A Faster Algorithm to Recognize Even-Hole-Free Graphs*

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Abstract

We study the problem of determining whether an n-node graph G has an *even hole*, i.e., an induced simple cycle consisting of an even number of nodes. Conforti, Cornuéjols, Kapoor, and Vušković gave the first polynomial-time algorithm for the problem, which runs in $O(n^{40})$ time. Later, Chudnovsky, Kawarabayashi, and Seymour reduced the running time to $O(n^{31})$. The best previously known algorithm for the problem, due to da Silva and Vušković, runs in $O(n^{19})$ time. In this paper, we solve the problem in $O(n^{11})$ time. Moreover, if G has even holes, our algorithm also outputs an even hole of G in $O(n^{11})$ time.

1 Introduction

A *hole* is an induced simple cycle consisting of at least four nodes. A hole is *even* (respectively, *odd*) if it consists of an even (respectively, odd) number of nodes. See Figure 1 for an illustration. Even-hole-free graphs have been extensively studied in the literature (see, e.g., [1, 12–14, 19, 20, 29, 37]). See Vušković [41] for a recent survey. This paper studies the problem of determining whether a graph has even holes. Let n (respectively, m) be the number of nodes (respectively, edges) of the input graph. Conforti, Cornuéjols, Kapoor, and Vušković [11, 15] gave the first polynomial-time algorithm for the problem, which runs in $O(n^{40})$ time [6]. Later, Chudnovsky, Kawarabayashi, and Seymour [6] reduced the running time to $O(n^{31})$. Chudnovsky et al. [6] also observed that the running time can be further reduced to $O(n^{15})$ as long as prisms can be detected efficiently, but Maffray and Trotignon [30] showed that detecting prisms is NP-hard. The best previously known algorithm for the problem, due to da Silva and Vušković [20], runs in $O(n^{19})$ time. We solve the problem in $O(n^{11})$ time, as stated in the following theorem.

*The current version slightly improves upon the preliminary version [3] appeared in SODA 2012: (a) The time complexity for recognizing even-hole-free n-node m-edge graphs G is reduced from $O(m^2n^7)$ to $O(m^3n^5)$, which is an improvement if $m = o(n^2)$; and (b) if G has even holes, the current version shows how to output an even hole of G also in $O(m^3n^5)$ time.

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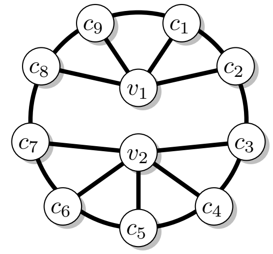


Figure 1: $C = v_1c_2c_3v_2c_7c_8v_1$ is a clean even hole of the 11-node graph G, since $M_G(C) = N_G^{2,2}(C) = \varnothing$. $C' = c_1c_2\cdots c_9c_1$ is an odd hole with $M_G(C') = \{v_2\}$.

Theorem 1.1. *It takes $O(m^3n^5)$ time to determine whether an n-node m-edge connected graph has even holes.*

Technical overview Throughout the paper, a *k-hole* (respectively, *k-cycle* and *k-path*) is a k-node hole (respectively, cycle and path). Our recognition algorithm for even-hole-free graphs consists of two phases. The first phase (see Lemma 2.3) either (1) ensures that the input graph G has even holes via the existence of a "beetle" (see §2 and Figure 2(a)) or a 4-hole in G or (2) produces a set $\mathbb{T}$ of "trackers" $(H, u_1u_2u_3)$, where H is a beetle-free and 4-hole-free induced subgraph of G that contains path $u_1u_2u_3$. $\mathbb{T}$ has the following even-hole-preserving property (see Property 1): If G has even holes, then $\mathbb{T}$ has a "lucky" tracker $(H, u_1u_2u_3)$ in that H has a shortest even hole C of G such that (a) C contains path $u_1u_2u_3$ and (b) the neighborhood of C in H is "super clean" (i.e., $M_H(C) = N_H^{2,2}(C) = N_H^{1,2}(C) = N_H^4(C) = \varnothing$ using notation to be defined in §2). The second phase applies an algorithm (see Lemma 2.4) on each tracker $(H, u_1u_2u_3) \in \mathbb{T}$ to either ensure that H has even holes or ensure that $(H, u_1u_2u_3)$ is not lucky. If all trackers in $\mathbb{T}$ are not lucky, the even-hole-preserving property of $\mathbb{T}$ implies that G is even-hole-free. Otherwise, G has an induced subgraph H containing an even hole, implying that G has even holes.

The recognition algorithm for beetle-free graphs (see the proof of Lemma 2.3) in the first phase is based on Chudnovsky and Seymour's three-in-a-tree algorithm [9] (see Lemma 3.1). If G has beetles or 4-holes, G has even holes. Otherwise, if G has even holes, the neighborhood of each shortest even hole C of G is "clean" (i.e., $N_G^{1,2}(C) = N_G^4(C) = \varnothing$, see the proof of Lemma 2.2). To further ensure that the neighborhood of C is super clean, we generate a set $\mathbb{S}$ of "super cleaners" $(S, u_1u_2u_3)$, where S is a node subset of G and $u_1u_2u_3$ is a path of G, such that at least one super cleaner $(S, u_1u_2u_3) \in \mathbb{S}$ satisfies $u_1u_2u_3 \subseteq C \subseteq H = G \setminus S$ and $M_H(C) = N_H^{2,2}(C) = \varnothing$ for some shortest even hole C of G (see the proof of Lemma 2.3). The set $\mathbb{T}$ consisting of the trackers $(G \setminus S, u_1u_2u_3)$ with $(S, u_1u_2u_2) \in \mathbb{S}$ has the required even-hole-preserving property.

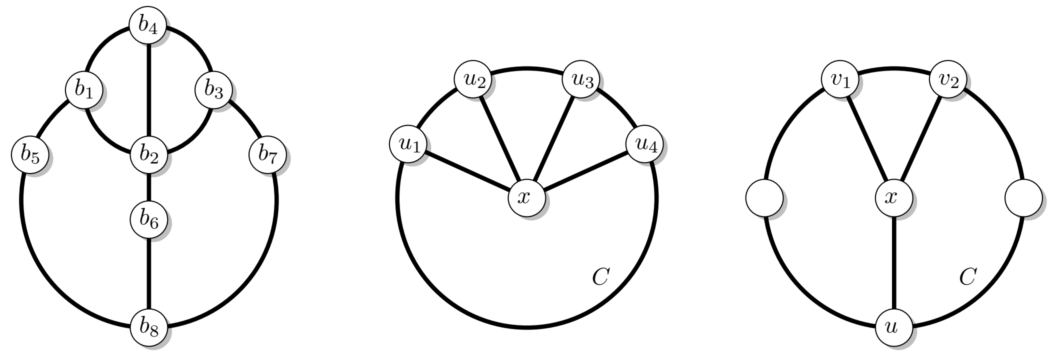
The algorithm applied on each tracker $T = (H, u_1u_2u_3) \in \mathbb{T}$ in the second phase is a decomposition algorithm (see, e.g., the categorization of Vušković [41, §4]) based upon an observation of da Silva and Vušković [20] (see Lemma 4.9) that if a connected graph H is even-hole-free, star-cutset-free, and non-path 2-join-free, then H is an extended clique tree. Since even holes can be efficiently detected in an extended clique tree (see Lemma 4.6, which is a slightly faster implementation of the algorithm of da Silva and Vušković [20]), our algorithm performs two stages of even-hole-preserving decompositions on H, first via star-cutsets and then via non-path 2-joins, until each of the resulting graph either is an extended clique tree or has $O(1)$ nodes. If all of the resulting graphs are even-hole-free, T is not lucky; otherwise, H has even holes. An immediate challenge for the first stage of decompositions is that there are no known polynomial-time detection algorithms for star-cutsets. Fortunately, as noted by Chvátal [10] (see Lemma 4.3), if H is dominated-node-free, a star-cutset of H has to be a full star-cutset of H, which can be efficiently detected. Thus, at the beginning of each decomposition in the first stage, we preprocess $(H, u_1u_2u_3)$ by deleting all dominated nodes of H and carefully updating nodes u_1, u_2, and u_3 such that the luckiness of $(H, u_1u_2u_3)$ is preserved (see Lemma 4.4). The correctness of this preprocessing step relies on the fact that H is beetle-free and the requirement for $(H, u_1u_2u_3)$ to be lucky that the neighborhood of some shortest even hole C in H with $u_1u_2u_3 \subseteq C$ is super clean. Path $u_1u_2u_3$ is crucial in the stage of decompositions via star-cutsets for the dominated-node-free graph H. Specifically, if S is a star-cutset of H, by merely examining the neighborhood of path $u_1u_2u_3$ in H, we can efficiently identify a connected component B of $H \setminus S$ such that $(H[S \cup B], u_1u_2u_3)$ preserves the luckiness of $(H, u_1u_2u_3)$ (see Step 3 in the proof of Lemma 4.1). We then let $H = H[C \cup B]$ and repeat the above procedure for $O(n)$ iterations until H is star-cutset-free. The second stage, i.e., decompositions via non-path 2-joins for star-cutset-free graphs, is based upon the detection algorithm for non-path 2-joins of Charbit et al. [4] (see Lemma 4.8). This stage decomposes an m-edge star-cutset-free graph into a set of $O(m)$ smaller graphs, each of which either consists of $O(1)$ nodes or is an extended clique tree (see the proof of Lemma 4.2).

