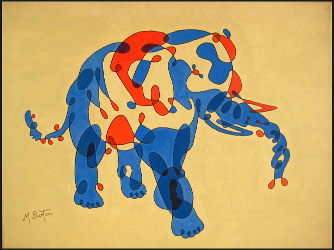
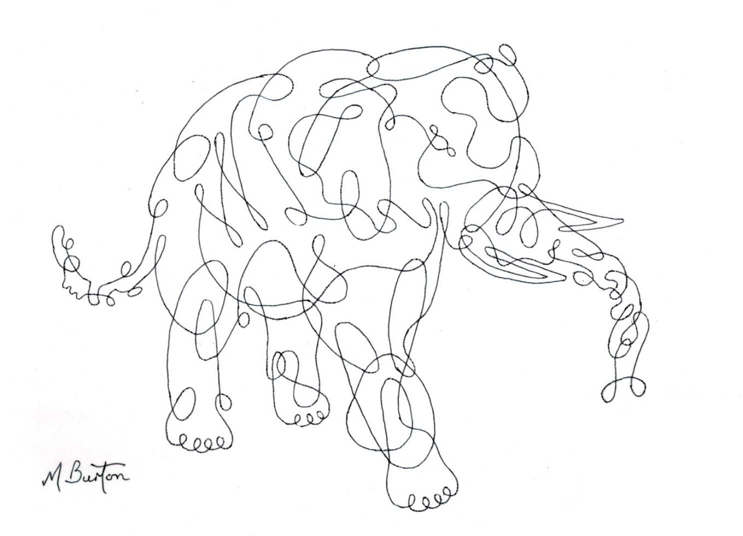
# Untangling Planar Curves



*Hsien-Chih Chang* & Jeff Erickson
University of Illinois at Urbana-Champaign

SoCG 2016, Boston

# How to simplify a doodle?




Draft of “Fluorescephant”, Mick Burton, 1973

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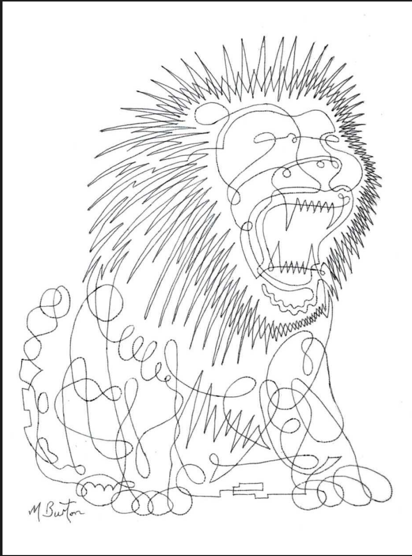
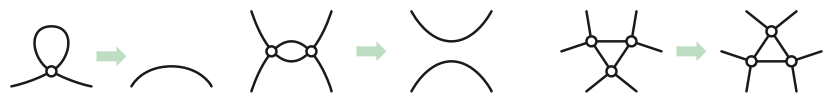



"Lion" in Continuous Line and Colour Sequence, Mick Burton, 2012

# Homotopy moves



1→0 2→0 3→3

# How many?

# Previous bounds

- $\mathbf{O(n^2)}$ moves are always enough
  - regular homotopy (no $1 \leftrightarrow 0$ moves) [Francis 1969]
  - electrical transformations [Steinitz 1916, Feo and Provan 1993] (close reading to [Truemper 1989, Noble and Welsh 2000])
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# Which one?

$$\boldsymbol{\Theta(n)?\quad \Theta(n^2)?}$$

# Our Result

$$\Theta\left(n^{3/2}\right)$$

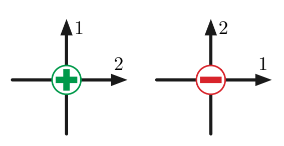
$\Omega(n^{3/2})$ homotopy moves

# Defect

[Arnold 1994, Aicardi 1994]

$$\delta(\gamma) \coloneqq -2 \sum_{x \between y} \operatorname{sgn}(x) \cdot \operatorname{sgn}(y) \qquad \text{[Polyak 1998]}$$

