# Unwinding Annular Curves and Electrically Reducing Planar Networks

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## 1 Introduction

Any continuous deformation of a closed curve on any surface can be decomposed into a finite sequence of *homotopy moves*, consisting of the following three operations and their inverses. A 1→0 move removes an empty loop; a 2→0 move removes an empty bigon, and a 3→3 move flips an empty triangle. A classical argument of Steinitz [6] implies that any planar curve with $n$ vertices can be simplified using $O(n^2)$ homotopy moves; using a later result of Hass and Scott [4] one can extend this upper bound to contractible curves on arbitrary surfaces. We prove that Hass and Scott's quadratic upper bound is tight.

**Theorem 1** *Simplifying a contractible curve in the annulus (or any surface that has the annulus as its covering space) requires $\Omega(n^2)$ homotopy moves in the worst case.*

This improves our previous $\Omega(n^{3/2})$ lower bound, which follows from an analysis of curves in the plane, and generalizes our previous $\Omega(n^2)$ lower bound for simplifying *non-contractible* curves on higher-genus surfaces [1].

Our second result concerns the reduction of plane graphs via *electrical transformations*: leaf reductions, loop reductions, series-parallel reductions, and $\Delta\leftrightarrow Y$ transformations. We distinguish between two types of electrical transformations in plane graphs: A loop reduction, parallel reduction, or $\Delta\to Y$ transformation is *facial* if the edges deleted by the operation bound a face in $G$, and *non-facial* otherwise. Dual pairs of facial electrical transformations correspond to local transformations in the medial graph of $G$, which we call *medial electrical moves*.

**Theorem 2** *Reducing an $n$-vertex plane graph with two terminals as much as possible requires $\Omega(n^2)$ facial electrical transformations in the worst case.*

The proof uses our quadratic homotopy lower bound for annular curves. This result matches the upper bound implied by Hass and Scott [4], and it strengthens and generalizes our earlier $\Omega(n^{3/2})$ lower bound for reducing plane graphs to a single vertex [1].

## 2 Unwinding Annular Curves

To simplify our analysis of annular curves, it is convenient to work in the punctured plane $\mathbb{R}^2 \setminus \{o\}$, where $o$ is an

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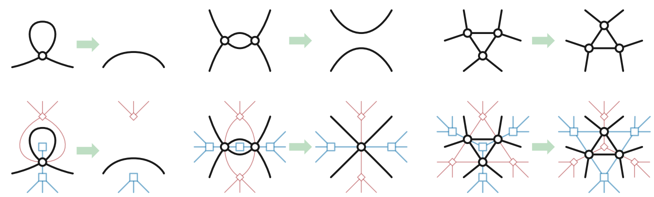

**Figure 1.** Top row: Homotopy moves 1→0, 2→0, and 3→3. Bottom row: Medial electrical moves 1→0, 2→1, and 3→3.

arbitrary point called the *origin*. In any homotopy in the punctured plane, homotopy moves that contract either the face containing the origin or the outer face of the curve are forbidden. It is precisely these forbidden homotopy moves that make the quadratic lower bound possible; if we only forbid homotopy moves on the outer face, then any curve can be simplified using at most $O(n^{3/2})$ moves [1].

Let $\gamma$ be an arbitrary oriented closed curve in the punctured plane, and let $p$ be any point outside the image of $\gamma$. The *winding number* $wind(\gamma, p)$ is the number of times $\gamma$ crosses a generic ray $\rho$ based at $p$ from right to left, minus the number of times $\gamma$ crosses $\rho$ from left to right. For any vertex $x$ of $\gamma$, the winding number $wind(\gamma, x)$ is defined as the average of the winding numbers around the four faces incident to $x$.

*Smoothing* $\gamma$ at a vertex $x$ replaces a small neighborhood of $x$ with two disjoint simple paths. There are two possible smoothings at each vertex, one of which splits $\gamma$ into two subcurves; let $\gamma_x^+$ and $\gamma_x^-$ denote the subcurves locally to the left of $x$ and locally to the right of $x$, respectively. We define the *type* of any vertex $x$ as $type(\gamma, x) := wind(\gamma_x^+, o)$. A vertex $x$ is *irrelevant* if either $type(\gamma, x) = 0$ or $type(\gamma, x) = wind(\gamma, o)$ and *relevant* otherwise. Two vertices $x$ and $y$ have *complementary* types if $type(\gamma, x) + type(\gamma, y) = wind(\gamma, o)$.

Case analysis shows that homotopy moves modify the types and winding numbers of vertices as follows: (a) Each 1↔0 move creates or destroys one irrelevant vertex. (b) Each 2↔0 move creates or destroys two vertices with complementary types and identical winding numbers. (c) Each 3↔3 move changes the winding numbers of the three vertices, each by exactly 1. (d) Otherwise, homotopy moves do not change the type or winding number of any vertex.

