

Spectral Aspects of Symmetric Matrix Signings*

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Abstract

The spectra of signed matrices have played a fundamental role in social sciences, graph theory, and control theory. In this work, we investigate the computational problems of *finding* symmetric signings of matrices with natural spectral properties. Our results are the following:

1. We characterize matrices that have an invertible signing: a symmetric matrix has an invertible symmetric signing *if and only if* the support graph of the matrix contains a perfect 2-matching. Further, we present an efficient algorithm to search for an invertible symmetric signing.
2. We use the above-mentioned characterization to give an algorithm to find a minimum increase in the support of a given symmetric matrix so that it has an invertible symmetric signing. We also show a quantitative version of the above-mentioned characterization by presenting a non-trivial lower bound on the number of invertible symmetric signed adjacency matrices.
3. We show NP-completeness of the following three problems: verifying whether a given matrix has a symmetric signing that is positive semi-definite/singular/has bounded eigenvalues. However, we also illustrate that the complexity could differ substantially for input matrices that are adjacency matrices of graphs.

We use combinatorial techniques in addition to classic results from matching theory. Our algorithm for solving the search problem mentioned in (1) might be helpful in identifying families of polynomials that admit a constructive proof for Combinatorial Nullstellensatz.

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1 Introduction

The spectra of several graph-related matrices such as the adjacency and the Laplacian matrices have become fundamental objects of study in computer science research. They have had a tremendous impact in several areas including machine learning, data mining, web search and ranking, scientific computing, and computer vision. In this work, we undertake a systematic and comprehensive investigation of the spectrum and the invertibility of *symmetric signings* of matrices. We study natural spectral properties of symmetric signed matrices and address the computational problems of finding and verifying the existence of symmetric signings with these spectral properties.

For a real symmetric $n \times n$ matrix M and an $n \times n$ matrix s taking values in $\{\pm 1\}$ —which we refer to as a *signing*—we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s . We say that s is a *symmetric signing* if s is a symmetric matrix and an *off-diagonal signing* if s takes value $+1$ on the diagonal. Signed adjacency matrices (respectively, Laplacians) correspond to the case where M is the adjacency matrix (respectively, Laplacian) of a graph. We recall that a real symmetric matrix is *positive semi-definite* (psd) if all its eigenvalues are non-negative. We study the following computational problems:

INVERTIBLESIGNING: Given a real symmetric matrix M , verify if there exists a symmetric signing s such that $M(s)$ is invertible (that is, non-singular).

SINGULARSIGNING: Given a real symmetric matrix M , verify if there exists an off-diagonal symmetric signing s such that $M(s)$ is singular.

BOUNDEDEVALUESIGNING: Given a real symmetric matrix M and a real number λ , verify if there exists an off-diagonal symmetric signing s such that the largest eigenvalue $\lambda_{\max}(M(s))$ is at most λ .

PSDSIGNING: Given a real symmetric matrix M , verify if there exists a symmetric signing s such that $M(s)$ is positive semi-definite.

1.1 Motivations

Invertible signings and Combinatorial Nullstellensatz. INVERTIBLESIGNING can be reformulated as verifying whether a natural multivariate polynomial associated with a given matrix is non-vanishing for some point in the domain $\{\pm 1\}^{n \times n}$. Combinatorial Nullstellensatz is a landmark result that gives a condition under which a multivariate-polynomial has a non-vanishing point within a domain of interest.

Theorem 1.1 (Combinatorial Nullstellensatz [4]). *Let F be an arbitrary field and f be a multivariate polynomial over F with variables x_1, \dots, x_n . Let t_1, \dots, t_n be non-negative integers such that the degree of f equals $\sum_{i=1}^n t_i$, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is non-zero. Given subsets S_1, \dots, S_n of F with $|S_i| > t_i$ for each i , there is a tuple (s_1, \dots, s_n) satisfying $f(s_1, \dots, s_n) \neq 0$, where each s_i is selected from S_i .*

Since its discovery by Alon [4], Combinatorial Nullstellensatz has served as a powerful algebraic tool to show the existence of certain combinatorial structures, similar to the Lovász Local Lemma [17] and Spencer’s Discrepancy Theorem [54]. However, in contrast to the Lovász Local Lemma and Spencer’s Discrepancy Theorem which have been made constructive [38, 43], all known proofs of Combinatorial Nullstellensatz are inherently non-constructive—that is, the proofs show the existence of a non-vanishing point without giving an efficient algorithm to find such a point [4, 33, 34, 41, 60]. A long-standing open problem is whether one can make Combinatorial Nullstellensatz constructive.

Namely, given a non-zero polynomial f , and a set $S := S_1 \times S_2 \times \cdots \times S_n$ satisfying the hypothesis of Theorem 1.1, is there an efficient algorithm to find a point $s \in S$ such that $f(s) \neq 0$?

In this work, we first use Combinatorial Nullstellensatz to obtain an *efficient combinatorial characterization* for the existence of an invertible symmetric signing of a given matrix. This result implies an efficient algorithm to solve INVERTIBLESIGNING. However, since the proof of the characterization relies on Combinatorial Nullstellensatz which is non-constructive in nature, we do not immediately obtain an efficient algorithm to *find* an invertible symmetric signing. This raises the following search problem as a special case of obtaining a constructive proof of Combinatorial Nullstellensatz:

SEARCHINVERTIBLESIGNING: Given a real symmetric matrix M , verify if there exists a symmetric signing s such that $M(s)$ is invertible and if so, *find such a signing*.

Our motivations behind investigating the complexity of SEARCHINVERTIBLESIGNING are twofold: First, it is natural to design an algorithm for a non-trivial application before attempting to design an algorithm for Combinatorial Nullstellensatz for broad families of polynomials (similar to how Moser’s algorithm for k -CNF satisfying the Lovász Local Lemma conditions [42] was generalized subsequently to obtain a constructive proof of the *general* Lovász Local Lemma [43]). Moreover, given the recent results of Belovs *et al.* [9] who show that Combinatorial Nullstellensatz over F_2 *under certain restricted input models* is complete for the complexity class PPA, the next natural question is to identify families of polynomials and input models under which Combinatorial Nullstellensatz admits a constructive proof.

As a main result of this work, we resolve the complexity of SEARCHINVERTIBLESIGNING favorably by designing an efficient algorithm. We believe that our algorithm could be of independent interest as a means to construct some of the combinatorial structures guaranteed to exist through Combinatorial Nullstellensatz (by adapting our algorithm suitably for the application).

Finding the solvability index of a signed matrix. The notion of *balance* of a symmetric signed matrix has been studied extensively in social sciences [24, 26, 27, 32]. A signed adjacency matrix is *balanced* if there is a partition of the vertex set such that all edges within each part are positive, and all edges in between two parts are negative (one of the parts could be empty). A number of works [3, 23, 32, 53, 62, 63] have explored the problem of minimally modifying signed graphs (or signed adjacency matrices) to convert it into a balanced graph.

In this work, we introduce a related problem regarding symmetric signed matrices: Given a symmetric matrix M , what is the smallest number of non-diagonal zero entries of M whose replacement by non-zeroes gives a symmetric matrix M' that has an invertible symmetric signing? We define this quantity to be the *solvability index*¹. Knowing this number might be helpful in studying systems of linear equations in signed matrices that might be ill-defined, and thus do not have a (unique) solution and in minimally modifying such matrices so that the resulting linear system becomes (uniquely) solvable. We use classic graph-theoretic techniques to show that solvability index is indeed computable efficiently.

Spectra of signed matrices and expanders. Marcus, Spielman, and Srivastava [40] showed that the adjacency matrix of every d -regular bipartite graph *has* a symmetric signing whose

¹Our definition of *solvability index* is similar to the well-known notion of *frustration index* [1, 25]. The *frustration index* of a matrix M is the minimum number of non-zero off-diagonal entries of M whose deletion results in a *balanced signed graph*. Computing the frustration index of a signed graph is NP-hard [29].

eigenvalues are at most $2\sqrt{d-1}$. Making this result *constructive* would immediately lead to an efficient algorithm to construct bipartite Ramanujan simple graphs of all degrees². However, making the result of Marcus *et al.* *constructive* requires an efficient algorithm to solve the sub-problem: find a symmetric signing for a given d -regular bipartite graph for which the largest eigenvalue of the signed adjacency matrix is at most $2\sqrt{d-1}$. This naturally raises the computational problem—is there an efficient algorithm to find a symmetric signing that minimizes the largest eigenvalue of a given matrix? Such an algorithm would also solve the sub-problem. This in turn motivates the need to investigate the complexity of BOUNDEDEVALUESIGNING which is the decision variant of the computational problem.

1.2 Our Results

We note that INVERTIBLESIGNING is a decision problem while SEARCHINVERTIBLESIGNING is a search problem. The following is our main result.

Theorem 1.2. *There exists a polynomial-time algorithm to solve SEARCHINVERTIBLESIGNING.*

Our proof of Theorem 1.2 leads to a structural characterization for the existence of invertible signings through the existence of *perfect 2-matchings* in the support graph of the matrix. We discuss this structural characterization and a quantitative version of it in the subsequent section.

Next, we define the *solvability index* of a real symmetric matrix M to be the smallest number of non-diagonal zero entries that need to be converted to non-zeroes so that the resulting *symmetric* matrix has an invertible symmetric signing. We give an efficient algorithm to find the solvability index of a given symmetric matrix M . We emphasize that the support-increase operation that we consider preserves symmetry, that is, if we replace the zero entry $A[i, j]$ by α , then the zero entry $A[j, i]$ is also replaced by α .