Related work Even-hole-free planar graphs [33] can be recognized in $O(n^3)$ time. It is NP-complete to determine whether a graph has an even (respectively, odd) hole that contains a given node [2]. The strong perfect graph theorem of Chudnovsky, Robertson, Seymour, and Thomas [7] states that a graph G is perfect if and only if both G and the complement of G are odd-hole-free. Although perfect graphs can be recognized in $O(n^9)$ time [5], the tractability of recognizing odd-hole-free graphs remains open (see, e.g., [25]). Polynomial-time algorithms for detecting odd holes are known for planar graphs [24], claw-free graphs [28, 36], and graphs with bounded clique numbers [16]. Graphs without holes (i.e., chordal graphs) can be recognized in $O(m+n)$ time [34, 35, 38]. Graphs without holes consisting of five or more nodes (i.e., weakly chordal graphs) can be recognized in $O(m^2+n)$ time [31, 32]. It takes $O(n^2)$ time to detect a hole that contains any $o((\log n/\log\log n)^{2/3})$ given nodes in an $O(1)$-genus graph [26, 27]. See [8, 17, 21, 42] for more results on odd-hole-free graphs.

Road map The rest of the paper is organized as follows. Section 2 gives the preliminaries and proves Theorem 1.1 by Lemmas 2.3 and 2.4. Section 3 proves Lemma 2.3. Section 4 proves Lemma 2.4. Section 5 concludes the paper by explaining how to augment Theorem 1.1 into an $O(m^3n^5)$-time algorithm that outputs an even hole of an n-node m-edge graph with even holes.

2 Preliminaries and the topmost structure of our proof

Unless clearly specified otherwise, all graphs throughout the paper are undirected and simple. Let $|S|$ be the cardinality of set S. Let G be a graph. Let $V(G)$ consist of the nodes in G. For any subgraph H of G, let $G[H]$ denote the subgraph of G induced by $V(H)$. Subgraphs H and H' of graph G are *adjacent* in G if some node of H and some node of H' are adjacent in G. For any subset U of $V(G)$, let $G \setminus U = G[V(G) \setminus U]$. For any subgraph H of G, let $N_G(H)$ consist of the nodes of $V(G) \setminus V(H)$ that are adjacent to H in G and let $N_G[H] = N_G(H) \cup V(H)$.



(a) (b) (c)

Figure 2: (a) A beetle B, where $B[\{b_1, b_2, b_3, b_4\}]$ is a diamond. (b) If $x \in N_G^4(C)$, then $G[C \cup \{x\}]$ is a beetle B, where $B[\{u_1, u_2, u_3, x\}]$ is a diamond. (c) A node $x \in N_G^{1,2}(C)$.

Let C be a hole of G. A node $x \in V(G) \setminus V(C)$ is a *major node* [6] of C in G if at least three distinct nodes of $N_C(x)$ are pairwise non-adjacent in G. Let $M_G(C)$ consist of the major nodes of C in G. For instance, in Figure 1, $M_G(C) = \varnothing$ and $M_G(C') = \{v_2\}$.

Lemma 2.1 (Chudnovsky et al. [6, Lemma 2.2]). *If C is a shortest even hole of graph G and $x \in M_G(C)$, $|N_C(x)|$ is even.*

If $x \in N_G(C) \setminus M_G(C)$, $1 \leq |N_C(x)| \leq 4$ and $C[N_C(x)]$ has at most two connected components. Moreover, if $C[N_C(x)]$ is not connected, each connected component of $C[N_C(x)]$ has at most two nodes. Let $N_G^i(C)$ with $1 \leq i \leq 4$ consist of the nodes $x \in N_G(C) \setminus M_G(C)$ such that $|N_C(x)| = i$ and $C[N_C(x)]$ is connected. Let $N_G^{i,j}(C)$ with $1 \leq i \leq j \leq 2$ consist of the nodes $x \in N_G(C) \setminus M_G(C)$ such that $C[N_C(x)]$ is not connected and the two connected components of $C[N_C(x)]$ has i and j nodes, respectively. We have

$$N_G(C) = N_G^1(C) \cup N_G^2(C) \cup N_G^3(C) \cup N_G^4(C) \cup N_G^{1,1}(C) \cup N_G^{1,2}(C) \cup N_G^{2,2}(C) \cup M_G(C). \quad (1)$$

We say that C is a *clean* hole of G if $M_G(C) = N_G^{2,2}(C) = \varnothing$. For any 3-path $u_1u_2u_3$ of G, a clean hole C of G that contains path $u_1u_2u_3$ is a $u_1u_2u_3$*-hole* of G if C is a shortest even hole of G. For instance, if G is as shown in Figure 1, $C = v_1c_2c_3v_2c_7c_8v_1$ is a $v_1c_2c_3$-hole of G. If H is an induced subgraph of G and $u_1u_2u_3$ is a 3-path of H, we call $(H, u_1u_2u_3)$ a *tracker* of G. A tracker $(H, u_1u_2u_3)$ of G is *lucky* if H contains a $u_1u_2u_3$-hole of G. If G has lucky trackers, G has even holes. The following even-hole-preserving property of $\mathbb{T}$ reduces the problem of determining whether G is even-hole-free to the problem of determining whether all trackers in $\mathbb{T}$ are not lucky.

Property 1. *If G has even holes, $\mathbb{T}$ contains a lucky tracker of G.*

An induced subgraph of G is a *beetle* of G if it consists of (1) a 4-cycle $b_1b_2b_3b_4b_1$ with exactly one chord b_2b_4 (i.e., a *diamond* [15, 29] of G) and (2) a tree I of $G \setminus \{b_4\}$ having exactly three leaves b_1, b_2, and b_3 with the property that $I \setminus \{b_1, b_2, b_3\}$ is an induced tree of G not adjacent to b_4. See Figure 2(a) for an illustration. At least one of the three cycles in $B \setminus \{b_2\}$, $B \setminus \{b_1, b_4\}$, and $B \setminus \{b_3, b_4\}$ is an even hole of G. Nodes b_5, b_6, b_7, and b_8 need not be distinct. For instance, as illustrated by Figure 2(b), if C is a hole of G and x is a node of $N_G^4(C)$, then $G[C \cup \{x\}]$ is a beetle of G.