- $x \between y$ means $x$ and $y$ are *interleaved* — $x, y, x, y$
- $\operatorname{sgn}(\cdot)$ follows Gauss convention



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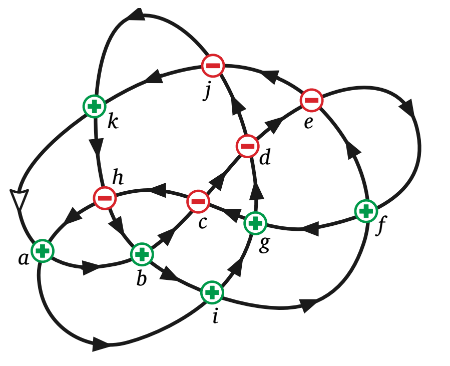

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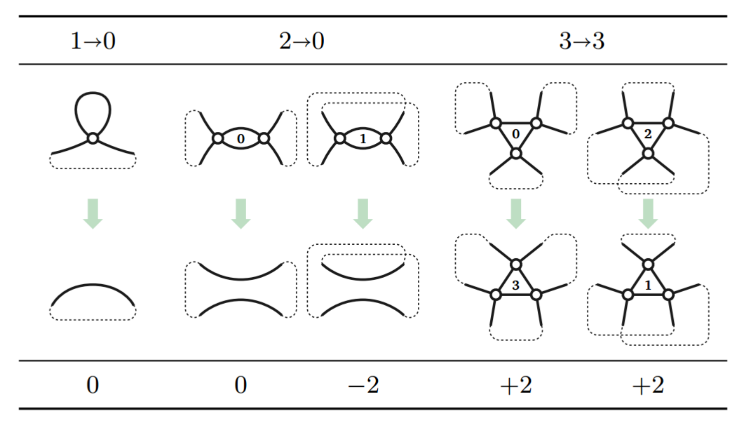
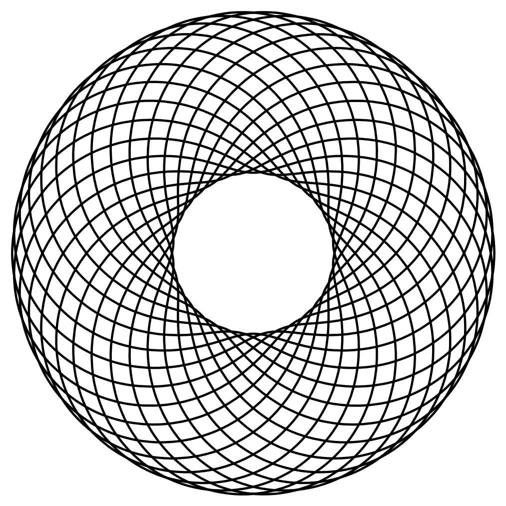