To prove Theorem 1, consider any contractible $\gamma$ and any homotopy that contracts $\gamma$ to a point, and let $x$ be a relevant vertex at any stage of the homotopy. We can follow $x$ through the homotopy to a 2→0 move that deletes $x$ and a complementary vertex; symmetrically, we can follow

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$x$ backward through the homotopy either to a vertex of $\gamma$ or to a 0→2 move that creates $x$ and a complementary vertex. Proceeding inductively, we obtain a matching between complementary relevant vertices of $\gamma$. Each matched pair is connected by a path through the homotopy that alternately moves forward and backward through the homotopy, switching direction and type at 2↔0 moves.

During the homotopy, the winding number of a vertex changes precisely when it participates in a 3→3 move. Thus, the path between two matched vertices $x$ and $y$ must pass through at least $\left|wind(\gamma,x) - wind(\gamma,y)\right|$ 3→3 moves, and the entire homotopy must contain at least $\sum_{x\sim y} \left|wind(\gamma,x) - wind(\gamma,y)\right|/3$ 3→3 moves, where the sum is over all matched pairs of vertices of $\gamma$.

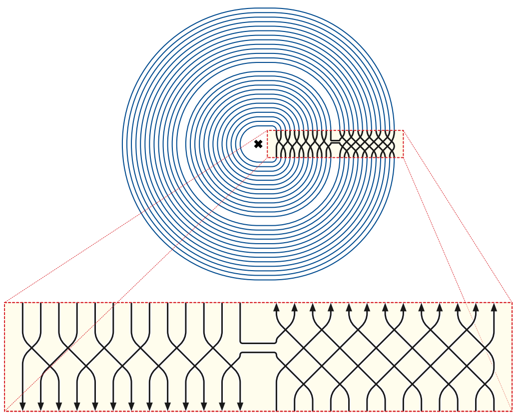

**Figure 2.** Our bad example curve $X_{13}$ in the punctured plane.

For any relatively prime integers $p$ and $q$, the *flat torus knot* $T(p,q)$ winds $|p|$ times around the origin and oscillates $|q|$ times between two concentric circles. For any odd integer $p$, let $X_p$ denote the connected sum of $T(-p,1)$ and $T(p,2)$, where the former curve is scaled to lie inside the innermost face of the latter. This curve is contractible and has $3(p-1)$ vertices. Analysis of the types and winding numbers of the vertices of $X_p$ implies that any homotopy that contracts $X_p$ contains at least $p(p-1)/6$ 3→3 moves.

## 3 Planar Graphs with Two Terminals

Most applications of electrical transformations designate two vertices as *terminals*. In this context, leaf reductions, series reductions, and $Y\rightarrow\Delta$ transformations at terminals are not proper electrical transformations. Every 2-terminal plane graph can be reduced by facial electrical transformations to a unique graph, which we call a *bullseye*, but not necessarily to a single edge. (The medial graph of a bullseye is the curve $T(p,1)$, for some even integer $p$.)

A *multicurve* is an immersion of one or more closed curves. For a plane graph $G$ with two terminals, the medial graph of $G$ is the image of a multicurve embedded in the annulus. Arguments of Truemper [7] and Noble and Welsh [5], described in detail in our earlier paper [1], imply that reducing a plane curve $\gamma$ using medial electrical moves requires at least as many steps as reducing $\gamma$ using homotopy moves. To prove Theorem 2, we extend these arguments to annular curves. Specifically, we show that the number of medial electrical moves required to reduce an annular curve is at least the number of homotopy moves required to reduce the same curve; the quadratic lower bound now follows directly from Theorem 1.

Two key ingredients of our proof may be of independent interest. First, we show that a multicurve $\gamma$ on *any* surface can be further reduced using medial electrical moves if and only if $\gamma$ can be further reduced using homotopy moves. We prove this fact using a classical result [3] that any multicurve can be simplified as much as possible via homotopy moves without ever increasing the number of vertices.

Let $\gamma$ be an arbitrary connected multicurve in the punctured plane. Let $X(\gamma)$ denote the minimum number of medial electrical moves required to reduce $\gamma$. Let $depth(\gamma)$ denote the minimum number of times a generic ray based at the origin crosses $\gamma$ (not considering the directions of crossings). Our second key observation, which relies on the first, is that the inequality

$$X(\overline{\gamma}) < X(\gamma) + \left(depth(\gamma) - depth(\overline{\gamma})\right)/2$$

holds for every connected proper smoothing $\overline{\gamma}$ of $\gamma$. This generalizes the simpler inequality $X(\overline{\gamma}) < X(\gamma)$ for connected multicurves in the plane [1].

As a final remark, by including one more operation called the *terminal leaf reduction* in addition to facial electrical transformations, any 2-terminal plane graph can be reduced to a single edge; indeed, existing electrical reduction algorithms for plane graphs rely exclusively on these operations [2, 7]. Unfortunately, our lower bound does *not* apply when terminal leaf reductions are allowed.

## References


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