Planar Emulators for Monge Matrices

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1 Abstract

We constructively show that any cyclic Monge distance matrix can be represented as the graph distances between vertices on the outer face of a planar graph. The structure of the planar graph depends only on the number of rows of the matrix, and the weight of each edge is a fixed linear combination of constantly many matrix entries. We also show that the size of our constructed graph is worst-case optimal among all planar graphs.

1.1 Related work

Sketching graph distances. Emulators—arbitrary graphs that preserve distances between terminals in the input graph—are known to exist in general [8,9,18]. But without additional assumptions on the input graph there is a linear lower-bound on the size of the emulator (with respect to the size of the input graph) when the number of terminals is a polynomial $\Theta(n^\alpha)$ for some range of $\alpha$ strictly less than 1 [18]. Chang, Gawrychowski, Mozes, and Weimann [14] constructed the first sub-linear size emulator for any undirected unweighted planar graph: given any $k$-terminal planar graph with $n$ vertices, an emulator of size $O(\min\{k^2, (kn)^{1/2}\})$ can be constructed in $O(n)$ time, which is optimal up to logarithmic factors.

A related structure, called a spanner, which preserves the distances approximately up to additive or multiplicative errors, is relatively well-understood for general graphs [9, 31, 43, 48, 50]. Spanners with stronger guarantees exist for geometrically/topologically constrained graphs [4, 13, 23, 37]. Similarly, distance oracles that answer distance queries exactly or approximately are known to exist for planar and surface graphs [1, 5, 15, 27, 35, 36, 41, 46, 47]. (See Ahmed et al. [3] for a recent survey on distance sketching.)

Circular planar graphs. One of the central problems in the theory of circular planar graphs considers the following problem: Given measures of effective resistances between all pairs of terminals, can we reconstruct a planar resistor network realizing the measures where the terminals lie on the boundary? Colin de Verdière et al. [16, 17] and Curtis et al. [20, 21] showed that the reconstruction problem can be solved precisely when the effective resistance matrix is totally non-negative. The problem sounds similar to ours

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in spirit; in fact, when looking closer, the planar emulator problem is equivalent to their reconstruction problem in the \((\min, +)\)-semiring instead of the standard \((+\times)\)-ring. The techniques involved in proving their theorem rely crucially on the fact that the weights are over a \((+\times)\)-ring and therefore do not apply to our problem.

1.2 Preliminaries

Monge properties. A matrix \(M\) satisfies the Monge property if for any two rows \(i < i'\) and two columns \(j < j'\), one has

\[
M[i, j] + M[i', j'] \leq M[i', j] + M[i, j'].
\]

Matrix \(M\) satisfies the anti-Monge property if the sign of the above inequality flipped. We often reorder the terms in the inequality to emphasize the monotonicity on the entry differences:

\[
M[i', j'] - M[i, j'] \leq M[i', j] - M[i, j].
\]

For the purpose of this paper we only consider distance matrices, where the diagonal entries are all zeros, the entries are symmetric and satisfy the triangle inequality. A distance matrix \(M\) is cyclic Monge\(^2\) if for any four indices \(i, i', j, j'\) in cyclic order (that is, \(i \leq i' \leq j \leq j'\) after some cyclic reordering of \([i, i', j, j']\), one has

\[
M[i, j] + M[i', j] \leq M[i', j'] + M[i, j'].
\]

(Notice the inequality sign flipped comparing to the standard Monge property.) Let \(M\) be a cyclic Monge distance matrix and let \(A\) and \(B\) be two disjoint sub-intervals of the index set of \(M\). Then the submatrix of \(M\) between \(A\) and \(B\) must be an (anti-)Monge matrix.

Planar emulators. Consider an undirected planar graph \(G\) with edge weights and let \(\partial G\) be the vertices on the boundary of the outer face of \(G\). We consider the distance matrix \(M\) between vertices in \(\partial G\): for any pair of vertices \(i\) and \(j\) in \(\partial G\), we set \(M[i, j]\) to be the distance between \(i\) and \(j\) in \(G\).