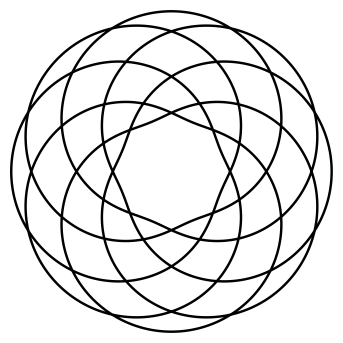

defect changes by at most 2 under any homotopy moves

# Flat torus knots $T(p, q)$

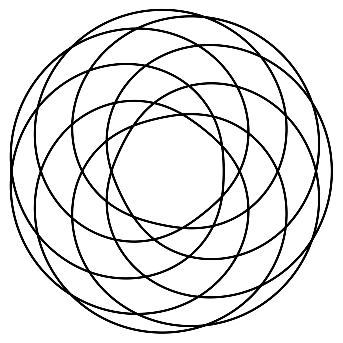


$(\mathbf{p}-\mathbf{1})\mathbf{q}$ intersection points

# Flat torus knots $T(p, q)$



$T(7, 8)$

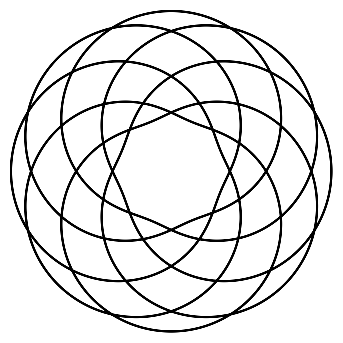


$T(8, 7)$

$\delta(\mathbf{T}(\mathbf{p}, \mathbf{p}+\mathbf{1})) = \mathbf{2}\binom{\mathbf{p}+\mathbf{1}}{\mathbf{3}}$
[Even-Zohar et al. 2014]

$\delta(\mathbf{T}(\mathbf{q}+\mathbf{1}, \mathbf{q})) = -\mathbf{2}\binom{\mathbf{q}}{\mathbf{3}}$
[Hayashi et al. 2012]

# Flat torus knots $T(p, q)$



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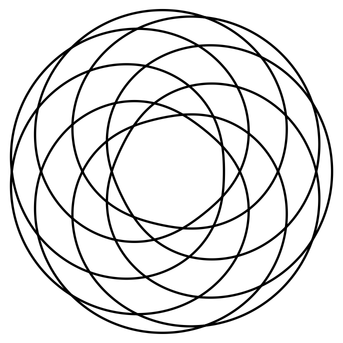


$T(8, 7)$

$\delta(T(p, p+1)) = 2\binom{p+1}{3}$

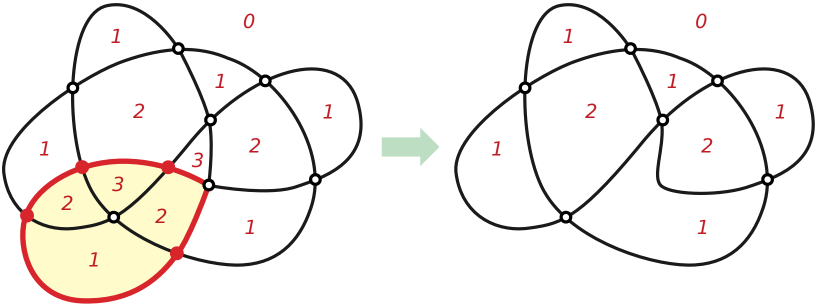
[Even-Zohar et al. 2014]

$\delta(T(q+1, q)) = -2\binom{q}{3}$

[Hayashi et al. 2012]

$\mathbf{O(n^{3/2})}$ homotopy moves

# Loop reductions



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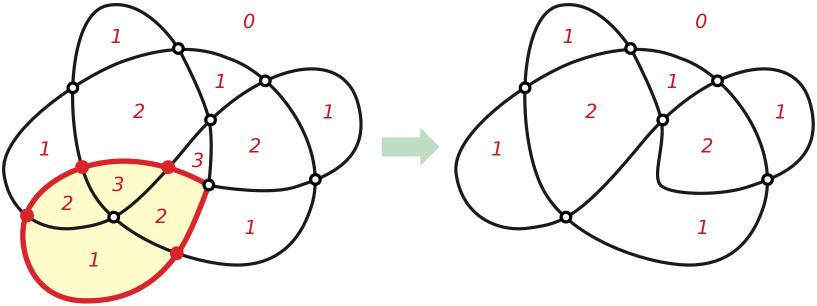


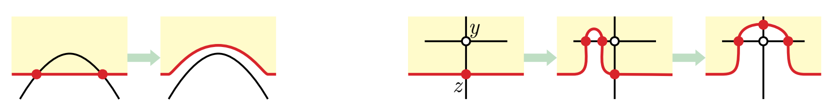


- at most $O(A)$ moves, where **A** is number of interior faces
- face-depth potential $\Phi$ decreases by at least $A$