Theorem 1.3. *There exists a polynomial-time algorithm to find the solvability index of a given real symmetric matrix.*

Theorems 1.2 and 1.3, in particular, imply that there exists an efficient algorithm to verify whether a given matrix has an invertible symmetric signing. We next focus on the remaining spectral properties mentioned in the introduction and consider the decision problem of verifying the existence of a signing to achieve these spectral properties. Intriguingly, the complexity of BOUNDEDEVALUESIGNING has not been studied in the literature even though it is widely believed to be a difficult problem in the graph sparsification community. We shed light on this problem by showing that it is NP-complete. Owing to the close connection between the maximum eigenvalue, positive semi-definiteness, and singularity (by suitable translations), we obtain that PSDSIGNING and SINGULARSIGNING are also NP-complete.

Theorem 1.4. BOUNDEDEVALUESIGNING, PSDSIGNING, and SINGULARSIGNING are NP-complete.

The hard instances generated by our proof of Theorem 1.4 are real symmetric matrices with non-zero diagonal entries and hence, it does not resolve the computational complexity of the problem of finding a signing of a given *graph-related* matrix (for example, the adjacency matrix) that minimizes its largest eigenvalue. Our next result provides some evidence that one might be able to design efficient algorithms to solve the NP-complete problems appearing in Theorem 1.4 for graph-related matrices. In particular, we show that SINGULARSIGNING and its search variant admit efficient algorithms when the input matrix corresponds to the adjacency matrix of a *bipartite* graph.

²While efficient construction of bipartite Ramanujan *multi-graphs* of all degrees is known [14], it still remains open to efficiently construct bipartite Ramanujan *simple* graphs of all degrees.

Theorem 1.5. *There exists a polynomial-time algorithm to verify if the adjacency matrix A_G of a given bipartite graph G has a symmetric signing s such that $A_G(s)$ is singular; and if so, find such a signing.*

1.2.1 Invertible Signings—Further Results

Theorem 1.2, in particular, implies that INVERTIBLESIGNING is solvable efficiently. In fact, our proof-techniques give an efficient characterization for the existence of an invertible signing. This characterization also leads to an alternative algorithm to solve INVERTIBLESIGNING. We believe that this characterization might be of independent interest and hence, describe it now.

Combinatorial characterization for invertible signings. The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $[n] := \{1, \dots, n\}$, and the edge set of G is $\{\{u, v\} \mid M[u, v] \neq 0\}$. We note that G could have self-loops depending on the diagonal entries of M . A *perfect 2-matching* in a graph G with edge set E is an assignment $x : E \rightarrow \{0, 1, 2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v). We show the following characterization:

Theorem 1.6. *Let M be a symmetric $n \times n$ matrix and let G be the support graph of M . The following are equivalent:*

1. *There exists a symmetric signing s such that the signed matrix $M(s)$ is invertible.*
2. *The support graph G contains a perfect 2-matching.*

Remark 1. For a subset S of vertices in a graph G , let $N_G(S)$ be the *non-inclusive neighborhood* of S , that is, $\{u \in V \setminus S \mid \{u, v\} \text{ is an edge of } G \text{ for some } v \text{ in } S\}$. A subset S of vertices is said to be *independent* if there are no edges between any pair of vertices in S . A subset S of vertices is said to be *expanding* in G if $|N_G(S)| \geq |S|$. Tutte [57] showed that the existence of a *non-expanding independent set* is equivalent to the absence of perfect 2-matchings in the graph, which in turn has been used in the study of independent sets [5, 13, 56]. Thus, Theorem 1.6 can be interpreted as a *spectral characterization* of graphs with non-expanding independent sets: a graph contains a non-expanding independent set if and only if every symmetric signed adjacency matrix of the graph is singular.

Remark 2. Tutte [57] gave an efficient algorithm to verify whether a given graph contains a perfect 2-matching. This algorithm along with Theorem 1.6 immediately leads to an efficient algorithm to solve INVERTIBLESIGNING.

Remark 3. We present a proof of Theorem 1.6 that is based on Combinatorial Nullstellensatz and is hence, non-constructive in nature. We are also aware of a proof through the DeMillo-Lipton-Schwartz-Zippel lemma [15, 52, 65] which is also non-constructive. These proofs do not immediately lead to an algorithm to solve SEARCHINVERTIBLESIGNING. In contrast, our algorithm for SEARCHINVERTIBLESIGNING mentioned in Theorem 1.2 is a *constructive proof* of Theorem 1.6. Moreover, our algorithm for SEARCHINVERTIBLESIGNING leads to a *constructive generalized characterization* for the existence of invertible signings for certain *asymmetric* sub-matrices of a given symmetric matrix. This general characterization is based on a generalization of perfect 2-matchings in graphs and is presented in Theorem 2.6.

Combinatorics of invertible signings. We next present a quantitative version of Theorem 1.6. We investigate the number of invertible signed adjacency matrices of graphs that contain a perfect 2-matching. One of the motivations behind studying the combinatorics is the design of randomized algorithms: if the number is sufficiently large, then it could reduce the amount of randomness in a popular algebraic algorithm for verifying the existence of a perfect matching in a bipartite graph due to Lovász [39].

It is well-known that flipping the signs on the edges of a *cut* preserves the spectrum of the signed adjacency matrix. Thus, the existence of one invertible signed adjacency matrix for a (connected) graph G also implies the existence of 2^{n-1} invertible signed adjacency matrices. In comparison, the lower bound obtained in our next result is much larger. We emphasize that our lower bound holds for general simple graphs (and not just bipartite graphs).

Theorem 1.7. *Let G be a simple graph with n vertices and m edges that has at least one perfect 2-matching. Then, the number of invertible signed adjacency matrices of G is at least $2^{m-n}/n!$.*

Theorem 1.7 shows that the fraction of invertible signed adjacency matrices of a graph G containing a perfect 2-matching is at least $2^{-O(n \log n)}$. An upper bound of $2^{-\Omega(n)}$ on the fraction is demonstrated by the graph that is a disjoint union of 4-cycles. While our bound in Theorem 1.7 does not reduce the amount of randomness needed in the algebraic algorithm, we believe that it could be of interest from the perspective of combinatorics. It is also worth noticing that the well-known DeMillo-Lipton-Schwartz-Zippel lemma [15, 52, 65] does not provide any non-trivial lower bound for this particular application.

1.3 Related work

Earlier applications of signed matrices. Signed adjacency matrices were introduced as early as 1953 by Harary [24], to model social relations involving disliking, indifference, and liking. They have since been used in an array of network applications such as finding “balanced groups” in social networks where members of the same group like or dislike each other [24] and reaching consensus among a group of competing or cooperative agents [7]. Studies of spectral properties of general signed matrices as well as signed adjacency matrices have led to breakthrough advances in multiple areas such as algorithms [2, 10, 18, 19, 35, 40], graph theory [46, 49, 55, 59], and control theory [12, 30, 44, 47, 50]. The *determinant* of signed adjacency matrices of graphs has also played a seminal role in answering fundamental questions concerning graphs and linear systems—for example, see the survey by Thomas [55]. While the literature on matrix signings is extensive [64], we are unaware of any previous work that address the computational problems of finding signings of matrices to achieve the spectral properties mentioned in this work.

Polynomial methods. Combinatorial Nullstellensatz is closely related to the DeMillo-Lipton-Schwartz-Zippel lemma [15, 52, 65] as well as the Alon-Tarsi lemma [6]. These lemmas also present conditions under which a non-vanishing point exists for a multi-variate polynomial. The algorithmic status of all these lemmas are similar—all known proofs are non-constructive. It is well-known that the DeMillo-Lipton-Schwartz-Zippel lemma considers arbitrary polynomials and hence, making it constructive deterministically would also derandomize the polynomial identity testing problem. Alon raised the possibility of making Combinatorial Nullstellensatz constructive. West [61] discussed the corresponding search problem of Combinatorial Nullstellensatz over F_2 and conjectured its relation to the complexity class PPA (Polynomial Parity Argument) defined by Papadimitriou [45]. Varga [58] showed that the search problem *under certain restricted input models* is in PPA. Earlier

this year, Belovs *et al.* [9] showed that the search problem, again under certain restricted input models, is indeed PPA-complete.

1.4 Organization

In Section 1.5, we review definitions and notations. In Section 2, we focus on our main results related to invertible signings. This includes the combinatorial characterization of matrices with invertible signings (Theorem 1.6) and an efficient algorithm to find an invertible signing (Theorem 1.2). We sketch a proof of these two results. Due to space constraints, we present all missing proofs in the appendix.

1.5 Preliminaries

Unless otherwise specified, all matrices are symmetric and take values over the reals. Since all of our results are for symmetric signings, we will just use the term *signing* to refer to a symmetric signing in the rest of this work. For convenience and consistency we denote the entry-wise product of two symmetric $n \times n$ matrices M and s as $M(s)$ (even when s is not necessarily a signing).

Let S_n be the set of permutations of n elements, M be a real symmetric $n \times n$ matrix, and s be a symmetric $n \times n$ signing. Then, the *permutation expansion* of the determinant of a signed matrix $M(s)$ is given by

$$\det M(s) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \cdot \prod_{i=1}^n M(s)[i, \sigma(i)].$$

A permutation σ in S_n has a unique cycle decomposition and hence corresponds to a vertex-disjoint union of directed cycles on n vertices. Removing the orientation gives an undirected graph which is a vertex disjoint union of cycles and matching edges.

2 Invertible Signings

In this section, we focus on invertible signings and the proofs of Theorems 1.2 and 1.6. We start by introducing the notation and terminologies.