It is not immediately clear that any cyclic Monge distance matrix \(M\) comes as a distance matrix generated from some planar graph \(G\). A planar emulator for a distance matrix \(M\) is a graph \(G\) whose vertex set \(V(G)\) contains the indices of \(M\) (and possibly others), and the graph distance \(d_G(u, v)\) between any pair of vertices \(u\) and \(v\) in \(G\) is equal to \(M[u, v]\). Planarity and the Jordan curve theorem ensures that any distance matrix \(M\) of a planar emulator must satisfy the cyclic Monge property. Our main result shows that the converse is also true: any cyclic Monge distance matrix admits a planar emulator.

In Section 2 we describe the construction and prove its correctness. We show that the size of the construction is optimal in Section 3, and conclude the paper in Section 4.

\(^2\)This is known as the Kalmanson matrix [22,33], which is slightly more restricted than a triangular Monge matrix [12] or the convex quadrangle inequality [26].

2 Constructing a planar emulator

The goal of this section is to construct planar emulators for arbitrary cyclic Monge distance matrices.

Theorem 1 Given any \(n \times n\) cyclic Monge distance matrix \(M\), there is a planar emulator for \(M\) with \(\binom{n}{2}\) edges.

For any given positive integer \(n\), we define a planar graph \(G^n\) as follows (see Figure 1). Let the vertices of \(G^n\) be the set \(\{v_i, j\}\), where \(i\) ranges in \([1 : n]\) and \(j\) ranges in \([1 : \min\{i, n - i + 1\}\]. Define terminal \(p_i\) to be \(v_{i, \min\{i, n - i + 1\}}\). The edges of \(G^n\) consist of horizontal edges and vertical edges. A horizontal edge \(e_{i,j}^{\downarrow}\) lies between each \(v_{i,j}\) and \(v_{i,j+1}\) where \(j\) ranges in \([1 : n/2]\) and \(i\) ranges in \([j : n - j]\). A vertical edge \(e_{i,j}^{\uparrow}\) lies between each \(v_{i,j}\) and \(v_{i,j+1}\) where \(j\) ranges in \([1 : \min\{i, n + 1 - i\} - 1]\) and \(i\) ranges in \([2 : n - 1]\).

![Figure 1: Graph \(G^6\).](image)

Consider a cyclic Monge distance matrix \(M\) and for brevity denote \(M_{i,j} := M[i, j]\). We define the graph \(G^n_M\) as an edge-weighted copy of \(G^n\), where the weight of a horizontal edge \(e_{i,j}^{\downarrow}\) is

\[
\omega(e_{i,j}^{\downarrow}) := \frac{1}{2} \left( M_{i+1,j} - M_{i,j} + M_{i, n-j+1} - M_{i+1, n-j+1} \right),
\]

and the weight of a vertical edge \(e_{i,j}^{\uparrow}\) is

\[
\omega(e_{i,j}^{\uparrow}) := \frac{1}{2} \left( M_{i, j+1} - M_{i+1,j+1} + M_{i, n-j} - M_{i, n-j} + M_{j+1, n-j} - M_{j, n-j+1} \right).
\]

(See Figure 2.) Henceforth, we will refer to the edge-weighted graph \(G^n_M\) as the canonical realization of \(M\).

For the rest of the section, we show that \(G := G^n_M\) is a planar emulator of \(M\). For this, it suffices to show that \(d_G(p_i, p_j) = M[i, j]\) for all pairs of terminals \(p_i\) and \(p_j\). First, we derive some properties of \(G\) using the fact that \(M\) is a cyclic Monge matrix.

Lemma 2 If \(M\) is a cyclic Monge matrix, then all edge weights of \(G^n_M\) are non-negative.

Proof. An edge of \(G^n_M\) is either horizontal or vertical. For any horizontal edge \(e_{i,j}^{\downarrow}\), the cyclic Monge property states that \(M_{i,j} + M_{i+1,n-j+1} \leq M_{i+1,j} + M_{i, n-j+1}\), and therefore \(2\omega(e_{i,j}^{\downarrow}) = M_{i+1,j} - M_{i,j} + M_{i, n-j+1} - M_{i+1, n-j+1} \geq 0\).
Let $M$ be a cyclic Monge distance matrix. The
and
of horizontal edges followed by a path
of vertical edges (both paths might be empty). When $\min\{i, n-i+1\} > \min\{j, n-j+1\}$ we define $\pi_{j,i} := \pi_{i,j}$.