Lemma 2.2. *If G is a beetle-free graph, $N_G(C) \subseteq N_G^{1,1}(C) \cup N_G^1(C) \cup N_G^2(C) \cup N_G^3(C)$ holds for any clean shortest even hole C of G.*

Proof. By $M_G(C) = N_G^{2,2}(C) = \varnothing$ and Equation (1), it suffices to show $N_G^{1,2}(C) = N_G^4(C) = \varnothing$. If $x \in N_G^4(C)$ as illustrated by Figure 2(b), then $G[C \cup \{x\}]$ is a beetle of G, a contradiction. If $x \in N_G^{1,2}(C)$, then let u, v_1, and v_2 be the nodes of $N_C(x)$ such that v_1 and v_2 are adjacent in C, as illustrated by Figure 2(c). Let P_1 be the path of $C \setminus \{v_2\}$ between u and v_1. Let P_2 be the path of $C \setminus \{v_1\}$ between u and v_2. Either $G[\{x\} \cup P_1]$ or $G[\{x\} \cup P_2]$ is an even hole of G shorter than C, a contradiction. The lemma is proved. □

2.1 Proving Theorem 1.1

Lemma 2.3. *It takes $O(m^3n^5)$ time to complete either one of the following tasks for any n-node m-edge graph G. Task 1: Ensuring that G has even holes. Task 2: (a) Ensuring that G has no beetles and (b) obtaining a set $\mathbb{T}$ of $O(m^2n)$ trackers of G that satisfies Property 1.*

Lemma 2.4. *Given a tracker $T = (H, u_1u_2u_3)$ of an n-node beetle-free graph G, it takes $O(mn^4)$ time to either ensure that H has even holes or ensure that T is not lucky.*

Proof of Theorem 1.1. We apply Lemma 2.3 on G in $O(m^3n^5)$ time. If Task 1 is completed, then the theorem is proved. If Task 2 is completed, then G has no beetles and we have a set $\mathbb{T}$ of $O(m^2n)$ trackers of G that satisfies Property 1. By Property 1 of $\mathbb{T}$ and Lemma 2.4, one can determine whether G has even holes in time $|\mathbb{T}| \cdot O(mn^4) = O(m^3n^5)$. The theorem is proved. □

3 Proving Lemma 2.3

A *clique* of G is a complete subgraph of G. A clique of G is *maximal* if it is not contained by other cliques of G. We need the following four lemmas to prove Lemma 2.3.

Lemma 3.1 (Chudnovsky and Seymour [9])**.** *It takes $O(n^4)$ time to determine whether an n-node graph has an induced tree that contains three given nodes.*

Lemma 3.2 (da Silva and Vušković [19, Section 2] and Farber [22, Proposition 2])**.** *If G is an n-node m-edge 4-hole-free graph, it takes $O(mn^2)$ time to either ensure that G has even holes or obtain all $O(n^2)$ maximal cliques of G.*

Lemma 3.3 (Chudnovsky, Kawarabayashi, and Seymour [6, Lemma 4.2])**.** *Any shortest even hole C of a 4-hole-free graph G contains an edge v_1v_2 with $N_G^{2,2}(C) \subseteq N_G(v_1) \cap N_G(v_2)$.*

Lemma 3.4. *For any shortest even hole C of a 4-hole-free graph G, if $G[M_G(C)]$ is not a clique of G, there is a node u of C with $M_G(C) \subseteq N_G(u)$.*

Before proving Lemma 3.4, we first prove Lemma 2.3 using Lemmas 3.1, 3.2, 3.3, and 3.4.

Proof of Lemma 2.3. We claim that G has beetles if and only if at least one of the $O(m^3n)$ choices of node b_4 and three distinct edges b_1b_5, b_2b_6, and b_3b_7 of G satisfies the following conditions.

- $G[\{b_1, b_2, b_3, b_4\}]$ is the 4-cycle $b_1b_2b_3b_4b_1$ with exactly one chord b_2b_4.
- The edges between $\{b_1, b_2, b_3\}$ and $\{b_5, b_6, b_7\}$ are exactly b_1b_5, b_2b_6, and b_3b_7.

- $\{b_5, b_6, b_7\} \cap \{b_1, b_2, b_3, b_4\} = \varnothing$, but nodes b_5, b_6, and b_7 need not be distinct.
- An induced tree I' of $G \setminus ((N_G[b_1] \cup \cdots \cup N_G[b_4]) \setminus \{b_5, b_6, b_7\})$ contains $\{b_5, b_6, b_7\}$.

The claim can be verified by seeing that if I'' is the minimal subtree of I' that contains $\{b_5, b_6, b_7\}$, then $I = I'' \cup \{b_1b_5, b_2b_6, b_3b_7\}$ is a tree of $G \setminus \{b_4\}$ with leaf set $\{b_1, b_2, b_3\}$ having the property that $I \setminus \{b_1, b_2, b_3\}$ is an induced tree of G not adjacent to b_4. By the claim and Lemma 3.1, it takes $O(m^3n^5)$ time to determine whether G has beetles. It takes $O(n^4)$ time to determine whether G has 4-holes. If G has 4-holes or beetles, G has even holes. The lemma is proved by completing Task 1 in $O(m^3n^5)$ time. The rest of the proof assumes that G is 4-hole-free and beetle-free.

By Lemma 3.2, it takes $O(mn^2)$ time to either ensure that G has even holes or obtain the $O(n^2)$ maximal cliques of G. If G has even holes, the lemma is proved by completing Task 1 in $O(mn^2)$ time. Otherwise, let $\mathbb{T}$ consist of the trackers of G that are in the form of $(G \setminus S_1, u_1u_2u_3)$ or $(G \setminus S_2, u_1u_2u_3)$ with

$$\begin{array}{lclcl} S_1 & = & S_1(u_1, u_2, u_3, v_1, v_2) & = & (N_G(v_1) \cap N_G(v_2)) \cup (N_G(u_2) \setminus \{u_1, u_3\}); \\ S_2 & = & S_2(u_1, u_2, K) & = & (N_G(u_1) \cap N_G(u_2)) \cup V(K), \end{array}$$

where u_1u_2 and v_1v_2 are edges of G and K is a maximal clique of G. We have $|\mathbb{T}| = O(m^2n) + O(mn^2) = O(m^2n)$. Since all $O(n^2)$ maximal cliques of G are available, $\mathbb{T}$ can be computed in time $O(m^2n) \cdot O(n+m) = O(m^3n)$ time. To ensure the completion of Task 2, it remains to show that $\mathbb{T}$ satisfies Property 1. Suppose that G has even holes. Let C be an arbitrary shortest even hole of G. The following case analysis shows that $\mathbb{T}$ contains lucky trackers of G.

Case 1: $M_G(C) \subseteq N_G(u_2)$ holds for a node u_2 of C. Let u_1 and u_3 be the neighbors of u_2 in C. By $M_G(C) \subseteq N_G(u_2) \setminus \{u_1, u_3\}$ and Lemma 3.3, there is an edge v_1v_2 of C with $M_G(C) \cup N_G^{2,2}(C) \subseteq S_1$. By the choices of u_1 and u_3, we have $(N_C(u_2) \setminus \{u_1, u_3\}) \cap C = \varnothing$. Since v_1v_2 is an edge of hole C, we have $N_G(v_1) \cap N_G(v_2) \cap C = \varnothing$. Thus, $S_1 \cap C = \varnothing$, implying that C is a clean hole of $G \setminus S_1$ containing path $u_1u_2u_3$. Since C is a shortest even hole of G, C is also a shortest even hole of $G \setminus S_1$. Therefore, C is a $u_1u_2u_3$-hole of $G \setminus S_1$.