Symmetric signings of asymmetric sub-matrices. Let M be a symmetric $n \times n$ matrix. For $X, Y \subseteq [n]$ being a subset of row and column indices of the same cardinality, let $M[X, Y]$ denote the submatrix of M obtained by picking the rows in X and the columns in Y . We note that $M[X, Y]$ is a square matrix, but it may *not* be symmetric even though M is symmetric. We are interested in finding a symmetric $n \times n$ signing s so that the square submatrix $M(s)[X, Y]$ is invertible. We emphasize that for a symmetric signing s , the (possibly asymmetric) matrix $M(s)[X, Y]$ is symmetric on $X \cap Y$, that is, the $[i, j]$ 'th and the $[j, i]$ 'th entries of the matrix $M(s)[X, Y]$ are the same for every $i, j \in X \cap Y$.

Perfect 2-matchings in subgraphs. Let G be a simple undirected graph with vertex set V and edge set E possibly containing self-loops. Let $X, Y \subseteq V$ be subsets of vertices. We consider the subgraph $G[X \cup Y]$ induced by $X \cup Y$. An (X, Y) -*cycle-cover* is a collection of edges of the subgraph $G[X \cup Y]$ that induce a vertex-disjoint union of paths and cycles (cycles could be loop edges) in $G[X \cup Y]$ such that (1) every cycle is a subgraph of $G[X \cap Y]$, and (2) every path either has one end-vertex in $X \setminus Y$ and the other end-vertex in $Y \setminus X$, or has both end-vertices in $X \cap Y$ with only one edge (see Figure 2.1 for an example). We note that (X, X) -cycle-covers correspond to

perfect 2-matchings in $G[X]$ and hence, (V, V) -cycle-covers correspond to perfect 2-matchings in G . It follows that in (X, X) -cycle-covers all paths are only a single edge in G . Furthermore, the existence of an (X, Y) -cycle-cover is possible only if $|X| = |Y|$.

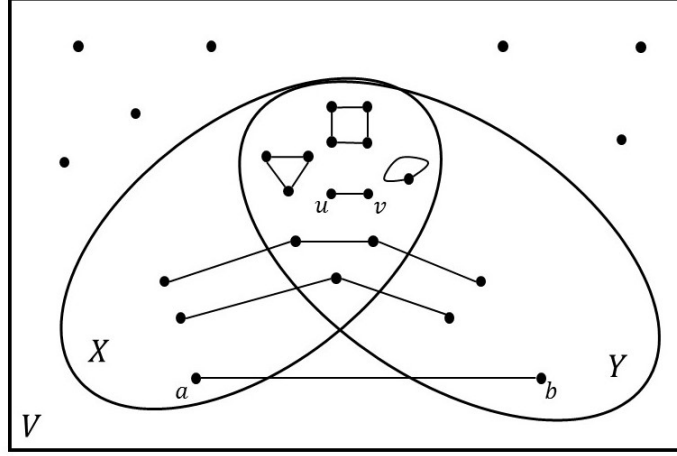


Figure 2.1: An (X, Y) -cycle-cover F . Furthermore, by our definitions below, the edge $\{a, b\}$ is in $\text{Paths}(F)$ while the edge $\{u, v\}$ is in $\text{Matchings}(F)$.

The following lemma shows that the existence of an (X, Y) -cycle-cover in a given graph can be verified efficiently. The lemma follows by a reduction to the perfect matching problem in bipartite graphs.

Lemma 2.1. *There exists a polynomial-time algorithm that decides if there is an (X, Y) -cycle-cover in a given graph G with vertex set V for given subsets $X, Y \subseteq V$ with $|X| = |Y|$.*

Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$ and s be a symmetric $n \times n$ matrix. We derive a convenient expression for $\det(M(s)[X, Y])$ that is based on (X, Y) -cycle-covers. For an (X, Y) -cycle-cover F , let $\text{Cycles}(F)$, $\text{Paths}(F)$, and $\text{Matchings}(F)$ denote the set of cycles in F , paths in F with end-vertices in $X \setminus Y$ and $Y \setminus X$, and paths in F that are contained in $G[X \cap Y]$, respectively. We also note that $\text{Cycles}(F)$, $\text{Paths}(F)$, and $\text{Matchings}(F)$ are all vertex-disjoint from one another and if $X = Y$ then $\text{Paths}(F) = \emptyset$. We define

$$\begin{aligned} M(s)_{\text{Cycles}(F)} &:= \prod_{C \in \text{Cycles}(F)} \prod_{\{u, v\} \in C} M(s)[u, v], \\ M(s)_{\text{Paths}(F)} &:= \prod_{P \in \text{Paths}(F)} \prod_{\{u, v\} \in P} M(s)[u, v], \text{ and} \\ M(s)_{\text{Matchings}(F)} &:= \prod_{\{u, v\} \in \text{Matchings}(F)} M(s)[u, v]^2. \end{aligned}$$

With this notation, we have the following claim that the determinant of $M(s)[X, Y]$ is a $\{\pm 1\}$ -linear combination of terms corresponding to (X, Y) -cycle-covers in G . The proof is based on the permutation expansion of the determinant of $M(s)[X, Y]$.

Lemma 2.2 ((X, Y) -cycle-cover expansion). *Let $M \in \mathbb{R}^{n \times n}$ be a symmetric $n \times n$ matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$, and s be a symmetric $n \times n$ matrix. Let G be the support graph of M*

and \mathcal{F} be the set of all (X, Y) -cycle-covers in G . Then, there exists $\lambda_F \in \{\pm 1\}$ for all $F \in \mathcal{F}$ such that

$$\det(M(s)[X, Y]) = \sum_{F \in \mathcal{F}} \lambda_F M(s)_{\text{Cycles}(F)} M(s)_{\text{Paths}(F)} M(s)_{\text{Matchings}(F)}.$$

Moreover, if $F_1, F_2 \in \mathcal{F}$ such that $\text{Cycles}(F_1) = \text{Cycles}(F_2)$ and $\text{Paths}(F_1) = \text{Paths}(F_2)$ then $\lambda_{F_1} = \lambda_{F_2}$.

2.1 Characterization

In this section, we characterize matrices with an invertible signing through the existence of a perfect 2-matching in their support graphs (Theorem 1.6). To prove Theorem 1.6 we identify a linear multivariate polynomial f such that it is the identically zero polynomial if and only if an associated graph has no perfect 2-matching.

Definition 2.3. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let s be a $n \times n$ symmetric matrix of variables. Let G be the support graph of M , \mathcal{F} be the set of all perfect 2-matchings in G , and $\lambda_F \in \{\pm 1\}$ for all $F \in \mathcal{F}$ be the signs guaranteed by Lemma 2.2. We define f as the polynomial

$$f(s) := \sum_{F \in \mathcal{F}} \lambda_F M(\mathbf{1})_{\text{Matchings}(F)} M(s)_{\text{Cycles}(F)}$$

where $\mathbf{1}$ is the all positive signing.

The following lemma characterizes the existence of a perfect 2-matching using the polynomial f defined above.

Lemma 2.4. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix and s be a $n \times n$ symmetric matrix of variables. Let G be the support graph of M and \mathcal{F} be the set of all perfect 2-matchings in G . With this notation, the polynomial $f(s)$ is the zero polynomial if and only if $\mathcal{F} = \emptyset$.

Theorem 1.6 now follows using Lemma 2.4 and by applying Combinatorial Nullstellensatz (Theorem 1.1) to the polynomial $f(s)$. While the above proof is non-constructive, Theorem 1.6 and Lemma 2.1 do lead to the following result about the complexity of INVERTIBLESIGNING.

Corollary 2.5. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then there is a polynomial-time algorithm that decides if there is a symmetric signing s such that $M(s)$ is invertible.

We mention that the authors are aware of multiple proofs of Theorems 1.6 that are non-constructive. In the next section we will present a constructive proof of this result. To the best of our understanding, the constructive proofs require additional tools such as (X, Y) -cycle-covers.

2.2 Finding Invertible Signings

In this section, we prove Theorem 1.2. For this, we show a constructive generalization of our characterization from Theorem 1.6. The following theorem characterizes the existence of invertible symmetric signings for potentially asymmetric submatrices of symmetric matrices.

Theorem 2.6. Let M be a real symmetric $n \times n$ matrix with support graph G and $X, Y \subseteq [n]$ with $|X| = |Y|$. The following are equivalent:

- (i) There exists an (X, Y) -cycle-cover in G .
- (ii) There exists a symmetric signing s such that $M(s)[X, Y]$ is invertible.

Moreover, there exists an efficient algorithm to verify if there exists a symmetric signing s such that $M(s)[X, Y]$ is invertible and if so, find such a signing.

Notation. We begin with the notations that will be used in the proof of Theorem 2.6. Let M be a real symmetric $n \times n$ matrix with support graph G and let V and E be the vertex and edge sets of G respectively. Let A and B be subsets of V . We define $E[A, B]$ to be the set of edges with one end-vertex in A and the other end-vertex in B . We use $E[A]$ to denote $E[A, A]$. Let e be an edge in G corresponding to the non-zero entry $M[u, v]$ ($= M[v, u]$). We define M^e as the matrix obtained by setting $M[u, v]$ and $M[v, u]$ to 0. For an $n \times n$ signing s and row and column indices $u, v \in [n]$, we can obtain another signing s' such that $s'[u, v] := -s[u, v]$, $s'[v, u] := -s[v, u]$ and $s'[i, j] := s[i, j]$ for every entry $(i, j) \in [n] \times [n] \setminus \{(u, v), (v, u)\}$. We call this operation as s' obtained from s by flipping on (u, v) .

Proof of Theorem 2.6. We first present a constructive proof of the characterization. We will then use the proof to design the algorithm. Lemma 2.2 immediately shows that (ii) implies (i): If we have a symmetric signing s such that $M(s)[X, Y]$ is invertible, then at least one of the terms in the (X, Y) -cycle-cover expansion of $\det(M(s)[X, Y])$ is non-zero. Hence, there exists an (X, Y) -cycle-cover in G .