**Lemma 3** Let $M$ be a cyclic Monge distance matrix. The weight of $\pi_{j,i}$ in $G^*_M$ is $M_{i,j}$.

**Proof.** We assume that $j \leq \lfloor n/2 \rfloor$ (the other case is symmetric). The vertex at the end of $\pi_{j,i}$ (and at the start of $\pi_{i,j}$) is $v_{i,j}$. Let $i' := \min\{i, n-i+1\}$, then the weight of $\pi_{i,j}$ is

$$\omega(\pi_{j,i}) = \sum_{x \in [i'-1]} \omega(e_{i,x} \oplus e_{x,i}) + \sum_{y \in [j'-1]} \omega(e_{j,y} \oplus e_{y,j})$$

$$= \frac{1}{2} ((M_{i,j} - M_{i,j'}) + (M_{j,i'} - M_{j,i'} + M_{i,j'} - M_{i,j} + M_{i,j' + 1} - M_{i,j}))$$

$$= \frac{1}{2} ((M_{i,j} + M_{i,j'} - M_{i,j'} + M_{i,j'}) + (M_{j,i'} - M_{j,i'} + M_{i,j'} - M_{i,j} + M_{i,j' + 1} - M_{i,j}))$$

where either $M_{i,j'} = 0$ and $M_{i,j'} + 1 = M_{i,n-j'} + 1$, or $M_{i,n-j'} + 1 = 0$ and $M_{i,n-j'} = M_{i,n-j} + 1$, so $\omega(\pi_{j,i}) = M_{i,j}$.

By Lemma 3 we have $d_G(p_i, p_j) \leq M_{i,j}$, so it remains to show that $d_G(p_i, p_j) \geq M_{i,j}$. Define the $y$-coordinate of a horizontal edge $e_{i,j}$ as $j$, and the $x$-coordinate of a vertical edge $e_{i,j}$ as $i$. We next show that $G$ contains a minimum-weight path from $p_i$ to $p_j$ whose horizontal edges all have the same $y$-coordinate. It follows that there is a minimum-weight path consisting of at most one subpath of horizontal edges.

**Lemma 4** Let $M$ be a cyclic Monge distance matrix. For any pair of terminals $p$ and $p'$, $G^*_M$ has a minimum-weight path from $p$ to $p'$ whose horizontal edges all have the same $y$-coordinate.

**Proof.** For a path $\pi$, let $\sigma(\pi)$ be the sum of $y$-coordinates of its horizontal edges. Let $\alpha$ be a minimum-weight path from $p$ to $p'$ that minimizes $\sigma(\alpha)$ (over all minimum-weight paths from $p$ to $p'$). We claim that all horizontal edges of $\alpha$ have the same $y$-coordinate. Suppose not, then $\alpha$ contains a two-edge subpath consisting of a vertical edge $e_{i,j}$ and a horizontal edge $e_{i,j+1}$ or $e_{i+1,j}$. We consider only the case where the subpath has edges $e_{i,j}$ and $e_{i,j+1}$ (the other case is symmetric). Consider the path $\beta$ obtained from $\alpha$ by replacing this subpath by $e_{i,j+1}$ and $e_{i+1,j}$. Then $\sigma(\beta) < \sigma(\alpha)$, so by assumption $\beta$ cannot be a minimum-weight path. However, Figure 3 shows that the weight of $\beta$ is at most that of $\alpha$, contradicting that $\alpha$ is a minimum-weight path that minimizes $\sigma$.

Finally, we show that there is a minimum-weight path for which additionally, its vertical edges all have the same $x$-coordinate. Together with the fact that all edge weights are non-negative (Lemma 2), it follows that $\pi_{j,i}$ is a minimum-weight path between $p_j$ and $p_i$.

**Lemma 5** Let $M$ be a cyclic Monge distance matrix. For any pair of terminals $p$ and $p'$, $G^*_M$ has a minimum-weight path from $p$ to $p'$ whose horizontal edges all have the same $y$-coordinate, and whose vertical edges all have the same $x$-coordinate.
Proof. By Lemma 4, there is a minimum-weight path from $p$ to $p'$ whose horizontal edges all have the same $y$-coordinate, and without loss of generality assume that this $y$-coordinate is maximal over all such paths. Because all edges have nonnegative weights by Lemma 2, we may assume that this path consists of a path of vertical edges (with decreasing $y$-coordinates), followed by a path of horizontal edges whose $x$-coordinates are increasing or decreasing, and finally a path of vertical edges with increasing $y$-coordinates. Suppose that the subpath of horizontal edges is surrounded by vertical edges $e_{i,j}$ and $e_{i',j}$ with $i < i'$ (the case $i > i'$ is symmetric). Let $\alpha$ be the path consisting of $e_{i,j}$, the edges $e_{x,j}$ for $i < x < i'$, and $e_{x,j}$; let $\beta$ be the path of edges $e_{x,j}$ for $i < x < i'$.

