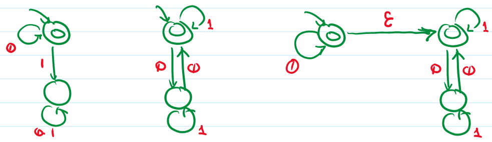


$$\delta((q,r),a) := \Big(\delta(q,a),\ \delta(r,a)\Big)$$



$A \circ B$:

example $L_A = 0^*$ $L_B = \{ w : \text{even } \#0s \}$ $\underbrace{00}_{x}\underbrace{1101}_{y}$ $L_A \circ L_B$



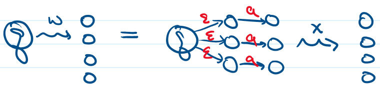
What if we augment DFA w/ $\varepsilon$-Transitions?

Finite Automata w/ $\varepsilon$-Transitions:

- $Q$, $s$, $A$.
- $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $\delta : 2^Q \times \Sigma_\varepsilon \to \underbrace{2^Q}_{\text{powerset of } Q}$

$$\varepsilon\text{-Reach}(q) := \{ r \in Q : q \overset{\varepsilon}{\rightsquigarrow} r \}$$

$$\delta^*(q,w) := \begin{cases} \varepsilon\text{-Reach}(q) & \text{if } w = \varepsilon \\ \delta^*\big(\delta(\varepsilon\text{-Reach}(q), a), x\big) & \text{if } w = ax \\ & a \in \Sigma,\ x \in \Sigma^*. \end{cases}$$

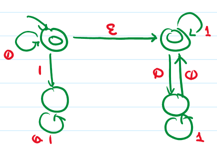


$$\left( \delta(S,a) := \bigcup_{q \in S} \delta(q,a), \quad \delta^*(S,w) := \bigcup_{q \in S} \delta^*(q,w) \right)$$

M accepts w if $\delta^*(s,w)$ contains at least one accepting state

"One of us is successful"

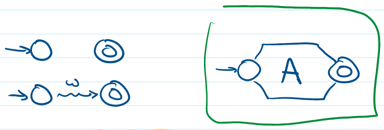
example $L_A = 0^*$ $L_B = \{ w : \text{even } \#0s \}$ 001101



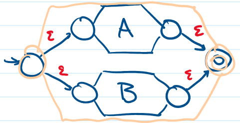
Proof.

$\phi$:

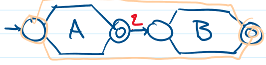
$\omega$:



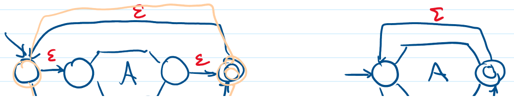
A+B:



$A \circ B$:

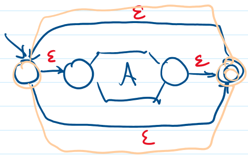


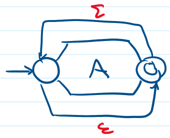
$A^*$



Why not?

$A^*$

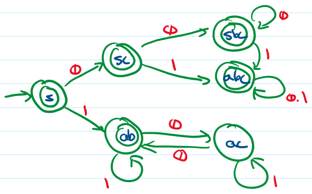



why not?

<u>Thm.</u> Every regular language is ε-automatic.

$$\begin{tikzcd} \text{R.E.} \arrow[r, dashed] \arrow[rr, bend right] & \text{DFA} \arrow[r] & \varepsilon\text{DFA} \arrow[l, bend right] \end{tikzcd}$$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