We show that (i) implies (ii). Suppose not. Among the counterexamples, consider the ones with $|X|$ minimum and among these, pick a matrix M with minimum number of non-zero entries. Without loss of generality, let M be an $n \times n$ matrix with support graph G and let $X, Y \subseteq [n]$ with $|X| = |Y|$. Since we chose a counterexample, we have that

- (A) there exists an (X, Y) -cycle-cover in G , but
- (B) there does not exist a symmetric signing s such that $M(s)[X, Y]$ is invertible.

We will arrive at a contradiction by showing that a signing s satisfying (ii) exists. We begin with the following claim about the counterexample. The proofs of Claims 2.7, 2.8, and 2.9 are omitted in the interest of space. Their proofs follow from viewing the determinant of $M(s)[X, Y]$ as a function of an entry in s that corresponds to a specific type of edge in an (X, Y) -cycle-cover in G and by exploiting the minimality of the counterexample. The proofs appear in the appendix (Section A.5).

Claim 2.7. $E[X \setminus Y, Y] = \emptyset$ and $E[Y \setminus X, X] = \emptyset$.

Now, if $X \setminus Y \neq \emptyset$ and there does not exist an edge $e \in E[X \setminus Y, Y]$, then there does not exist an (X, Y) -cycle-cover in G , a contradiction to assumption (A) about the counterexample. Hence, $X \setminus Y = \emptyset$. Similarly, $Y \setminus X = \emptyset$. Thus, we have $X = Y$ in the counterexample. We next show that the counterexample cannot have any self-loop edges.

Claim 2.8. *There does not exist a self-loop edge in $E[X]$.*

Our next claim strengthens this further by showing that the counterexample has no (X, Y) -cycle-cover with cycle edges.

Claim 2.9. *Every (X, X) -cycle-cover in G has no cycles.*

By Claims 2.7 and 2.8, the counterexample has $X = Y$ with no loop edges in $E[X]$. Furthermore, by Claim 2.9, every (X, X) -cycle-cover in G has no cycles. By definition of (X, X) -cycle-covers, it follows that each (X, X) -cycle-cover in G corresponds to a perfect matching in $G[X]$. Let N be an (X, X) -cycle-cover in G .

Claim 2.10. *N is the unique (X, X) -cycle-cover in G .*

Proof. Let e be an arbitrary edge in N . Suppose there exists an (X, X) -cycle-cover N' in $G - e$. Then, the above claim implies that N' is also a perfect matching in $G[X]$. We consider $N'' := N \cup N'$. Since N and N' are perfect matchings in $G[X]$, the set of edges N'' induces a node-disjoint union of edges and even-cycles in $G[X]$. Hence, N'' is an (X, X) -cycle-cover in G . Furthermore, since $e \in N \setminus N'$, it follows that N'' contains at least one cycle. This contradicts Claim 2.9. Thus, every edge $e \in N$ belongs to every (X, X) -cycle-cover in G . Consequently, N is the unique (X, X) -cycle-cover in G . \square

Since N is the unique (X, X) -cycle-cover in G , by Lemma 2.2, we have that

$$\det(M(s)[X, X]) = (-1)^{|N|} \prod_{\{u,v\} \in N} M(s)[u, v]^2$$

which is non-zero for every signing s . Thus, there exists a symmetric signing s such that

$$\det(M(s)[X, X]) \neq 0,$$

a contradiction to assumption (B) about the counterexample. This completes the proof of the characterization. We note that the above proof of the characterization is constructive and immediately leads to the algorithm FINDSIGNING(M, X, Y) in Figure 2.2.

FINDSIGNING(M, X, Y):

Input: $M \in \mathbb{R}^{n \times n}$ with support graph $G = (V, E)$, $X, Y \subseteq [n]$ satisfying $|X| = |Y|$.

Output: A symmetric signing $s \in \{\pm 1\}^{n \times n}$ such that $M(s)[X, Y]$ is invertible if such a signing exists.

1. If there exists no (X, Y) -cycle-cover then return “No Invertible Signing”.
2. If $E[X \setminus Y, Y] \cup E[Y \setminus X, X] \neq \emptyset$:
 - 2.1. Pick $e = \{u, v\} \in E[X \setminus Y, Y]$ such that $u \in X \setminus Y$ and $v \in Y$.
 - 2.2. If there is no $(X - u, Y - v)$ -cycle-cover in G :
 - 2.2.1 $s \leftarrow$ FINDSIGNING(M^e, X, Y).
 - 2.3. Else: (there would be an $(X - u, Y - v)$ -cycle-cover in G in this case)
 - 2.3.1 $s \leftarrow$ FINDSIGNING($M, X - u, Y - v$).
 - 2.4. If $M(s)[X, Y]$ is invertible then return s .
 - 2.5. Else return s' obtained from s by flipping on (u, v) .
3. Else: (sets X and Y would be identical in this case)
 - 3.1. If there exists an (X, Y) -cycle-cover in G with a cycle edge $e = \{u, v\}$:
 - 3.1.1. $s \leftarrow$ FINDSIGNING($M, X - u, Y - v$).
 - 3.1.2. If $M(s)[X, Y]$ is invertible then return s .
 - 3.1.3. Else return s' obtained from s by flipping on (u, v) .
 - 3.2. Else: (all (X, Y) -cycle-covers would be perfect matchings in $G[X]$ in this case)
 - 3.2.1 Return **1** (all positive signing).

Figure 2.2: The algorithm FINDSIGNING(M, X, Y).

We now describe an efficient implementation of the non-trivial steps in FINDSIGNING. Lemma 2.1 implies that Steps 1 and 2.2 can be implemented to run in polynomial time. We recall that any cycle edge in an (X, X) -cycle-cover must be a cycle edge in some perfect 2-matching in $G[X]$. The following claim shows that Step 3.1 can be implemented to run in polynomial time. Finally, the recursive algorithm terminates in polynomial time since each recursive call reduces either $|X \cup Y|$ or the number of non-zero entries in M . \square

Claim 2.11. *There is a polynomial-time algorithm that given a graph, finds an edge that belongs to a cycle in some perfect 2-matching of the graph or decides that no such edge exists.*

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A Appendix

A.1 Proof of Lemma 2.1

Proof of Lemma 2.1. We recall that determining if a bipartite graph has a perfect matching can be done in polynomial time. Hence, it will suffice to show that deciding if an (X, Y) -cycle-cover exists in G can be reduced in polynomial time to deciding if a perfect matching exists in a bipartite graph.

Let $L = \{v_l \mid v \in X\}$ and $R = \{v_r \mid v \in Y\}$. Let H be the bipartite graph with vertex set $L \cup R$ and an edge set $\{\{v_l, u_r\} \mid \{v, u\} \in (X \times Y) \cap E\}$. The following claim completes the proof. □

Claim A.1. *There is an (X, Y) -cycle-cover in G if and only if there is a perfect matching in H .*

Proof. Let F be an (X, Y) -cycle-cover in G . Consider the edge set $M = \{\{v_l, u_r\} \mid \{v, u\} \in (X \times Y) \cap F\}$. We note that for each $v \in X \setminus Y$ the degree of v_l in M is one. Likewise, for each $v \in Y \setminus X$ the degree of v_r in M is one. Finally, if $v \in X \cap Y$ then the degrees of v_l and v_r in M is one or two. It follows that M is a perfect 2-matching in H and hence there must be a perfect matching in H .

Now let M be a perfect matching in H . Consider the edge set $F = \{\{v, u\} \mid \{v_l, u_r\} \in M\}$. We note that F contains no edge between vertices in $Y \setminus X$. Likewise, F contains no edge between vertices in $X \setminus Y$. Moreover, for each $v \in X \setminus Y \cup Y \setminus X$ the degree of v is exactly one. Finally, for each $v \in X \cap Y$ the degree of v is one or two. It follows that F is an (X, Y) -cycle-cover in G . □

A.2 Proof of Lemma 2.2

Proof of Lemma 2.2. For simplicity, we denote $M' = M[X, Y]$. Let $k := |X|$ and let S_k denote the set of permutations on k elements. Then, by the permutation expansion of the determinant, we have

$$\det(M'(s)) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^k s[u, \sigma(i)] M'[i, \sigma(i)].$$

We recall that $\text{sgn}(\sigma) \in \{\pm 1\}$. Moreover, if $\sigma_1, \sigma_2 \in S_k$ such that σ_1 and σ_2 have the same cycle structure then $\text{sgn}(\sigma_1) = \text{sgn}(\sigma_2)$. Now, we note that there is a one-to-one correspondence between S_k and bijections from X to Y . So, we may view $\sigma \in S_k$ as a bijection $\sigma' : X \rightarrow Y$. Now, consider the graph $H_{\sigma'}$ on vertex set $X \cup Y$ and edge set $F_{\sigma'} := \{\{u, v\} : \sigma'(u) = v\}$. Since σ' is a bijection, it follows that $F_{\sigma'}$ is an (X, Y) -cycle-cover in the complete graph on vertex set $X \cup Y$. Moreover, there is a one-to-one correspondence between S_k and (X, Y) -cycle-covers. Hence,

$$\begin{aligned} \prod_{i=1}^n s[i, \sigma(i)] M'[i, \sigma(i)] &= \prod_{u \in X} s[u, \sigma(u)] M[u, \sigma'(u)] \\ &= M(s)_{\text{Cycles}(F_{\sigma'})} M(s)_{\text{Paths}(F_{\sigma'})} M(s)_{\text{Matchings}(F_{\sigma'})}. \end{aligned}$$

The above-term is non-zero only if $F_{\sigma'}$ is an (X, Y) -cycle-cover in the support graph of G . Furthermore, if $F_1, F_2 \in \mathcal{F}$ such that $\text{Cycles}(F_1) = \text{Cycles}(F_2)$ and $\text{Paths}(F_1) = \text{Paths}(F_2)$ then $\lambda_{F_1} = \lambda_{F_2}$ since the corresponding permutations would have the same cycle structure. \square

A.3 Proof of Lemma 2.4

We first note that the polynomial $f(s)$ defined in Section 2.1 is a linear multivariate polynomial in the entries of s and every term

$$\lambda_F M(\mathbf{1})_{\text{Matchings}(F)} M(s)_{\text{Cycles}(F)}$$

is a non-zero multiple of the monomial

$$\prod_{C \in \text{Cycles}(F)} \prod_{\{u, v\} \in C} s[u, v].$$

We will make use of this observation in the proof of Lemma 2.4 as well as in the proof of Theorem 1.6.