Apply cyclic Monge property twice, one can show that $2M_{i,j} + 2M_{i,j+1} + 2M_{i+1,j+1} + 2M_{i+1,j} - 2M_{i+1,n-j} - M_{i,n-j} \geq M_{i',j+1} + M_{i',n-j}$, which implies that the weight of $\beta$ is at most that of $\alpha$, so replacing $\alpha$ by $\beta$ yields a shortest path whose horizontal edges all have the same $y$-coordinate, but one bigger than that of the horizontal edges of $\alpha$, which is a contradiction. (See Figure 4.)

As an immediate corollary of Lemmas 2, 3, and 5, every $n \times n$ cyclic Monge distance matrix has a planar emulator of size $\binom{n}{2}$, proving Theorem 1.

3 Lower bound on the size of planar emulators

In this section we show that some Monge distance matrices require $\binom{n}{2}$ edges in any of its planar emulator. A similar result by Cossarini [19] says that any planar emulator of some cyclic Monge matrix requires $\binom{n}{2}$ edges. Therefore, our canonical realization is worst-case optimal in size.

Theorem 6 Some $n \times n$ Monge distance matrices have no planar emulator with fewer than $\binom{n}{2}$ edges.

Proof. Let $M$ be a Monge distance matrix. The vector $(M_{i,j})_{i,j} \in \mathbb{R}^{(n)}$ completely determines $M$ since $M_{i,j} = 0$ and $M_{i,j} = M_{i,j}$ as $d$ is a graph metric on the canonical realization of $M$. The set of such vectors over all Monge distance matrices yields a convex polytope $\mathcal{P}$, as it is bounded only by the hyperplanes arising from the linear inequalities of the triangle inequality and cyclic Monge property. We show that $\mathcal{P}$ is $(\binom{n}{2})$-dimensional.

For this, we define a family of $\binom{n}{2}$ sets $(E^i_{e_{i,j}})_{e_{i,j} \in E(G)}$ of edges indexed by the edges of $G^M$. For each horizontal edge $e_{i,j}$, let $E^i_{e_{i,j}} := \{e_{i',j} \mid j' \leq j\}$. For each vertical edge $e_{i,j}$, let $E^i_{e_{i,j}} := \{e_{i,j}' \} \cup E^i_{e_{i,j}} \cup E^i_{e_{i,j}}$. For each edge $e$, define the weight function $\omega_e$ as the characteristic function of $E_e$; in other words, let $\omega_e : E \to \{0, 1\}$, with $\omega_e(e') = 1$ if $e' \in E_e$, and $\omega_e(e') = 0$ otherwise. We show that the $(\binom{n}{2})$ weight functions $(\omega_e)_{e_{i,j} \in E(G)}$ are linearly independent. For each horizontal edge $e_{i,j}$, $\omega_{e_{i,j}}$ sets only the weight of edge $e_{i,j}$ to one, and all other edges to zero. Similarly, for each horizontal edge $e_{i,j}$ with $j > 1$, $e \mapsto \omega_{e_{j}}(e) - \omega_{e_{j+1}}(e)$ sets only the weight of edge $e_{i,j}$ to one. Finally, for each vertical edge $e_{i,j}$, $e \mapsto \omega_{e_{i,j}}(e) - \omega_{e_{i,j}}(e)$ sets only the weight of edge $e_{i,j}$ to one. Since each of the $(\binom{n}{2})$ edges can be set to weight one while all other edges are set to zero, the defined weight functions are linearly independent, and
Theorem 7 Some $n \times n$ unit-Monge distance matrices have no planar emulator with fewer than $n^2/8 + n/2$ edges.