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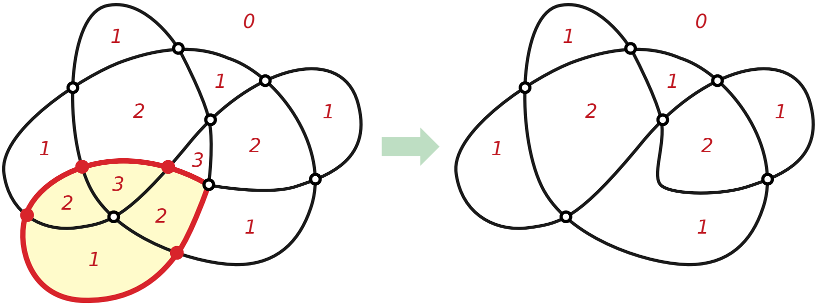


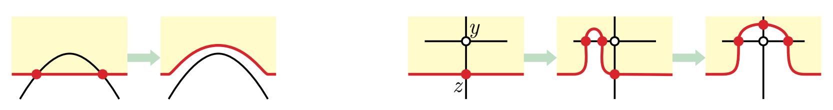


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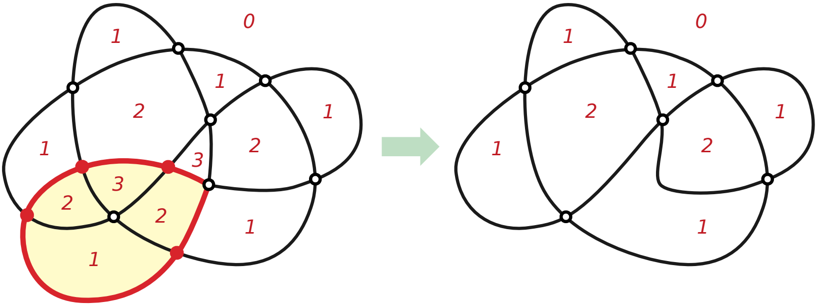


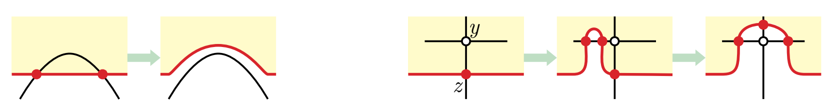


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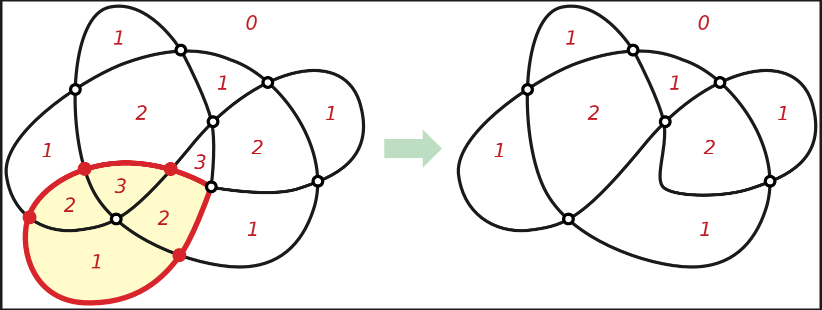
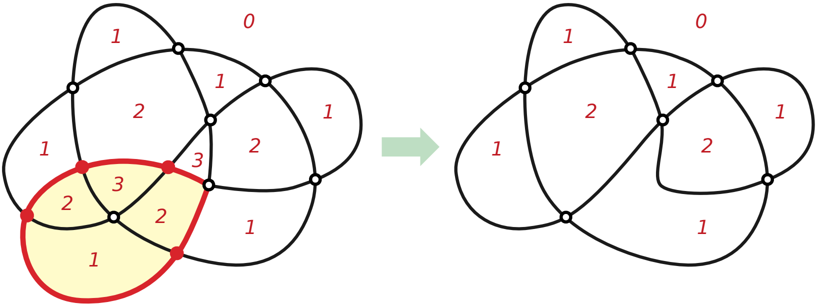



- $O(\Phi) = O(n^2)$ homotopy moves
- Why does the depth matter?