Proof of Lemma 2.4. If $\mathcal{F} = \emptyset$ then $f(s)$ is the zero polynomial by definition of $f(s)$. We now show that if $\mathcal{F} \neq \emptyset$ then there exists a monomial in $f(s)$ that has non-zero coefficient. Let F be a perfect 2-matching in G . We will now show that the monomial

$$g(s) := \prod_{C \in \text{Cycles}(F)} \prod_{\{u, v\} \in C} s[u, v]$$

has non-zero coefficient in $f(s)$. It is sufficient to prove that there exists at least one term in $f(s)$ that is a non-zero multiple of $g(s)$ and every term in $f(s)$ that is a non-zero multiple of $g(s)$ has the same sign. Since F is a perfect 2-matching in G we know from the definition of $f(s)$ that

$$\lambda_F M(\mathbf{1})_{\text{Matchings}(F)} M(s)_{\text{Cycles}(F)}$$

is a term in $f(s)$ and is a non-zero multiple of $g(s)$. Now suppose that there exists another $F' \in \mathcal{F}$ such that

$$\lambda_{F'} M(\mathbf{1})_{\text{Matchings}(F')} M(s)_{\text{Cycles}(F')}$$

is a non-zero multiple of $g(s)$. We note that

$$\{\{u, v\} \mid \{u, v\} \in C, C \in \text{Cycles}(F)\} = \{\{u, v\} \mid \{u, v\} \in C, C \in \text{Cycles}(F')\}.$$

That is, every edge in a cycle of F is also in a cycle of F' . Hence, $\text{Cycles}(F) = \text{Cycles}(F')$. We recall that $\text{Paths}(F) = \text{Paths}(F') = \emptyset$ since F and F' are both perfect 2-matchings in G . Hence, by Lemma 2.2 we know that $\lambda_F = \lambda_{F'}$ and thus every term in $f(s)$ that is a non-zero multiple of $g(s)$ has the same sign as the term

$$\lambda_F M(\mathbf{1})_{\text{Matchings}(F)} M(s)_{\text{Cycles}(F)}.$$

Hence, $f(s)$ is not the zero polynomial. \square

A.4 Proof of Theorem 1.6

In addition to the observation made at the beginning of Section A.3, we also note that $f(s) = \det(M(s))$ for all symmetric $n \times n$ signing s since $s[u, v]^2 = 1$ for all $u, v \in [n]$.

Proof of Theorem 1.6. Suppose there exists a perfect 2-matching in G . We will apply Combinatorial Nullstellensatz (Theorem 1.1) by taking the field F to be the reals. Let s be a $n \times n$ symmetric matrix of variables. Then by Lemma 2.4 the linear multivariate polynomial $f(s)$ is not identically zero. That is, the polynomial $f(s)$ has a term

$$\prod_{u=1}^n \prod_{v=u}^n s[u, v]^{t_{\{u, v\}}}$$

with non-zero coefficient such that $t_{\{u, v\}} \in \{0, 1\}$ for all $u, v \in [n]$ where $v \leq u$ and

$$\sum_{u=1}^n \sum_{v=u}^n t_{\{u, v\}}$$

is the degree of $f(s)$. Now consider $S_{\{u, v\}} := \{\pm 1\}$ for every $u, v \in [n]$ with $v \leq u$. We immediately have that $|S_{\{u, v\}}| \geq t_{\{u, v\}}$ for all $u, v \in [n]$ where $v \leq u$. Hence, by Combinatorial Nullstellensatz (Theorem 1.1), there must exist an $n \times n$ matrix z where every entry of z takes a value from $\{\pm 1\}$ with $z[u, v] = z[v, u]$ for all $u, v \in [n]$, and such that $f(z) \neq 0$. That is, there must exist a symmetric $n \times n$ signing z such that $\det(M(z)) = f(z) \neq 0$.

Now suppose there exists a symmetric $n \times n$ signing s' such that $M(s')$ is invertible. Then it follows that $f(s') = \det(M(s')) \neq 0$. Hence, $f(s)$ is not the zero polynomial and by Lemma 2.4 we know that there must exist a perfect 2-matching in G . \square

A.5 Proof of Claims in Theorem 2.6

Proof of Claim 2.7. Suppose there exists an edge $e \in E[X \setminus Y, Y]$. Let $e = \{u, v\}$ with $u \in X \setminus Y$ and $v \in Y$. Then there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X, Y]$ can be expressed as a linear function of $s[u, v]$:

$$\det(M(s)[X, Y]) = s[u, v] M[u, v] \alpha \det(M(s)[X - u, Y - v]) + \det(M^{\bar{e}}(s)[X, Y]). \quad (1)$$

Case 1: Suppose there exists an (X, Y) -cycle-cover F containing e . We observe that $F - e$ is an $(X - u, Y - v)$ -cycle-cover in G . Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M(s)[X - u, Y - v]) \neq 0$. Since $\det(M(s)[X, Y])$ is a linear function of $s[u, v]$, it follows that $\det(M(s)[X, Y]) \neq 0$ or $\det(M(s')[X, Y]) \neq 0$ where s' is a signing obtained from s by flipping on (u, v) . Hence, we have a contradiction to assumption (B) about the counterexample.

Case 2: Suppose that every (X, Y) -cycle-cover in G does not contain e . Then there is no $(X - u, Y - v)$ -cycle-cover in G . Since (ii) implies (i), it follows that $\det(M(s)[X - u, Y - v]) = 0$ for every symmetric signing s . Let F be an (X, Y) -cycle-cover in G (as promised to exist by (A)). Then F is an (X, Y) -cycle-cover in $G - e$. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\bar{e}}(s)[X, Y]) \neq 0$. By (1), we observe that $\det(M(s)[X, Y]) \neq 0$. Thus, the symmetric signing s is a contradiction to assumption (B) about the counterexample.

Hence, $E[X \setminus Y, Y] = \emptyset$. Similarly $E[Y \setminus X, X] = \emptyset$. \square

Proof of Claim 2.8. Suppose there exists a self-loop edge in $E[X]$. Let $e = \{u, u\}$ for some $u \in X$. Then, we again have that $\det(M(s)[X, Y])$ is a linear function of $s[u, u]$:

$$\det(M(s)[X, X]) = s[u, u]M[u, u]\det(M(s)[X - u, X - u]) + \det(M^{\bar{e}}(s)[X, X]). \quad (2)$$

We arrive at a contradiction by proceeding similar to the proof of the previous claim. We avoid restating the proof in the interests of brevity. \square

Proof of Claim 2.9. Suppose there exists an (X, X) -cycle-cover F in G with a cycle C induced by F . Let $e = \{u, v\}$ be an edge in the cycle. By Claim 2.8, we know that $u \neq v$. We observe that $\det(M(s)[X, X])$ is a quadratic function of $s[u, v]$, i.e., there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X, X]$ can be expressed as

$$\begin{aligned} \det(M(s)[X, X]) &= -s[u, v]^2 M[u, v]^2 \det(M(s)[X - \{u, v\}, X - \{u, v\}]) \\ &\quad + 2s[u, v]M[u, v]\alpha \det(M^{\bar{e}}(s)[X - u, X - v]) + \det(M^{\bar{e}}(s)[X, X]). \end{aligned} \quad (3)$$

Furthermore, $F - e$ is an $(X - u, X - v)$ -cycle-cover in G . Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\bar{e}}(s)[X - u, X - v]) \neq 0$. We now define the quadratic function

$$\begin{aligned} f(x) &:= -x^2 M[u, v]^2 \det(M(s)[X - \{u, v\}, X - \{u, v\}]) \\ &\quad + 2xM[u, v]\alpha \det(M^{\bar{e}}(s)[X - u, X - v]) + \det(M^{\bar{e}}(s)[X, X]), \end{aligned}$$

and consider the roots of the quadratic equation $f(x) = 0$. Since $\det(M^{\bar{e}}(s)[X - u, X - v]) \neq 0$, the sum of the roots of this quadratic equation is non-zero. Since the real roots of a quadratic function are symmetric about the extreme point of the parabola defined by the function (i.e., symmetric about $\arg \min f(x)$), there exists $x \in \{\pm 1\}$ that is *not* a root of $f(x)$. Hence, either $\det(M(s)[X, Y]) \neq 0$ or $\det(M(s')[X, Y]) \neq 0$ where s' is a signing obtained from s by flipping on (u, v) . Thus, either s or s' contradict assumption (B) about the counterexample. \square

Proof of Claim 2.11. To prove the claim we consider $\text{FINDCYCLEEDGE}(G)$ in Figure A.1.