Proof. Let $M$ be a distance matrix defined as follows. Consider a rectangular grid graph with vertex set $\{0, \ldots, w\} \times \{0, \ldots, h\}$ and edges between vertices at distance 1, so that vertex $(x, y)$ (unit-weight) edges to $(x \pm 1, y)$ and $(x, y \pm 1)$. For all $x$ and $k$, we have $d(0, y), (w, y + k)) = w + k$, and symmetrically $d((x, 0), (x, h)) = h + k$ for all $x$ and $k$. Let $M$ be the distance matrix from the set of vertices $\{(x, 0)\} \cup \{(0, y)\}$ to the set of vertices $\{(x, h)\} \cup \{(w, y)\}$; distance matrix $M$ must be unit-Monge.

Consider an arbitrary planar emulator $G$ of $M$. Let $d_G$ denote the shortest-path metric on $G$. For vertices $i, j, k, \ell$ in clockwise-order along the outer face, we have $d_G(i, \ell) + d_G(j, k) \leq d_G(i, k) + d_G(j, \ell)$. On the other hand, for any pair of points $p$ and $q$ where $p$ is on a shortest path from $i$ to $\ell$ and $q$ on a shortest path from $j$ to $k$, we have $d_G(i, \ell) + d_G(j, k) + 2d_G(p, q) \geq d_G(i, k) + d_G(j, \ell)$.

Denote by $\pi_{y}^{x}$ a shortest path in $G$ between $(0, y)$ and $(w, y)$, and by $\pi_{x}^{k}$ a shortest path in $G$ between $(x, 0)$ and $(x, h)$. We will show that the paths $\pi_{y}^{x}$ are disjoint and have $h$ edges each. Recall that $d_G(i, \ell) + d_G(j, k) + 2d_G(p, q) \geq d_G(i, k) + d_G(j, \ell)$, so

$$\|\pi_{y}^{x}\| + \|\pi_{y}^{x}\| + 2d_G(\pi_{y}^{x}, \pi_{y}^{x}) = 2w + 2d_G(\pi_{y}^{x}, \pi_{y}^{x}) \geq d_G((0, y), (w, y + k)) + d_G((0, y + k), (w, y)) = 2w + k,$$

and thus any pair of points $p \in \pi_{y}^{x}$ and $q \in \pi_{x}^{k}$ on distinct paths have distance at least $k \geq 1$, so different such paths are vertex-disjoint. Any path $\pi_{i}^{j}$ must cross all the (vertex-disjoint) paths $\pi_{0}^{1}, \ldots, \pi_{h}^{x}$, and thus have at least $h$ edges (not shared with any path $\pi_{y}^{x}$) of length at least 1. Therefore, the paths $\pi_{x}^{j}$ and $\pi_{y}^{x}$ (over all $x$ and by symmetric argument $y$) contain at least $(w + 1)h + (h + 1)w$ edges. We have $n = 2(w + h)$; by taking $w = h = n/4$, this yields a lower bound of $2(n/4 + 1)(n/4) = n^2/8 + n/2$

edges for any planar emulator of $M$. □

We remark that the argument of Theorem 7 depends only on distances between opposite sides of the grid, and can be made to depend only on the linearly many distances $d((0, y), (w, y + k))$ and $d((x, 0), (x + k, h))$ with $k \in \{-1, 0, 1\}$. Cossarini [19] proved that any planar emulator for some $n \times n$ cyclic unit-Monge matrix must have at least $\binom{n}{2}$ edges.

Our result, while slightly weaker in comparison, applies to general unit-Monge matrices, which can be viewed as the directed version of the problem.

4 Discussion

In this paper we have shown that any cyclic Monge distance matrix admits a quadratic-size planar emulator. Our construction is universal in the sense that the underlying graph does not depend on the entries of the matrix. And there are metrics for which each edge must be used by some shortest path. We also showed that already for planar emulators of unit-Monge distance matrices (which can be represented in linear space), $\Omega(n^2)$ edges are sometimes necessary.

The cyclic-Monge distance matrices considered in this paper are closely connected to the set of intrinsic metrics of topological disks. In particular, a given metric on points in a circle can be realized as a metric intrinsic to a topological disk bounded by that circle if and only if the metric is a cyclic-Monge distance matrix. We conclude with an open problem.

- Under what conditions do surfaces other than the disk (such as the Möbius strip, or a torus with holes) realize a given metric between points on their boundary? Do such surfaces also have a universal emulator, and if so, one with at most $\binom{n}{2}$ edges?

References