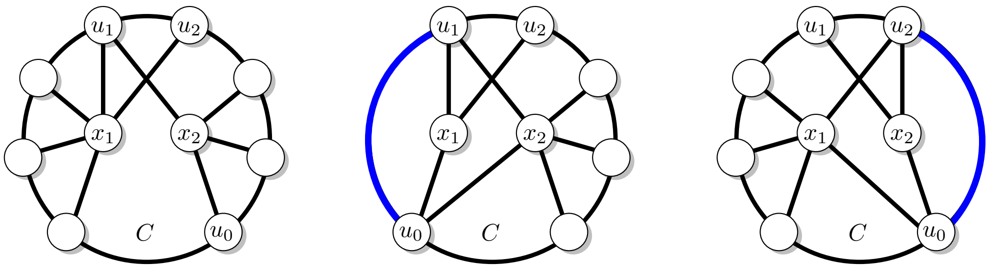
Case 2: $M_G(C) \not\subseteq N_G(u)$ holds for all nodes u of C. By Lemma 3.4, $G[M_G(C)]$ is a clique of G. Let K be a maximal clique of G with $M_G(C) \subseteq V(K)$. Combining with Lemma 3.3, there is an edge u_1u_2 of C with $M_G(C) \cup N_G^{2,2}(C) \subseteq S_2$. We have $V(K) \cap C = \varnothing$ or else $M_G(C) \cap C = \varnothing$ implies $M_G(C) \subseteq V(K) \setminus \{u\} \subseteq N_G(u)$ for any node $u \in V(K) \cap C$, a contradiction. Since u_1u_2 is an edge of C, we have $N_G(u_1) \cap N_G(u_2) \cap C = \varnothing$. Thus, $S_2 \cap C = \varnothing$, implying that C is a clean hole of $G \setminus S_2$ containing path $u_1u_2u_3$, where u_3 is the neighbor of u_2 in C other than u_1. Since C is a shortest even hole of G, C is also a shortest even hole of $G \setminus S_2$. Therefore, C is a $u_1u_2u_3$-hole of $G \setminus S_2$. □

The rest of the section proves Lemma 3.4. An edge u_1u_2 of hole C is a *gate* [6] of C with respect to major nodes x_1 and x_2 of C if the following conditions hold:

Condition G1: There are two edges u_1x_2 and u_2x_1 and at least one of edges u_1x_1 and u_2x_2.

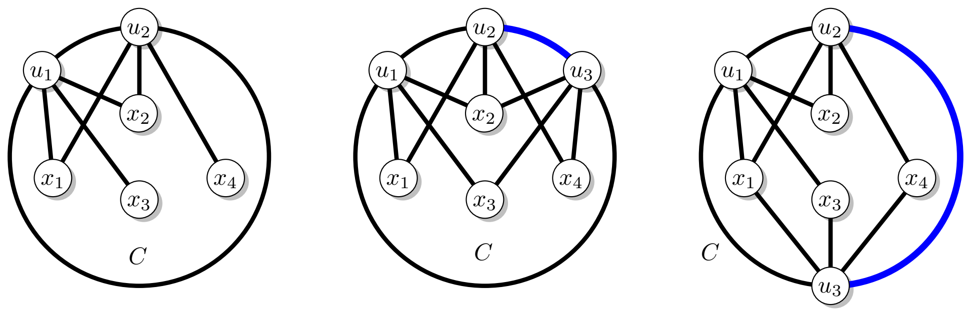
Condition G2: There is a node u_0 of $C \setminus \{u_1, u_2\}$ such that x_1 (respectively, x_2) is not adjacent to $C \setminus V(P_1)$ (respectively, $C \setminus V(P_2)$), where P_1 (respectively, P_2) is the path of C between u_2 (respectively, u_1) and u_0 that contains u_1 (respectively, u_2).

See Figure 3 for an illustration.



(a) (b) (c)

Figure 3: (a) Edge u_1u_2 is a gate of the 8-hole C induced by nodes other than x_1 and x_2, which are the major nodes of C. (b) and (c) Illustrations for the proof of Lemma 3.4.



(a) (b) (c)

Figure 4: Illustrations for the proof of Lemma 3.4.

Lemma 3.5 (Chudnovsky et al. [6, Lemmas 2.3 and 2.4])**.** *The following statements hold for any shortest even hole C of a 4-hole-free graph G.*

1. *If x_1 and x_2 are non-adjacent nodes of $M_G(C)$, there is a gate of C with respect to x_1 and x_2 in G.*
2. *If X is a subset of $M_G(C)$ with $|X| = 3$ such that $G[X]$ has at most one edge, $X \subseteq N_G(u)$ holds for some node u of C.*

Proof of Lemma 3.4. Let x_1 and x_2 be two non-adjacent nodes of $M_G(C)$. Let U consist of the nodes u of C that are adjacent to both of x_1 and x_2. By Lemma 3.5(1), there is a gate u_1u_2 of C with respect to x_1 and x_2. We have $\varnothing \neq U \subseteq \{u_1, u_2, u_0\}$, where u_0 is a node of C ensured by Condition G2. Assume $u_0 \in U$ for contradiction. By Condition G1, u_0 is adjacent u_1 or u_2 in G or else one of $u_1x_1u_0x_2u_1$ and $u_2x_1u_0x_2u_2$ would be a 4-hole of G. If u_0 is adjacent to u_1 as illustrated by Figure 3(b), then Condition G2 implies $N_C(x_1) = \{u_0, u_1, u_2\}$, which contradicts with $x_1 \in M_G(C)$. If u_0 is adjacent to u_2 as illustrated by Figure 3(c), then Condition G2 implies $N_G(x_2) = \{u_0, u_1, u_2\}$, which contradicts with $x_2 \in M_G(C)$. We have $u_0 \notin U$, and thus $U \subseteq \{u_1, u_2\}$. The lemma holds trivially if $|M_G(C)| = 2$. To prove the lemma for $|M_G(C)| \geq 3$, we first show the claim: "*Each node $x \in M_G(C) \setminus \{x_1, x_2\}$ is adjacent to U.*" If one of x_1 and x_2 is not adjacent to x, the claim follows from Lemma 3.5(2). If both of x_1 and x_2 are adjacent to x, each node $u \in U$ is adjacent to x in G or else ux_1xx_2u is a 4-hole, a contradiction. The claim is proved.

By the above claim, the lemma holds if $|M_G(C)| = 3$ or $|U| = 1$. It remains to consider the cases with $|M_G(C)| \geq 4$ and $U = \{u_1, u_2\}$ (thus, there are edges u_1x_1 and u_2x_2) by showing that either u_1

or u_2 is adjacent to each node $x \in M_G(C)$. Assume $x_3 \in M_G(C) \setminus N_G(u_2)$ and $x_4 \in M_G(C) \setminus N_G(u_1)$ for contradiction. By the above claim, G has edges u_1x_3 and u_2x_4. We know $x_3 \notin N_G(x_4)$ or else $u_1u_2x_4x_3u_1$ is a 4-hole. See Figure 4(a). Observe that x_4 cannot be adjacent to both of x_1 and x_2 or else $u_1x_1x_4x_2u_1$ is a 4-hole. *Case 1: x_4 is not adjacent to x_2.* By Lemma 3.5(2), a node u_3 of C is adjacent to all of x_2, x_3, and x_4. Since u_3 is adjacent to both of x_3 and x_4, we have $u_3 \notin \{u_1, u_2\}$. See Figure 4(b). If C has edge u_2u_3, $u_1x_3u_3u_2u_1$ is a 4-hole; otherwise, $u_2x_2u_3x_4u_2$ is a 4-hole, a contradiction. *Case 2: x_4 is not adjacent to x_1.* By Lemma 3.5(2), a node u_3 of C is adjacent to all of x_1, x_3, and x_4. Since u_3 is adjacent to both of x_3 and x_4, we have $u_3 \notin \{u_1, u_2\}$. See Figure 4(c). If C has edge u_2u_3, $u_1x_3u_3u_2u_1$ is a 4-hole; otherwise, $u_2x_1u_3x_4u_2$ is a 4-hole, a contradiction. The lemma is proved. □

4 Proving Lemma 2.4

Subset S of $V(H)$ is a *star-cutset* [10] of graph H if $S \subseteq N_H[s]$ holds for some node s of S and the number of connected components of $H \setminus S$ is more than that of H.

Lemma 4.1. *For any tracker $T = (H, u_1u_2u_3)$ of an n-node m-edge beetle-free connected graph G, it takes $O(mn^3)$ time to complete one of the following three tasks. Task 1: Ensuring that H has even holes. Task 2: Ensuring that T is not lucky. Task 3: Obtaining a star-cutset-free induced subgraph H' of H such that if T is lucky, H' has even holes.*

Lemma 4.2. *It takes $O(mn^4)$ time to determine if an n-node m-edge star-cutset-free graph has even holes.*

Proof of Lemma 2.4. We apply Lemma 4.1 on the input tracker $T = (H, u_1u_2u_3)$ of G in $O(mn^3)$ time. If Task 1 or 2 is completed, the lemma is proved. If Task 3 is completed, since H' is star-cutset-free, Lemma 4.2 implies that it takes $O(mn^4)$ time to determine whether H' has even holes. Since H' is an induced subgraph of H, if H' has even holes, so does H; otherwise, T is not lucky. The lemma is proved. □

Subsection 4.1 proves Lemma 4.1. Subsection 4.2 proves Lemma 4.2.