If at any point we find a perfect 2-matching with a cycle then we return an edge from it. Hence, it only remains to show the correctness of Steps 5 and 6. Let F be a perfect 2-matching with no cycle edge. Suppose there exists a perfect 2-matching N_e for some edge e with no cycle edge. Then

FINDCYCLEEDGE(G):

Input: A graph G with vertex set V .

Output: An edge e that is a cycle edge in some perfect 2-matching in G if one exists.

1. If there exists no perfect 2-matching in G then return “No edge”.
2. Let F be a perfect 2-matching in G .
3. If F contains a cycle C then return any edge in C .
4. For $e \in F$:
 - 4.1. Let N_e be a perfect 2-matching in $G - e$ if one exists.
 - 4.2. If N_e exists and has a cycle C then return any edge in C .
5. If Step 4 finds N_e for some $e \in F$, then return e .
6. Else return “No edge”.

Figure A.1: The algorithm FINDCYCLEEDGE(G).

N_e and F are both perfect matchings in G . It follows that $N_e \cup F$ will be a perfect 2-matching where e is in a cycle and hence Step 5 is correct to return e . Now suppose that for all e there is no perfect 2-matching N_e . It follows that G has one unique perfect 2-matching F that is a perfect matching and hence Step 6 correctly returns that no cycle edge exists.

Using the algorithm from Lemma 2.1 we can perform Steps 1, 2 and 4.1 in polynomial time. Thus, FINDCYCLEEDGE(G) runs in polynomial time. \square

B Invertible Signings Continued

In this section, we again focus on invertible signings and present the proofs of Theorems 1.3 and 1.7.

B.1 Minimum Support Increase to Obtain an Invertible Signing

In this section, we study the problem of computing the solvability index of real symmetric matrices, thus proving Theorem 1.3. We recall the following definition: For a real symmetric matrix M , the *solvability index* of M is the smallest number of non-diagonal zero entries that need to be converted to non-zeroes so that the resulting *symmetric* matrix has an invertible signing. We remind the reader that the support-increase operation preserves symmetry.

By our characterization in Theorem 1.6, computing the solvability index of a matrix reduces to the following edge addition problem:

EDGEADD: Given a graph G (possibly with self-loops) with vertex set V and edge set E , find

$\min \{ |F| \mid F \text{ is a set of non-edges of } G \text{ with no loops such that } G + F \text{ has a perfect 2-matching} \}.$

In the above, $G + F$ denotes the graph obtained by adding the edges in F to G . In the rest of the section, we will show that EDGEADD can be solved efficiently, which will imply Theorem 1.3.

Theorem B.1. *There is a polynomial-time algorithm to solve EDGEADD.*

We need some terminology from matching theory. Let G be a graph on vertex set V and edge set E . For a subset S of vertices, denote the *induced subgraph* of G on S as $G[S]$ and the *non-inclusive neighborhood* of S in G by $N_G(S)$. We recall that a *matching* M in G is a subset of edges where each vertex is incident to at most one edge in M . Let $\nu(G)$ denote the cardinality of a *maximum matching* in G and let

$$\nu_f(G) := \max \left\{ \sum_{e \in E} x_e \mid \sum_{e \in \delta(v)} x_e \leq 1, \text{ and } x_e \geq 0 \text{ for all } e \in E \right\}$$

denote the value of a *maximum fractional matching* in G . For a matching M , we define a vertex u to be M -*exposed* if none of the edges of M are incident to u , and a vertex v to be an M -*neighbor* of u if edge $\{u, v\}$ is in M . A vertex u in V is said to be *inessential* if there exists a maximum cardinality matching M in G such that u is M -exposed, and is said to be *essential* otherwise. A graph H is *factor-critical* if there exists a perfect matching in $H - v$ for every vertex v in H . The following result is an immediate consequence of the odd-ear decomposition characterization of Lovász [37].

Lemma B.2 (Lovász [37]). *If G is a factor-critical graph, then G has a perfect 2-matching.*

The *Gallai-Edmonds decomposition* [16, 20, 21] of a graph G is a partition of the vertex set of G into three sets (B, C, D) , where B is the set of inessential vertices, $C := N_G(B)$, and $D := V \setminus (B \cup C)$. Let B_1 denote the set of isolated vertices in $G[B]$ and $B_{\geq 3} := B \setminus B_1$. For notational convenience, we will denote the Gallai-Edmonds decomposition as $(B = (B_1, B_{\geq 3}), C, D)$. The Gallai-Edmonds decomposition of a graph is unique and can be found efficiently [16]. The following theorem summarizes the properties of the Gallai-Edmonds decomposition that we will be using (properties (i) and (ii) are well-known and can be found in Schrijver [51] while property (iii) follows from results due to Balas [8] and Pulleyblank [48]—see Bock *et al.* [11] for a proof of property (iii)):

Theorem B.3. *Let $(B = (B_1, B_{\geq 3}), C, D)$ be the Gallai-Edmonds decomposition of a graph G . We have the following properties:*

- (i) *Each connected component in $G[B]$ is factor-critical.*
- (ii) *Every maximum matching M in G contains a perfect matching in $G[D]$ and matches each vertex in C to distinct components in $G[B]$.*
- (iii) *Let M be a maximum matching that matches the largest number of B_1 vertices. Then there are $2(\nu_f(G) - \nu(G))$ M -exposed vertices in $B_{\geq 3}$.*

We observe that G contains a perfect 2-matching if and only if $\nu_f(G) = |V|/2$. Therefore, adding edges to get a perfect 2-matching in G is equivalent to adding edges to increase the maximum fractional matching value to $|V|/2$.

Proof of Theorem B.1. We will assume that G has no isolated vertices and no self-loops in the rest of the proof. We make this assumption here in order to illustrate the main idea underlying the algorithm. This assumption can be relaxed by a case analysis in the algorithm as well as the proof of correctness. We defer the details of the case analysis to the full-version of the paper.

EDGEADD(G):

Input: A graph G with no isolated vertices and no self-loops.

Output: A set F of non-edges of G such that $G + F$ contains a perfect 2-matching.

1. Find the Gallai-Edmonds decomposition $(B = (B_1, B_{\geq 3}), C, D)$ of G .
2. Find a maximum matching M that matches the largest number of B_1 vertices.
3. Let $S := \{u \in B_1 \mid u \text{ is } M\text{-exposed}\}$.
4. If $|S|$ is even:
 - Pick an arbitrary pairing of the vertices in S .
5. If $|S|$ is odd:
 - Consider a vertex s in S , pick a vertex t in $N_G(s)$ and let u be the M -neighbor of t .
 - Pair up u with s and pick an arbitrary pairing of the vertices in $S \setminus \{s\}$.
6. Return the set of pairs F .

Figure B.1: The algorithm EDGEADD(G).

We use the algorithm $\text{EDGEADD}(G)$ given in Figure B.1. We briefly describe an efficient implementation for Step 2, since it is easy to see that other steps can be implemented efficiently. In order to find a maximum matching that matches the largest number of B_1 vertices (as mentioned in property (iii) of Theorem B.3), we first find the Gallai-Edmonds decomposition and a maximum matching M . Then, we repeatedly augment M by searching for M -alternating paths (of even-length) from M -exposed B_1 vertices. This approach can be implemented to run in polynomial time. Alternatively, Step 2 can also be implemented by solving a maximum weight matching with suitably chosen weights.

We now argue the correctness of the algorithm. We first show that if $|S|$ is odd, then there is a choice of vertices t and u as described in the algorithm $\text{EDGEADD}(G)$: this is because, G has no isolated vertices and hence there exists a vertex t in $N_G(s)$. Moreover, by Theorem B.3, since s is in B_1 , it follows that t is in C and thus t is matched by M to a vertex u in B . Now, Claim B.4 proves feasibility and bounds the size of the returned solution F while Claim B.5 proves the optimality. \square

Claim B.4. *The algorithm $\text{EDGEADD}(G)$ returns a set F of non-edges of G such that (1) $G + F$ contains a perfect 2-matching, and (2) $|F| = \lceil |V|/2 - \nu_f(G) \rceil$.*

Proof. By property (ii) of Theorem B.3, the set F is a set of non-edges of G . We will construct a perfect 2-matching in $G + F$. By property (i) of Theorem B.3, every component in $G[B_{\geq 3}]$ is factor-critical. By Lemma B.2, every component K in $G[B_{\geq 3}]$ contains a perfect 2-matching x^K . Let N_K denote the support of x^K . Let \mathcal{K} denote the components in $G[B_{\geq 3}]$ that contain an M -exposed vertex. We have two cases:

Case 1: Suppose $|S|$ is even. Let N denote the set of edges of M that do not match any vertices in $\bigcup_{K \in \mathcal{K}} V(K)$. Now, the set of edges induced by $(\bigcup_{K \in \mathcal{K}} N_K) \cup N \cup F$ has a perfect 2-matching. A perfect 2-matching x in $G + F$ can be obtained by assigning $x(e) := x^K(e)$ for edges e in $\bigcup_{K \in \mathcal{K}} N_K$, $x(e) := 2$ for edges e in $N \cup F$, and $x(e) := 0$ for the remaining edges in $G + F$.

Case 2: Suppose $|S|$ is odd. Let N denote the set of edges of $M \setminus \{t, u\}$ that do not match any vertices in $\bigcup_{K \in \mathcal{K}} V(K)$. Now, $(\bigcup_{K \in \mathcal{K}} N_K) \cup N \cup (F \setminus \{s, u\}) \cup \{t, u\}, \{s, t\}, \{s, u\}$ has a perfect 2-matching. We note that the edges $\{t, u\}, \{s, t\}$ were already present in the graph owing to the choice of c and u while the edge $\{s, u\}$ was added as an edge from F . A perfect 2-matching x in $G + F$ can be obtained by assigning $x(e) := x^K(e)$ for edges $e \in \bigcup_{K \in \mathcal{K}} N_K$, $x(e) := 1$ for edges e in $\{t, u\}, \{s, t\}, \{s, u\}$, $x(e) := 2$ for edges e in $N \cup (F \setminus \{s, u\})$, and $x(e) := 0$ for the remaining edges in $G + F$.