# Loop reductions

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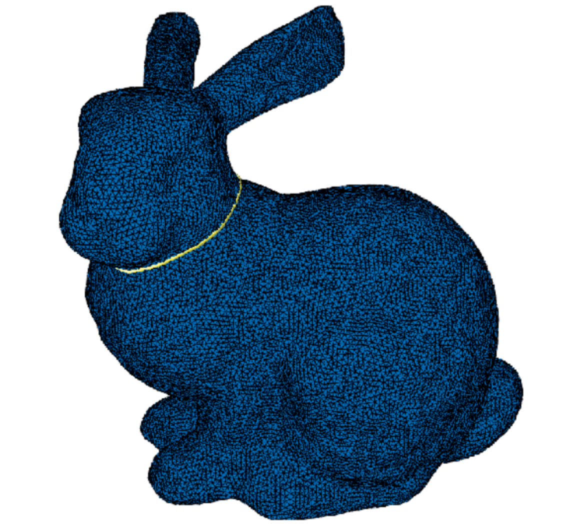
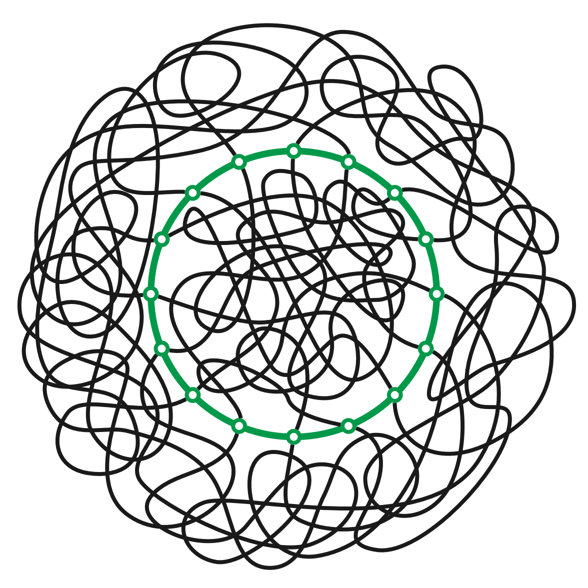
# Useful cycle technique



From "Choking Loops on Surfaces", Feng and Tong, 2013

# Tangle



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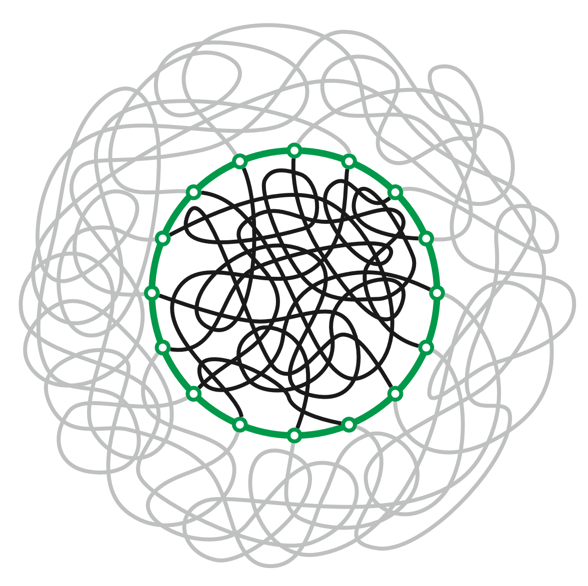