4.1 Proving Lemma 4.1

A star-cutset S of graph H is *full* if $S = N_H[s]$ holds for some node s of S. No polynomial-time algorithms are known for detecting star-cutsets (see, e.g., [15]), but full star-cutsets in an n-node m-edge graph can be detected in $O(mn)$ time. Node x *dominates* node y in graph H if $x \neq y$ and $N_H[y] \subseteq N_H[x]$. Node y is *dominated* in H if some node of H dominates y in H. We need the following three lemmas to prove Lemma 4.1.

Lemma 4.3 (Chvátal [10, Theorem 1])**.** *A graph without dominated nodes and full star-cutsets has no star-cutsets.*

Lemma 4.4. *If $T = (H, u_1u_2u_3)$ is a tracker of an n-node m-edge beetle-free connected graph G, it takes $O(mn^2)$ time to obtain a tracker $T' = (H', u'_1u'_2u'_3)$ of G, where H' is a dominated-node-free induced subgraph of H, such that if T is lucky, so is T'.*

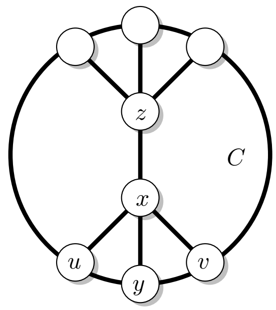


Figure 5: An illustration for the proof of Lemma 4.4.

Proof. We first prove the following claim for any beetle-free graph H: *"If a node x of H dominates a node y of a clean shortest even hole C of H, then $C' = H[C \cup \{x\} \setminus \{y\}]$ is a clean shortest even hole of H."* Let u and v be the neighbors of y on C. Since C is a hole and $y \in C$, we know $x \notin C$, implying $x \in N_H(C)$. Since x dominates y and $|N_C[y]| = 3$, there is a connected component of $C[N_C(x)]$ with at least 3 nodes. By Lemma 2.2, we have $x \in N_H^3(C)$, implying $N_C(x) = \{u, y, v\}$. Thus, C' is a shortest even hole of H. Assume $z \in M_H(C') \cup N_H^{2,2}(C')$ for contradiction. By $y \in N_H^3(C')$, $z \neq y$. By $C \setminus \{y\} = C' \setminus \{x\}$, exactly one of x and y is adjacent to z in H or else $z \in M_H(C) \cup N_H^{2,2}(C)$, contradicting to the fact that C is clean. *Case 1:* $z \in N_H^{2,2}(C')$. If $z \in N_H(y) \setminus N_H(x)$, we have $z \in M_H(C)$, contradicting to the assumption that C is a clean hole of H. If $z \in N_H(x) \setminus N_H(y)$, we have $z \in N_H^{1,2}(C)$, contradicting to Lemma 2.2. *Case 2:* $z \in M_H(C')$. By $|N_{C'}(z)| \geq 3$ and Lemma 2.1, $|N_{C'}(z)| \geq 4$. By $M_H(C) = N_H^{2,2}(C) = \varnothing$ and Lemma 2.2, $|N_C(z)| \leq 3$. By $C \setminus \{x\} = C \setminus \{y\}$, we have $z \in N_H(x) \setminus N_H(y)$, $|N_C(z)| = 3$, and $|N_{C'}(z)| = 4$. By Lemma 2.2, $z \in N_H^3(C)$. See Figure 5 for an illustration. Thus, $C[N_C(z)]$ is a 3-path, implying that $H[C' \cup \{z\}]$ is a beetle B of H in which $B[N_B[z]]$ is a diamond, a contradiction. The claim is proved.

The algorithm first iteratively updates $(H, u_1u_2u_3)$ by the following steps until H has no dominated nodes, and then outputs the resulting $(H, u_1u_2u_3)$ as $(H', u'_1u'_2u'_3)$.

Step 1: Let x and y be two nodes of H such that x dominates y.
Step 2: If there is an $i \in \{1, 2, 3\}$ with $y = u_i$, then let $u_i = x$.
Step 3: Let $H = H \setminus \{y\}$.

It takes $O(mn)$ time to detect nodes x and y such that x dominates y. Each iteration of the loop decreases $|V(H)|$ by one via Step 3. Therefore, the overall running time is $O(mn^2)$. Graph H' is a dominated-node-free induced subgraph of the initial H. It suffices to ensure that if the tracker $T = (H, u_1u_2u_3)$ of G at the beginning of an iteration is lucky, the tracker at the end of the iteration, denoted $T' = (H', u'_1, u'_2, u'_3)$, remains lucky. Let C be a $u_1u_2u_3$-hole of H. If $y \notin C$, C is a $u'_1u'_2u'_3$-hole of $H' = H \setminus \{y\}$. If $y \in C$, the above claim ensures that $C' = H[C \cup \{x\} \setminus \{y\}]$ is a clean shortest even hole of H. Since x dominates y, hole C' contains path $u'_1u'_2u'_3$. Thus, C' is a $u'_1u'_2u'_3$-hole of H'. Either way, $(H', u'_1u'_2u'_3)$ is lucky. The lemma is proved. □

Lemma 4.5. *If $(H, u_1u_2u_3)$ is a lucky tracker of graph G and S is a full star-cutset of H, one of the following two conditions holds.*

Condition B1: For each $u_1u_2u_3$-hole C of H, there exists a connected component B of $H \setminus S$ satisfying $C \subseteq H[B \cup S]$.

Condition B2: There are two non-adjacent nodes s_1 and s_2 of S and two connected components B_1 and B_2 of $H \setminus S$ with $\{s_1, s_2\} \subseteq N_H(B_1)$ and $\{s_1, s_2\} \subseteq N_H(B_2)$.

Proof. Let s be a node of S with $N_H[s] = S$. Let C be a $u_1u_2u_3$-hole of H. Assume that Condition B1 does not hold. There exist two distinct connected components B_1 and B_2 of $H \setminus S$ such that $V(C) \cap V(B_1) \neq \varnothing$ and $V(C) \cap V(B_2) \neq \varnothing$. Thus, $C[S]$ has at least two connected components. Let s_1 and s_2 be two nodes in distinct connected components of $C[S]$. By $\{s_1, s_2\} \subseteq N_H[s]$, we have $s \notin C$ or else s, s_1, and s_2 are in the same connected component of $C[S]$. By Lemma 2.2, we have $s \in N_H^{1,1}(C)$, implying $\{s_1, s_2\} = V(C) \cap S$. It follows that both s_1 and s_2 are adjacent to both B_1 and B_2. Let paths P_1 and P_2 be the two connected components of $C \setminus \{s_1, s_2\}$. B_1 has to contain one of P_1 and P_2 and B_2 has to contain other one of P_1 and P_2. Therefore, Condition B2 holds. The lemma is proved. □

Proof of Lemma 4.1. Let T_0 be the initial given tracker $(H, u_1u_2u_3)$ of G. The algorithm iteratively updates $(H, u_1u_2u_3)$ by the following three steps until Task 1, 2, or 3 is completed.

Step 1: Apply Lemma 4.4 in $O(mn^2)$ time on tracker $T = (H, u_1u_2u_3)$ to obtain a tracker $T' = (H', u'_1u'_2u'_3)$ of G, where H' is a dominated-node-free induced subgraph of H, such that if T is lucky, so is T'. Determine in $O(mn)$ time whether H' has full star-cutsets. If H' has full star-cutsets, then let $(H, u_1u_2u_3) = (H', u'_1u'_2u'_3)$ and proceed to Step 2; Otherwise, complete Task 3 by outputting H'.