Next we find the size of the set F returned by the algorithm. We observe that $|F| = \lceil |S|/2 \rceil$. It remains to bound $|S|$. For this, we count the number of vertices in the graph using the matched and exposed vertices. We have that $|V| = 2|M| + |S| + \{\text{number of } M\text{-exposed vertices in } B_{\geq 3}\}$. By property (iii) of Theorem B.3 and the choice of the matching M , we have $|V| = 2|M| + |S| + 2(\nu_f(G) - \nu(G))$. Since M is a maximum cardinality matching, we know that $|M| = \nu(G)$ and hence, $|S| = |V| - 2\nu_f(G)$. \square

Our next claim shows a lower bound on the optimal solution that matches the upper bound and hence proves the optimality of the returned solution.

Claim B.5. *Let F' be a set of non-edges of G . Suppose $G + F'$ has a perfect 2-matching. Then $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$.*

Proof. We first note that the addition of a non-edge can increase the value of the maximum fractional matching by at most one. That is, for every graph H and every non-edge e of H , we have $\nu_f(H + e) - \nu_f(H) \leq 1$ (this can be shown by considering the dual problem, namely the minimum fractional vertex cover). Now, consider an arbitrary ordering of the edges in the solution F' and let F'_i denote the set of first i edges according to this order and let $F'_0 = \emptyset$. Then,

$$\nu_f(G + F') - \nu_f(G) = \sum_{i=1}^{|F'|} (\nu_f(G + F_i) - \nu_f(G + F_{i-1})) \leq |F'|.$$

Thus, we have $|F'| \geq \nu_f(G + F') - \nu_f(G)$. We observe that if $G + F'$ has a perfect 2-matching, then $\nu_f(G + F') = |V|/2$. Hence, $|F'| \geq |V|/2 - \nu_f(G)$. Finally, we observe that $|F'|$ has to be an integer and hence, $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$. \square

B.2 Number of Invertible Signings

In this section, we show a lower bound on the number of invertible signings of adjacency matrices of graphs with at least one perfect 2-matching, thus proving Theorem 1.7.

Proof of Theorem 1.7. Let A denote the adjacency matrix of G while n and m be the number of vertices and edges in G respectively. For a vector $s \in \{\pm 1\}^{E(G)}$ that gives a sign for the edges of G , we define $A(s)$ to be the signed adjacency matrix of G obtained by taking the entry-wise product of A and the $n \times n$ symmetric signing whose entry on (i, j) (and by symmetry, (j, i)) is $s(\{i, j\})$ if $\{i, j\}$ is an edge in G , and $+1$ otherwise. We will abuse notation for the purposes of this proof only and call such a vector s to be a *signing* of the graph G . Let

$$\begin{aligned} \Gamma_1 &:= \left\{ s \in \{\pm 1\}^{E(G)} \mid A(s) \text{ is invertible} \right\}, \text{ and} \\ \Gamma_0 &:= \left\{ s \in \{\pm 1\}^{E(G)} \mid A(s) \text{ is singular} \right\}. \end{aligned}$$

For a signing s of G in Γ_0 , we have

$$\sum_{\omega \in S_n} A_\omega(s) = 0.$$

Assume for the sake of contradiction that the set Γ_1 has size less than $2^{m-n}/n!$.

By assumption there is a perfect 2-matching in G . Hence there exists a permutation $\tau \in S_n$ for which $A_\tau \neq 0$ and hence $A_\tau \in \{\pm 1\}$. Fix a permutation $\tau \in S_n$ with the least number of cycles for which $A_\tau \neq 0$ (i.e., the perfect 2-matching corresponding to τ having the largest number of matching edges). Let

$$\Omega_0 := \{ \sigma \in S_n : \text{Cycles}(\sigma) = \text{Cycles}(\tau) \}.$$

Then for every σ in Ω_0 and every signing s , we have $A_\sigma(s) = A_\tau(s)$. Now let

$$\mathcal{R} := \left\{ s \in \{\pm 1\}^{E(G)} \mid s(e) = +1 \text{ for every edge } e \text{ in } \text{Cycles}(\tau) \right\}.$$

Consider the absolute value of double-sum

$$\left| \sum_{s \in \mathcal{R}} \sum_{\omega \in S_n} A_\omega(s) \right|.$$

We have the upper bound

$$\left| \sum_{s \in \mathcal{R}} \sum_{\omega \in S_n} A_\omega(s) \right| = \left| \sum_{s \in \mathcal{R} \cap \Gamma_1} \sum_{\omega \in S_n} A_\omega(s) \right| \leq \sum_{s \in \mathcal{R} \cap \Gamma_1} n! \leq |\Gamma_1| \cdot n! < 2^{m-n}.$$

Now we will show a lower bound to the double-sum that contradicts the upper bound. We first note that

$$\sum_{s \in \mathcal{R}} \sum_{\omega \in S_n} A_\omega(s) = \sum_{s \in \mathcal{R}} \sum_{\omega \in \Omega_0} A_\omega(s) + \sum_{s \in \mathcal{R}} \sum_{\omega \in S_n \setminus \Omega_0} A_\omega(s).$$

Since τ is chosen to have the fewest number of cycles, it follows that any permutation in $S_n \setminus \Omega_0$ has at least one cycle edge that is not in τ , which implies that the signings in \mathcal{R} can be paired up such that each pair of signings differ only on that specific edge not in τ . Therefore for each permutation $\omega \in S_n \setminus \Omega_0$, we have

$$\sum_{s \in \mathcal{R}} A_\omega(s) = 0.$$

Hence,

$$\sum_{s \in \mathcal{R}} \sum_{\omega \in S_n \setminus \Omega_0} A_\omega(s) = 0.$$

Now fix any $s_0 \in \mathcal{R}$ and we have

$$\left| \sum_{s \in \mathcal{R}} \sum_{\omega \in S_n} A_\omega(s) \right| = \left| \sum_{s \in \mathcal{R}} \sum_{\omega \in \Omega_0} A_\omega(s) \right| = |\mathcal{R}| |\Omega_0| |A_\tau(s_0)| \geq |\mathcal{R}|.$$

The second equation above is because $A_\omega(s) = A_\tau(s_0)$ for every $\omega \in \Omega_0$ and every $s \in \mathcal{R}$. The last inequality above is because $\tau \in \Omega_0$ and hence $|\Omega_0| \geq 1$ and moreover, $A_\tau(s_0) \in \{\pm 1\}$.

Thus the cardinality of \mathcal{R} is a lower bound for the absolute value of the double-sum of interest. Now we note that \mathcal{R} is the set of all signings on edges for which $s(e) = +1$ on every edge e in $\text{Cycles}(\tau)$. The total number of edges in $\text{Cycles}(\tau)$ is at most n and hence $|\mathcal{R}| \geq 2^{m-n}$. Thus, we have

$$\left| \sum_{s \in \mathcal{R}} \sum_{\omega \in S_n} A_\omega(s) \right| \geq |\mathcal{R}| \geq 2^{m-n},$$

a contradiction to the upper bound on the absolute value of the double-sum. \square

C Singular Signings

In this section, we focus on singular signings. We will prove that `SINGULARSIGNING` for arbitrary matrices is NP-complete and that `SINGULARSIGNING` is solvable for adjacency matrices of bipartite graphs (Theorem 1.5). The former result will be used to complete the proof of Theorem 1.4.

C.1 Hardness of SingularSigning

In order to show the NP-completeness result, we reduce from the partition problem, which is a well-known NP-complete problem [31]. We recall the problem below:

PARTITION: Given an n -dimensional vector b of non-negative integers, determine if there is a ± 1 -signing vector z such that the inner product $\langle b, z \rangle$ equals zero.

We use the notion of *Schur complement*. The following lemma summarizes the definition and the relevant properties of the Schur complement.

Lemma C.1 (Horn and Johnson [28]). *Suppose $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{q \times q}$ are matrices such that A is invertible and the matrix*

$$D := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

is a symmetric matrix. Then the Schur complement of C in matrix D is defined to be

$$D_C := C - BA^{-1}B^T.$$

We have the following properties:

- (i) *Suppose A is positive definite. Then, D is positive semi-definite if and only if the Schur complement of C in D , namely D_C , is positive semi-definite.*
- (ii) $\det(D) = \det(A) \cdot \det(D_C)$.

Lemma C.2. SINGULARSIGNING is NP-complete.

Proof. SINGULARSIGNING is in NP since if there is an (off-diagonal) signing of the given matrix that is positive semi-definite or singular, then this signing gives the witness. In particular, we can verify if a given (off-diagonal) symmetric signed matrix is positive semi-definite or singular in polynomial time by computing its spectrum [22].