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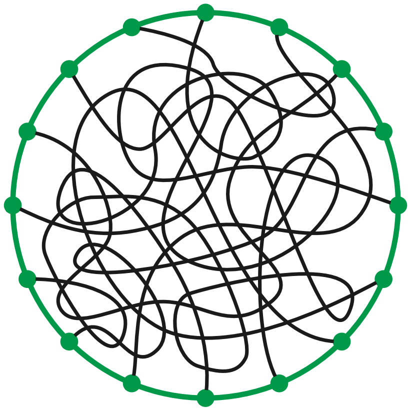
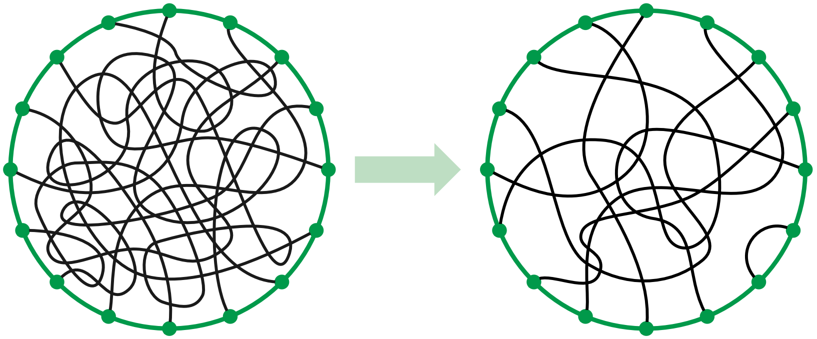


**m** vertices, **s** strands, max-depth **d**

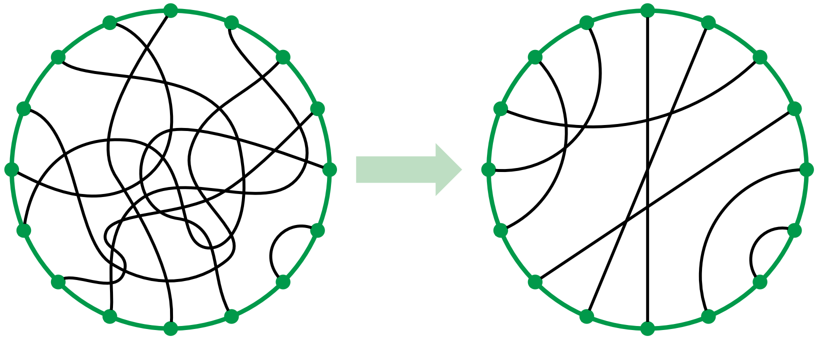
# Tangle reductions

- First, remove all the self-loops in $\mathbf{O(md)}$ moves



# Tangle reductions

- Second, straighten all strand in $\mathbf{O}(\mathbf{ms})$ moves



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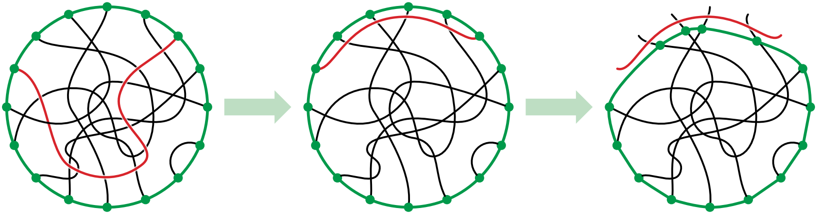
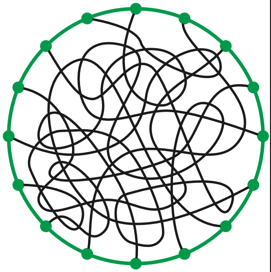


# Useful tangle

- A tangle is *useful* if $\mathbf{s} \leq \mathbf{m}^{1/2}$ and $\mathbf{d} = \mathbf{O}(\mathbf{m}^{1/2})$
- At least $\mathbf{\Omega}(\mathbf{m})$ vertices removed
- Tightening one useful tangle:
  $\mathbf{O}(\mathbf{md} + \mathbf{ms}) = \mathbf{O}(\mathbf{m}^{3/2})$ moves