Step 2: Let S be a full star-cutset of H. If Condition B2 of Lemma 4.5 holds, then complete Task 1 by outputting that G has even holes. Otherwise, proceed to Step 3.

Step 3: If either one of the following statements hold for $U = \{u_1, u_2, u_3\}$:

- $U \subseteq S$ and a connected component B of $H \setminus S$ is adjacent to both u_1 and u_3;
- $U \nsubseteq S$ and $U \subseteq B \cup S$ holds for a connected component B of $H \setminus S$,

then let $H = H[B \cup S]$ and proceed to the next iteration of the loop. Otherwise, complete Task 2 by outputting that T_0 is not lucky.

Step 1 does not increase $|V(H)|$. If Step 3 updates H, then $|V(H)|$ is decreased by at least one, since $H \setminus S$ has more than one connected component. The algorithm halts in $O(n)$ iterations. Step 1 takes $O(mn^2)$ time. Step 2 takes $O(mn^2)$ time: For any two non-adjacent nodes s_1 and s_2 in S, it takes $O(m)$ time to determine whether s_1 and s_2 have two or more common neighboring connected components of $H \setminus S$. Step 3 takes $O(m)$ time. The overall running time is $O(mn^3)$.

We first show the following claim for each iteration of the algorithm: *"If the $(H, u_1u_2u_3)$ at the beginning of an iteration is a lucky tracker of G, then (1) the intermediate $(H, u_1u_2u_3)$ throughout the iteration remains a lucky tracker of G, and (2) Step 3, if reached, proceeds to the next iteration."* It suffices to consider the situation that Step 3 is reached and focus on the update operation that replaces H with $H[B \cup S]$ via Step 3. By definition of Step 2, Condition B2 does not hold. By Lemma 4.5, Condition B1 holds. That is, there is a connected component B^* of $H \setminus S$ such that $H[B^* \cup S]$ contains some $u_1u_2u_3$-hole C of H. We prove the claim by showing that B^* has to be the connected component B of $H \setminus S$ in Step 3. Since $B^* = B$ holds trivially for the case $\{u_1, u_2, u_3\} \nsubseteq S$, we assume $\{u_1, u_2, u_3\} \subseteq S$. If $s \in C$, there are exactly two nodes of C that are adjacent to s in H;

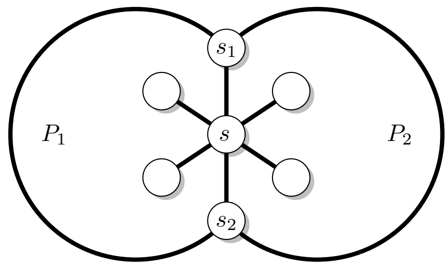


Figure 6: An illustration for the proof of Lemma 4.1.

otherwise, Lemma 2.2 implies that s has at most three neighbors of H in C. Either way, we have $|V(C) \cap S| \leq 3$. Since $u_1u_2u_3$ is a path of even hole C, nodes u_1 and u_3 are not adjacent in H. Since Condition B2 does not hold, at most one connected component of $H \setminus S$ can be adjacent to both u_1 and u_3 in H. By $V(C) \subseteq B^* \cup S$ and $|V(C) \cap S| \leq 3$, we have $(N_C(u_1) \cup N_C(u_3)) \setminus \{u_2\} \subseteq B^*$, implying $B^* = B$. The claim is proved.

For the correctness of the algorithm, we consider the three possible steps via which the algorithm halts. Step 1: Since H' is dominated-node-free and full-star-cutset-free, Lemma 4.3 implies that H' has no star-cutsets. By the above claim, Task 3 is completed. Step 2: Condition B2 holds. Let P_1 be a shortest path between s_1 and s_2 in $H[B_1 \cup \{s_1, s_2\}]$. Let P_2 be a shortest path between s_1 and s_2 in $H[B_2 \cup \{s_1, s_2\}]$. Since s_1 and s_2 are not adjacent, at least one of the three cycles of graph $P_1 \cup P_2 \cup \{ss_1, ss_2\}$ is an even hole of H. Since H is an induced subgraph of G, G has even holes. See Figure 6 for an illustration. Task 1 is completed. Step 3: By the above claim, if T_0 is lucky, Step 3 always proceeds to the next iteration of the loop. Thus, Task 2 is completed. The lemma is proved. □

4.2 Proving Lemma 4.2

4.2.1 Extended clique trees

Graph H is an *extended clique tree* [20] if there is a set S of two or less nodes of H such that each biconnected component of $H \setminus S$ is a clique. da Silva and Vušković [20, §2.3] described an $O(n^5)$-time algorithm to determine whether an n-node extended clique tree contains even holes, which can actually be implemented to run in $O(n^4)$ time.

Lemma 4.6. *It takes $O(n^4)$ time to determine whether an n-node extended clique tree has even holes.*

Proof. Let H_0 be the n-node extended clique tree. Let x and y be two nodes of H_0 such that each biconnected component of $H = H_0 \setminus \{x, y\}$ is a clique. For nodes u and v of H, let $P(u, v)$ be the shortest path of H between u and v and let $p(u, v)$ be the number of edges in $P(u, v)$. We spend $O(n^4)$ time to store the following information in a table M_1 for every two nodes u and v of H: (i) $p(u, v)$ and (ii) whether or not $P(u, v) \setminus \{u, v\}$ is adjacent to x (respectively, y). With M_1, it takes $O(n^2)$ time to determine whether H_0 has an even hole that contains y but not x: $H_0 \setminus \{x\}$ has an even hole if and only if there are two non-adjacent neighbors u and v of y in H such that $p(u, v)$ is even and $P(u, v) \setminus \{u, v\}$ is not adjacent to y. Similarly, with M_1, it takes $O(n^2)$ time to determine whether H_0 has an even hole that contains x but not y.

To determine whether H_0 has an even hole containing both x and y, we store in a table M_2 for every four nodes u_1, v_1, u_2, v_2 whether or not $P(u_1, v_1)$ and $P(u_2, v_2)$ are both disjoint and

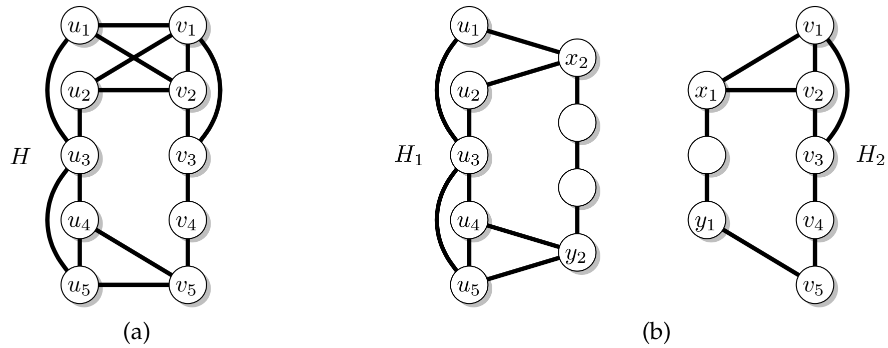


Figure 7: (a) A connected non-path 2-join $V_1|V_2$ of graph H with split (X_1, Y_1, X_2, Y_2), where $X_1 = \{u_1, u_2\}$, $X_2 = \{v_1, v_2\}$, $Y_1 = \{u_4, u_5\}$, $Y_2 = \{v_5\}$, $V_1 = X_1 \cup Y_1 \cup \{u_3\}$, and $V_2 = X_2 \cup Y_2 \cup \{v_3, v_4\}$. (b) The parity-preserving blocks of decomposition H_1 and H_2 of H for the connected 2-join $V_1|V_2$ with respect to the split (X_1, Y_1, X_2, Y_2).