We show NP-hardness of SINGULARSIGNING by reducing from PARTITION. Let the n -dimensional vector $b := (b_1, \dots, b_n)^T$ be the input to PARTITION, where each b_i is a non-negative integer. We construct a matrix M as an instance of SINGULARSIGNING as follows: Consider the following $(n+2) \times (n+2)$ -matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b \rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n -dimensional column vector of all ones. Claim C.3 proves the correctness of the reduction to SINGULARSIGNING. \square

Claim C.3. *The matrix M has a symmetric off-diagonal signing s such that $M(s)$ is singular if and only if there is a vector $z \in \{\pm 1\}^n$ such that the inner product $\langle b, z \rangle$ is zero.*

Proof. Construct the Schur complement M'_C of C of M' as in Claim D.2. Using property (ii) of Lemma C.1, we have that

$$\det M' = \det(I_n) \cdot \det(M'_C) = \det(I_n) \cdot \det \left(\begin{bmatrix} 0 & -\langle \hat{b}, z \rangle \\ -\langle \hat{b}, z \rangle & 0 \end{bmatrix} \right) = -\langle \hat{b}, z \rangle^2.$$

Therefore, $\det M' = 0$ if and only if $\langle \hat{b}, z \rangle = 0$. We note that $\langle \hat{b}, z \rangle = 0$ if and only if there is a ± 1 -vector z' such that $\langle b, z' \rangle = 0$. \square

C.2 Finding Singular Signings of Bipartite Graphs

In this section, we characterize bipartite graphs whose signed adjacency matrix is invertible for all signings. We use this characterization to prove Theorem 1.5. We will use the following results by Little [36] for our characterization. Lemma C.4 stated below is a slight extension of the original result by Little.

Lemma C.4 (Little [36]). *Let G be a graph with adjacency matrix A_G . Then $\det(A_G(s))$ is even for all signings s if and only if G has an even number of perfect matchings.*

Proof. Suppose G has n vertices. We recall that the permutation expansion of the determinant of $A_G(s)$ for a signing s is given by

$$\det A_G(s) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \cdot \prod_{i=1}^n A_G(s)[i, \sigma(i)].$$

In order to prove the claim, it suffices to show that every perfect 2-matching in G that is not a perfect matching must contribute an even number of distinct non-zero terms of the same sign to the determinant of $A_G(s)$ for all signings s . We show this next.

Let M be a perfect 2-matching in G that is not a perfect matching. Let C be the set of vertex disjoint cycles in M . Then M corresponds to $2^{|C|}$ distinct permutations of S_n and thus $2^{|C|}$ distinct terms in the permutation expansion of the determinant of $A_G(s)$ for all signings s . Since all such permutations have the same cycle structure it follows that each term must have the same sign. Thus each perfect 2-matching that is not a perfect matching in G contributes an even number to the permutation expansion of the determinant of $A_G(s)$. \square

Theorem C.5 (Little [36]). *Let G be a graph. Then G has an even number of perfect matchings if and only if there is a set $S \subseteq V(G)$ such that every vertex in G has even number of neighbors in S . Moreover, if G has an even number of perfect matchings, then such a set S can be found in polynomial time.*

We now have the ingredients to characterize bipartite graphs whose signed adjacency matrix is invertible for all signings.

Lemma C.6. *Let G be a bipartite graph and let A_G be the adjacency matrix of G . Then $\det(A_G(s)) \neq 0$ for all signings s if and only if G has an odd number of perfect matchings.*

Proof. Suppose G has an odd number of perfect matchings. By Lemma C.4, we have that $\det(A_G(s)) \neq 0$ for all signings s .

Now suppose that G has an even number of perfect matchings. By Theorem C.5, there exists a set $S \subseteq V(G)$ such that $|N_G(v) \cap S|$ is even for all $v \in V(G)$. We observe that the subgraph $G[S]$ induced by S is bipartite with every vertex having even degree. Thus, any closed walk on $G[S]$ has even number of edges and every connected component in $G[S]$ has an Eulerian tour with even number of edges. Let C be a connected component of $G[S]$ with m edges and let $T := (e_1, e_2, \dots, e_m)$ be an ordering of the edges that represents an Eulerian tour of C . Then we sign edge e_i to be positive if i is even and negative otherwise. Every vertex $v \in V(G) \setminus S$ has even number of edges between v and vertices in S . We partition the edges incident to v into two arbitrary parts of equal size and sign all the edges in one part to be positive and the rest of the edges in the other part to be negative. Let \hat{s} denote the resulting signing.

Under the signing \hat{s} every vertex v of G has an equal number of positive and negative edges to vertices in S . Thus, the sum of the column vectors corresponding to the vertices in S will be zero and hence $\det(A_G(\hat{s})) = 0$. \square

We note that the proof of Lemma C.6 is constructive since we can find a set S for which every vertex has even number of neighbors in S in polynomial time by Theorem C.5. Thus, Theorem 1.5 follows from Theorem C.5 and Lemma C.6.

D Hardness of Eigenvalue Problems

In this section we prove that PSDSIGNING and BOUNDEDVALUESIGNING are NP-complete. Together with Lemma C.2 this completes the proof of Theorem 1.4.

D.1 Hardness of Positive Semi-definite Signing Problem

In order to show the NP-completeness of PSDSIGNING, we again reduce from PARTITION [31]. The proof has a similar outline to the NP-completeness proof of SINGULARSIGNING (Lemma C.2).

Lemma D.1. PSDSIGNING is NP-complete.

Proof. PSDSIGNING is in NP since if there is an (off-diagonal) signing of the given matrix that is positive semi-definite, then this signing gives the witness. In particular, we can verify if a given (off-diagonal) symmetric signed matrix is positive semi-definite in polynomial time by computing its spectrum [22].

We show NP-hardness of PSDSIGNING by reducing from PARTITION. Let the n -dimensional vector $b := (b_1, \dots, b_n)^T$ be the input to the PARTITION problem, where each b_i is a non-negative integer. We construct a matrix M as an instance of PSDSIGNING as follows: Consider the following $(n+2) \times (n+2)$ -matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b \rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n -dimensional column vector of all ones. Claim D.2 proves the correctness of the reduction to PSDSIGNING. \square

Claim D.2. The matrix M has a signing s such that $M(s)$ is positive semi-definite if and only if there is a ± 1 -vector z such that the inner product $\langle b, z \rangle$ is zero.

Proof. We may assume that any signed matrix $M(s)$ that is positive semi-definite may not have negative entries in the diagonal because a positive semi-definite matrix will not have negative entries on its diagonal. Hence, we will only consider symmetric off-diagonal signing s of the matrix M of the following form:

$$M' := M(s) = \begin{bmatrix} I_n & \hat{b} & z \\ \hat{b}^T & \langle b, b \rangle & 0 \\ z^T & 0 & n \end{bmatrix},$$

where the n -dimensional vector z takes values in $\{\pm 1\}^n$ and $\hat{b} = (\hat{b}_1, \dots, \hat{b}_n)^T$, where \hat{b}_i takes value in $\{\pm b_i\}$ for every i . Let

$$\begin{aligned} A &:= I_n, \\ B &:= \begin{bmatrix} \hat{b} & z \end{bmatrix}, \text{ and} \\ C &:= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix}. \end{aligned}$$

Since $A = I_n$ is invertible, the Schur complement of C in M' is well-defined and is given by

$$\begin{aligned} M'_C &= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \hat{b}^T \\ z^T \end{bmatrix} I_n^{-1} \begin{bmatrix} \hat{b} & z \end{bmatrix} \\ &= \begin{bmatrix} \langle b, b \rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \langle \hat{b}, \hat{b} \rangle & \langle \hat{b}, z \rangle \\ \langle \hat{b}, z \rangle & \langle z, z \rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\langle \hat{b}, z \rangle \\ -\langle \hat{b}, z \rangle & 0 \end{bmatrix}, \end{aligned}$$

where the last equation follows because we have $\langle \hat{b}, \hat{b} \rangle = \langle b, b \rangle$ and $\langle z, z \rangle = n$.

We note that $A = I_n$ is positive definite. Therefore, by property (1) of Lemma C.1, the matrix M' is positive semi-definite if and only if M'_C is positive semi-definite. Therefore, M' is positive semi-definite if and only if $\langle \hat{b}, z \rangle = 0$. Finally, we note that $\langle \hat{b}, z \rangle = 0$ if and only if there is a ± 1 -vector z' such that $\langle b, z' \rangle = 0$. \square

D.2 Hardness of Bounded Eigenvalue Signing Problem

To prove that BOUNDEDVALUESIGNING is NP-complete, we consider the following problem that is closely related to PSDSIGNING:

NSDSIGNING: Given a real symmetric matrix M , verify if there exists a signing s such that $M(s)$ is negative semi-definite.

We observe that a real symmetric $n \times n$ matrix is positive semi-definite if and only if $-M$ is negative semi-definite. Lemma D.1 and this observation lead to the following corollary.

Corollary D.3. NSDSIGNING is NP-complete.

We next reduce NSDSIGNING to BOUNDEDVALUESIGNING which also completes the proof of Theorem 1.4.

Lemma D.4. BOUNDEDVALUESIGNING is NP-complete.

Proof. BOUNDEDVALUESIGNING is in NP since if there is an off-diagonal signing of a given matrix that has all eigenvalues bounded above by a given real number λ , then this signing gives the witness. We can verify if all eigenvalues of a given off-diagonal symmetric signed matrix are at most λ in polynomial time by computing the spectrum of the matrix.

We show NP-hardness of BOUNDEDVALUESIGNING by reducing from NSDSIGNING which is NP-complete by Corollary D.3. Let the real symmetric $n \times n$ matrix M be the input to the NSDSIGNING problem. We construct an instance of BOUNDEDVALUESIGNING by considering $\lambda = 0$ and the matrix M' obtained from M as follows (where $|a|$ denotes the magnitude of a):

$$M'[i, j] = \begin{cases} M[i, j] & \text{if } i \neq j, \\ -|M[i, j]| & \text{if } i = j. \end{cases}$$

We observe that every negative semi-definite signing of M has to necessarily have negative values on the diagonal. Hence, there is a signing s such that that $M(s)$ is negative semi-definite if and only if there is an off-diagonal signing t such that $\lambda_{\max}(M'(t)) \leq \lambda = 0$. \square