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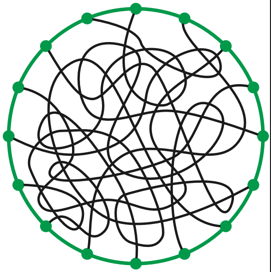

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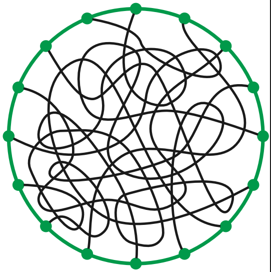
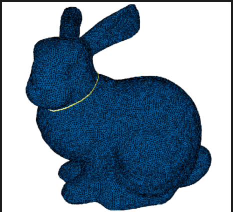


# Amortized analysis

- Algorithm: Tighten *any* useful tangle until the curve is simple



- In total $\mathbf{O}(\mathbf{n}^{3/2})$ homotopy moves

- How do we know that there is always a useful tangle?

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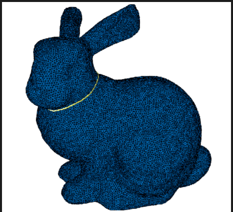

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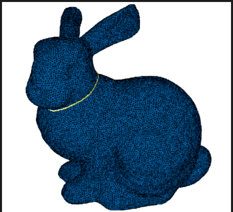

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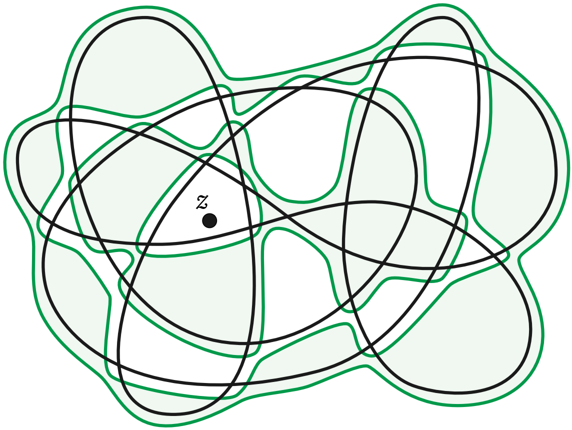
# Finding useful tangle

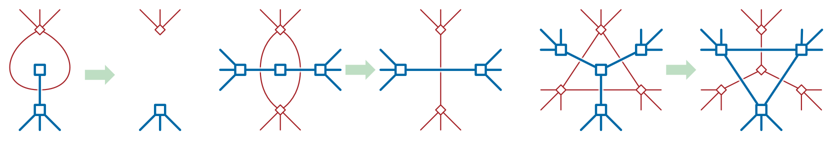



- Either one of them is useful, or the max-depth is $O(n^{1/2})$

# Future work & open questions

# Electrical transformations

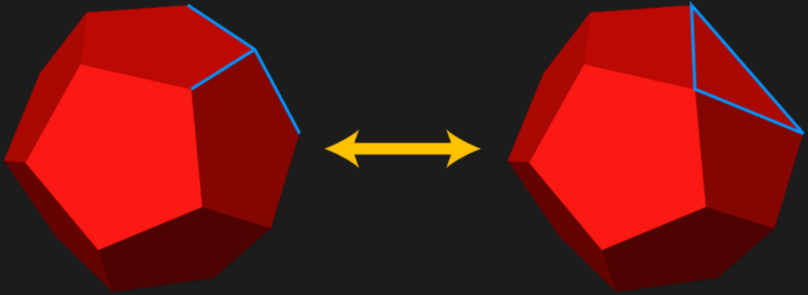
[Kennelly 1899]



degree-1 series-parallel ΔY transformation

# Steinitz's theorem

[Steinitz 1916, Steinitz and Rademacher 1934]