non-adjacent. It takes $O(n^2)$ time to compute the connected components of $H \setminus N_H[P(u_1, v_1)]$. Paths $P(u_1, v_1)$ and $P(u_2, v_2)$ are both disjoint and non-adjacent if and only if u_2 and v_2 are in the same connected component of $H \setminus N_H[P(u_1, v_1)]$. Therefore, M_2 can also be computed in $O(n^4)$ time. With tables M_1 and M_2, it takes $O(n^4)$ time to determine whether H_0 has an even hole containing both x and y: *Case 1: x and y are adjacent in H_0.* H_0 has an even hole containing both x and y if and only if there are nodes u and v such that (1) $H_0[\{u, x, y, v\}]$ is path $uxyv$, (2) $p(u, v)$ is odd, and (3) $P(u, v) \setminus \{u, v\}$ is not adjacent to $\{x, y\}$. *Case 2: x and y are not adjacent in H_0.* H_0 has an even hole containing both x and y if and only if there are nodes u_x, v_x, u_y, v_y of H such that (1) $H_0[\{u_x, x, v_x\}]$ is path $u_x x v_x$ and $H_0[\{u_y, y, v_y\}]$ is path $u_y y v_y$, (2) $p(u_x, u_y) + p(v_x, v_y)$ is even, and (3) $P(u_x, u_y)$ and $P(v_x, v_y)$ are both disjoint and non-adjacent. The lemma is proved. □

4.2.2 2-joins and non-path 2-joins

We say that $V_1|V_2$ is a *2-join* [18, 40] of a graph H with *split* (X_1, Y_1, X_2, Y_2) if (1) V_1 and V_2 form a disjoint partition of $V(H)$ with $|V_1| \geq 3$ and $|V_2| \geq 3$, (2) X_1 and Y_1 (respectively, X_2 and Y_2) are disjoint non-empty subsets of V_1 (respectively, V_2), and (3) each node of X_1 is adjacent to each node of X_2, each node of Y_1 is adjacent to each node of Y_2, and there are no other edges between V_1 and V_2. See Figure 7(a) for an example.

Lemma 4.7 (Trotignon et al. [40, Lemma 3.2])**.** *If $V_1|V_2$ is a 2-join of a star-cutset-free graph H with split (X_1, Y_1, X_2, Y_2), then the following statements hold for each $i \in \{1, 2\}$.*

1. *Each connected component of $H[V_i]$ contains at least one node in X_i and at least one node in Y_i.*
2. *Each node of V_i has a neighbor in V_i.*
3. *Each node of X_i has a non-neighbor in Y_i. Each node of Y_i has a non-neighbor in X_i.*
4. *$|V_i| \geq 4$.*

A 2-join $V_1|V_2$ of H with split (X_1, Y_1, X_2, Y_2) is a *non-path 2-join* [39] of H if $H[V_1]$ is not a path between a node of X_1 and a node of Y_1 and $H[V_2]$ is not a path between a node of X_2 and a node of Y_2. For instance, the 2-join in Figure 7(a) is a non-path 2-join. (Non-path 2-joins are called 2-joins by da Silva and Vušković [20, §1.3].)

Lemma 4.8 (Charbit et al. [4, Theorem 4.1]). *Given an n-node connected graph H, it takes $O(n^4)$ time to either output a non-path 2-join of H together with a split or ensure that H has no non-path 2-joins.*

Lemma 4.9 (da Silva and Vušković [20, Corollary 1.3]). *A connected graph that is even-hole-free, star-cutset-free, and non-path-2-join-free is an extended clique tree.*

Combining Lemmas 4.6, 4.8, and 4.9, we have the following lemma.

Lemma 4.10. *Given an n-node star-cutset-free graph H, it takes $O(n^4)$ time to either (a) determine whether H has even holes or (b) obtain a non-path 2-join of H with a split.*

Proof. It takes $O(n^4)$ time to determine whether the graph H is an extended clique tree: For any set S of two or less nodes of H, it takes $O(n^2)$ time to obtain the biconnected components of subgraph $H \setminus S$ [23] and determine whether all of them are cliques. If H is an extended clique tree, Lemma 4.6 implies that it takes $O(n^4)$ time to determine whether H has even holes. If H is not an extended clique tree, Lemma 4.8 implies that it takes $O(n^4)$ time to either obtain a non-path 2-join of H with a split or ensure that H has no non-path 2-joins. If H has no non-path 2-joins, Lemma 4.9 implies that H has even holes. □

4.2.3 Parity-preserving blocks of decomposition for connected 2-joins

A 2-join $V_1|V_2$ with split (X_1, Y_1, X_2, Y_2) is *connected* [40] if, for each $i \in \{1, 2\}$, there is an induced path P_i of $H[V_i]$ between a node x_i of X_i and a node y_i of Y_i such that $V(P_i) \setminus \{x_i, y_i\} \subseteq V_i \setminus (X_i \cup Y_i)$. For instance, the 2-join $V_1|V_2$ in Figure 7(a) is connected. By Lemma 4.7(1), any 2-join of a star-cutset-free graph is connected with respect to any split.

Let $V_1|V_2$ be a connected 2-join of graph H with split (X_1, Y_1, X_2, Y_2). For each $i \in \{1, 2\}$, let P_i be a shortest induced path P_i of $H[V_i]$ between a node x_i of X_i and a node y_i of Y_i with $V(P_i) \setminus \{x_i, y_i\} \subseteq V_i \setminus (X_i \cup Y_i)$. If $|V(P_i)|$ is even (respectively, odd), then let $p_i = 4$ (respectively, $p_i = 5$). The *parity-preserving blocks of decomposition* [40] of H for 2-join $V_1|V_2$ with respect to split (X_1, Y_1, X_2, Y_2) are the following graphs H_1 and H_2.

- H_1 consists of (a) $H[V_1]$, (b) a p_2-path between nodes x_2 and y_2, (c) edges x_2x for all nodes x of X_1, and (d) edges y_2y for all nodes y of Y_1.
- H_2 consists of (a) $H[V_2]$, (b) a p_1-path between nodes x_1 and y_1, (c) edges x_1x for all nodes x of X_2, and (d) edges y_1y for all nodes y of Y_2.

See Figure 7(b) for an example of H_1 and H_2.

Lemma 4.11 (Trotignon and Vušković [40, Lemma 3.8]). *If $V_1|V_2$ is a connected 2-join of a star-cutset-free graph H with split (X_1, Y_1, X_2, Y_2), the parity-preserving blocks of decomposition H_1 and H_2 of H for $V_1|V_2$ with respect to (X_1, Y_1, X_2, Y_2) are star-cutset-free graphs such that H is even-hole-free if and only if both H_1 and H_2 are even-hole-free.*

Lemma 4.12. *Let H be an n-node m-edge star-cutset-free graph. Either one of the parity-preserving blocks H_1 and H_2 of decomposition for an arbitrary non-path 2-join of H with respect to an arbitrary split has at most n nodes and $m - 1$ edges.*

Proof. We prove the lemma for H_1. The proof for H_2 is similar. Let $V_1|V_2$ be the non-path 2-join. Let (X_1, Y_1, X_2, Y_2) be the split. Let P_2 be a shortest path of $H[V_2]$ between a node of X_2 and a node of Y_2. For the case that $|V(P_2)|$ is even, we have $p_2 = 4$. By Lemma 4.7(4), $|V_2| \geq 4$, implying $|V(H_1)| = n - |V_2| + p_2 \leq n$. By the following case analysis, H_1 has at most $m - 1$ edges.

- $|V(P_2)| \geq 6$: By $P_2 \subseteq H[V_2]$, $H[V_2]$ has at least five edges. Thus, H_1 has at most $m-2$ edges.
- $|V(P_2)| = 4$: Since $V_1|V_2$ is a non-path 2-join of H, $P_2 \subsetneq H[V_2]$. If $V(P_2) = V_2$, $H[V_2]$ has at least four edges. If $V(P_2) \subsetneq V_2$, Lemma 4.7(2) implies that $H[V_2]$ has at least four edges. Either way, H_1 has at most $m-1$ edges.
- $|V(P_2)| = 2$: Lemma 4.7(3) ensures $|X_2| \geq 2$ and $|Y_2| \geq 2$. Lemma 4.7(1) implies that $H[V_2]$ has at least two edges. By $|X_2| \geq 2$ and $|Y_2| \geq 2$, the number of edges between V_1 and V_2 in H is at least two more than the number of edges between V_1 and $V(H_1)\setminus V_1$ in H_1. Therefore, H_1 has at most $m-1$ edges.

As for the case that $|V(P_2)|$ is odd, we have $p_2 = 5$. The following case analysis shows that H_1 has at most n nodes and at most $m-1$ edges.

- $|V(P_2)| \geq 5$: By $|V_2| \geq 5$, we have $|V(H_1)| \leq n$. P_2 has at least four edges. Since $V_1|V_2$ is a non-path 2-join of H, $P_2 \subsetneq H[V_2]$. If $V(P_2) = V_2$, then $H[V_2]$ has at least five edges. If $V(P_2) \subsetneq V_2$, then Lemma 4.7(2) implies that $H[V_2]$ has at least five edges. Either way, H_1 has at most $m-1$ edges.
- $|V(P_2)| = 3$: By Lemma 4.7(4), the proper subset $Z = V_2\setminus V(P_2)$ of V_2 is non-empty. We know $Z \cap (X_2 \cup Y_2) \neq \varnothing$ or else $V(P_2)$ would be a star-cutset of H. Assume $Z \cap X_2 \neq \varnothing$ without loss of generality. Let B be an arbitrary connected component of $H[Z]$ with $B \cap X_2 \neq \varnothing$. We know that B is adjacent to Y_2 in H or else $N_H[x] \setminus Z$ would be a star-cutset of H, where x is the endpoint of P_2 in X_2. Since P_2 is a shortest path between a node of X_2 and a node of Y_2, at least one node of B is not in $X_2 \cup Y_2$. Therefore, $|V_2| \geq 5$, implying $|V(H_1)| \leq n$. Moreover, $H[V_2]$ has at least four edges. By $|X_2| \geq 2$, the number of edges between V_1 and V_2 in H is at least one more than the number of edges between V_1 and $V(H_1) \setminus V_1$ in H_1. Thus, H_1 has at most $m-1$ edges.

The lemma is proved. □

4.2.4 Proving Lemma 4.2

We now prove Lemma 4.2 by Lemmas 4.10, 4.11 and 4.12.

Proof of Lemma 4.2. Assume without loss of generality that the given n-node m-edge star-cutset-free graph H_0 is connected. Let set $\mathbb{H}$ initially consist of a single graph H_0. We then repeat the following loop until $\mathbb{H} = \varnothing$ or we output that H_0 has even holes. Let H be a graph in $\mathbb{H}$. *Case 1: H has at most 11 edges.* It takes $O(1)$ time to determine whether H has even holes. If H has even holes, we output that H_0 has even holes. Otherwise, we delete H from $\mathbb{H}$. *Case 2: H has at least 12 edges.* We first delete H from $\mathbb{H}$ and then apply Lemma 4.10 on H. If H has even holes, we output that H_0 has even holes. If we obtain a non-path 2-join $V_1|V_2$ of H with split (X_1, Y_1, X_2, Y_2), we add to $\mathbb{H}$ the parity-preserving blocks H_1 and H_2 of decomposition for $V_1|V_2$ with respect to (X_1, Y_1, X_2, Y_2). If the above loop stops with $\mathbb{H} = \varnothing$, we output that H_0 has no even holes.

The correctness of our algorithm follows immediately from Lemma 4.11. By Lemma 4.12, each graph ever in $\mathbb{H}$ throughout our algorithm has at most n nodes. By Lemma 4.10, each iteration of the loop takes $O(n^4)$ time. It remains to show that the loop halts in $O(m)$ iterations. Observe that each iteration increases the overall number of edges of the graphs in $\mathbb{H}$ by no more than 10.

Let $f(m)$ be the maximum number of iterations of the above loop in which Lemma 4.10 is applied. Lemma 4.12 implies

$$f(m) \leq \begin{cases} 0 & \text{if } m \leq 11 \\ \max\{1 + f(m_1) + f(m_2) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\} & \text{if } m \geq 12. \end{cases}$$

By induction on m, we show $f(m) \leq \max(m - 11, 0)$, which clearly holds for $m = 1, 2, \ldots, 11$. If $m \geq 12$, the inductive hypothesis implies

$$\begin{aligned} f(m) &\leq \max\{1 + \max(m_1 - 11, 0) + \max(m_2 - 11, 0) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\} \\ &\leq \max\{\max(m_1 + m_2 - 21, m_1 - 10, m_2 - 10, 1) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\} \\ &\leq \max(m - 11, m - 11, m - 11, 1) \\ &= \max(m - 11, 0). \end{aligned}$$

By $f(m) = O(m)$, the number of iterations of the above loop is $O(m)$. The lemma is proved. $\square$

5 Concluding remarks

For any class $\mathbb{G}$ of induced subgraphs, one can augment a recognition algorithm for $\mathbb{G}$-free graphs into a $\mathbb{G}$-detection algorithm for an n-node graph G with a factor-$O(n)$ increase in the time complexity by a node-deletion method: (1) Let $H = G$. (2) For each node v of G, if $H \setminus \{v\}$ is not $\mathbb{G}$-free, then let $H = H \setminus \{v\}$. (3) Output the resulting graph H. See, e.g., [41, §4] for the case that $\mathbb{G}$ consists of even holes. Thus, Theorem 1.1 immediately yields a detection algorithm that runs in time $O(m^3n^6) = O(n^{12})$. However, our $O(m^3n^5)$-time recognition algorithm can be augmented into an even-hole-detection algorithm without increasing the time complexity.

The combination of the proofs of Theorem 1.1 and Lemma 2.3 actually gives two algorithms. The first algorithm determines if G is both beetle-free and 4-hole-free. The second algorithm determines if a beetle-free and 4-hole-free graph G is also even-hole-free. We first describe how to augment the first algorithm into an $O(m^3n^5)$-time detection algorithm. Since it takes $O(n^4)$ time to detect a 4-hole in G, it suffices to show how to detect an even hole in a graph G with beetles in $O(m^3n^5)$ time. As stated in the proof of Lemma 2.3, for each of the $O(m^3n)$ choices of node b_4 and edges b_1b_5, b_2b_6, and b_3b_7, it takes $O(n^4)$ time via Lemma 3.1 to determine if b_4, b_1b_5, b_2b_6, and b_3b_7 are contained by a beetle B in which $\{b_1, b_2, b_3, b_4\}$ induces a diamond. Once we know that there exists a beetle B containing a particular choice of b_4, b_1b_5, b_2b_6, and b_3b_7, by an augmented version of Lemma 3.1 via the above node-deletion method, it takes $O(n^5)$ time to actually detect such a beetle B. Therefore, if G has beetles, it takes $O(m^3n) \cdot O(n^4) + O(n^5) = O(m^3n^5)$ time to find a beetle of G, in which an even hole of G can be detected in $O(n)$ time.

The second algorithm can also be augmented into an $O(m^3n^5)$-time detection algorithm for a beetle-free graph G that has even holes. By Lemma 2.3, we obtain in $O(m^3n^5)$ time a set $\mathbb{T}$ of $O(m^2n)$ trackers that satisfies Property 1. Since G has even holes, there must be a tracker $(H, u_1u_2u_2)$ of $\mathbb{T}$ such that H contains an even hole of G, which according to Lemma 2.4 can be found in time $O(m^2n) \cdot O(mn^4) = O(m^3n^5)$. By the proof of Lemma 2.4, H is ensured to have even holes in two ways. (1) If it is ensured through completing Task 1 of Lemma 4.1, the proof of Lemma 4.1 actually gives a constructive proof for the existence of an even hole of H, which is also an even hole of G. (2) If it is ensured through completing Task 3 of Lemma 4.1 and then by

Lemma 4.2, we have a star-cutset-free induced subgraph H' of H that has even holes. We then apply the above node-deletion method on H' using Lemma 4.2 to detect in $O(mn^5)$ time an even hole of H', which is also an even hole of H and G. Therefore, if G is a 4-hole-free and beetle-free graph that has even holes, it takes time $O(m^3n^5)+O(mn^5) = O(m^3n^5)$ to output an even hole of G. Combining the above two detection algorithms, we have an $O(m^3n^5)$-time algorithm to output an even hole in an n-node m-edge graph with even holes.

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