From page "Steinitz's theorem" in Wikipedia, David Eppstein

# Many more applications

- Shortest paths and maximum flows [Akers, Jr. 1960]
- Estimating network reliability [Lehman 1963];
- Multicommodity flows [Feo 1985]
- Kernel on surfaces [Schrijver 1992]
- Construct link invariants [Goldman and Kauffman 1993]
- Counting spanning trees, perfect matchings, and cuts [Colbourn et al. 1995]
- Evaluation of spin models in statistical mechanics [Jaeger 1995]
- Solving generalized Laplacian linear systems [Gremban 1996, Nakahara and Takahashi 1996]
- Kinematic analysis of robot manipulators [Staffelli and Thomas 2002]
- Flow estimation from noisy measurements [Zohar and Gieger 2007]

# Previous bounds on electrical transformations

- Finite [Epifanov 1966, Feo 1985]
- A simple $\mathbf{O}(\mathbf{n}^3)$ algorithm
  - grid embedding [Truemper 1989]
- $\mathbf{O}(\mathbf{n}^2)$ steps are always enough
  - bigon reduction [Steinitz 1916]
  - depth-sum potential [Feo and Provan 1993]

# Feo and Provan Conjecture

$$\Theta(\mathbf{n}^{3/2})$$

# Higher genus surfaces

- How many homotopy moves needed to reduce curves on surfaces?
  - homotopic to simple curve: $O(n^2)$ moves [Hass and Scott 1985]
  - $\Omega(n^2)$ moves for non-contractible curves

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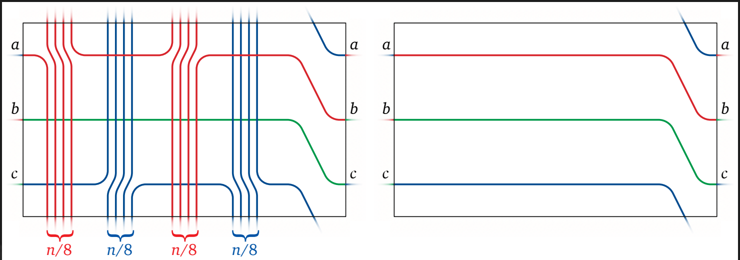

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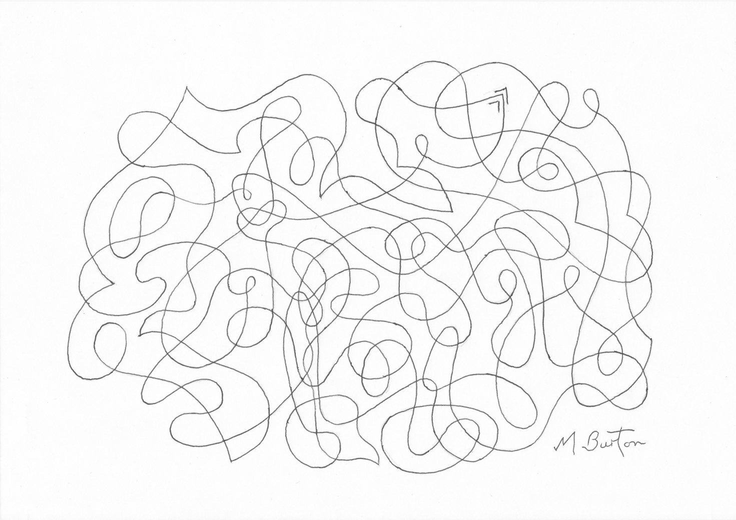
  - $\Omega(n^2)$ moves for non-contractible curves
  - no polynomial bound in general

# Higher genus surfaces

- How many homotopy moves needed to reduce curves on surfaces?

- **Conjecture.**
  - contractible: $\mathbf{O(n^{3/2})}$ moves
  - general: $\mathbf{O(n^2)}$ moves

# Questions?



# Thank you!