

# From Proximity to Utility: A Voronoi Partition of Pareto Optima\*

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## Abstract

We present an extension of Voronoi diagrams where when considering which site a client is going to use, in addition to the site distances, other site attributes are also considered (for example, prices or weights). A cell in this diagram is then the locus of all clients that consider the same set of sites to be relevant. In particular, the precise site a client might use from this candidate set depends on parameters that might change between usages, and the candidate set lists all of the relevant sites. The resulting diagram is significantly more expressive than Voronoi diagrams, but naturally has the drawback that its complexity, even in the plane, might be quite high. Nevertheless, we show that if the attributes of the sites are drawn from the same distribution (note that the locations are fixed), then the expected complexity of the candidate diagram is near linear.

To this end, we derive several new technical results, which are of independent interest. In particular, we provide a high-probability, asymptotically optimal bound on the number of Pareto optima points in a point set uniformly sampled from the  $d$ -dimensional hypercube. To do so we revisit the classical backward analysis technique, both simplifying and improving relevant results in order to achieve the high-probability bounds.

## 1. Introduction

**Informal description of the candidate diagram.** Suppose you open your refrigerator one day to discover it is time to go grocery shopping.<sup>1</sup> Which store you go to will be determined by a number of different factors. For example, what items you are buying, and do you want the cheapest price or highest quality, and how much time you have for this chore. Naturally the distance to the store will also be a factor. On different days which store is the best to go to will differ based on that day's preferences. However, there are certain stores you will never shop at. These are stores which are worse in every way than some other store (further, more expensive, lower quality, etc). Therefore, the stores that are relevant and in the *candidate set* are those that are not strictly worse

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<sup>1</sup>Unless you are feeling adventurous enough that day to eat the frozen mystery food stuck to the back of the freezer, which we strongly discourage you from doing.

1 in every way than some other store. Thus, every point in the plane is mapped to a set of stores  
2 that a client at that location might use. The *candidate diagram* is the partition of the plane into  
3 regions, where each candidate set is the same for all points in the same region. Naturally, if your  
4 only consideration is distance, then this is the (classical) Voronoi diagram of the sites. However,  
5 here deciding which shop to use is an instance of multi-objective optimization — there are multiple,  
6 potentially competing, objectives to be optimized, and the decision might change as the weighting  
7 and influence of these objectives mutate over time (in particular, you might decide to do your  
8 shopping in different stores for different products). The concept of relevant stores discussed above  
9 is often referred as the *Pareto optima*.

10 **Pareto optima in welfare economics.** Pareto efficiency, named after Vilfredo Pareto, is a core  
11 concept in economic theory and more specifically in welfare economics. Here each point in  $\mathbb{R}^d$   
12 represents the corresponding utilities of  $d$  players for a particular allocation of finite resources. A  
13 point is said to be *Pareto optimal* if there is no other allocation which increases the utility of any  
14 individual without decreasing the utility of another. The *First Fundamental Theorem of Welfare*  
15 *Economics* states that any competitive equilibrium (when supply equals demand) is Pareto optimal.  
16 The origins of this theorem date back to 1776 with Adam Smith’s famous (and controversial) work,  
17 “The Wealth of Nations,” but was not formally *proven* until the 20th century by Lerner, Lange,  
18 and Arrow (see [Fel08]). Naturally such proofs rely on simplifying (and potentially unrealistic)  
19 assumptions such as perfect knowledge, or absence of externalities. The *Second Fundamental*  
20 *Theorem of Welfare Economics* states that any Pareto optimum is achievable through lump-sum  
21 transfers (that is, taxation and redistribution). In other words each Pareto optima is a “best  
22 solution” under some set of societal preferences, and is achievable through redistribution in one  
23 form or another (see [Fel08] for a more in depth discussion).

24 **Pareto optima in computer science.** In computational geometry such Pareto optima points  
25 relate to the *orthogonal convex hull* [OSW84], which in turn relates to the well known convex hull  
26 (the input points that lie on the orthogonal convex hull is a super set of those which lie on the convex  
27 hull). Pareto optima are also of importance to the database community [BKS01, HTC13], in which  
28 context such points are called *maximal* or *skyline points*. Such points are of interest as they can be  
29 seen as the relevant subset of the (potentially much larger) result of a relational database query.  
30 The standard example is querying a database of hotels for the cheapest and closest hotel, where  
31 naturally hotels which are farther and more expensive than an alternative hotel are not relevant  
32 results. There is a significant amount of work on computing these points, see Kung *et al.* [KLP75].  
33 More recently, Godfrey *et al.* [GSG07] compared various approaches for the computation of these  
34 points (from a databases perspective), and also introduced their own new external algorithm.<sup>2</sup>

35 **Modeling uncertainty.** Recently, there is a growing interest in modeling uncertainty in data.  
36 As real data is acquired via physical measurements, noise and errors are introduced. This can  
37 be addressed by treating the data as coming from a distribution (e.g., a point location might be  
38 interpreted as a center of a Gaussian), and computing desired classical quantities adapted for such  
39 settings. Thus, a nearest-neighbor query becomes a probabilistic question — what is the expected  
40 distance to the nearest-neighbor? What is the most likely point to be the nearest-neighbor? (See  
41 [AAH<sup>+</sup>13] and references therein.)

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<sup>2</sup>There is of course a lot of other work on Pareto optimal points, from connections to Nash equilibrium to scheduling. We resisted the temptation of including many such references which are not directly related to our paper.

1 This in turn gives rise to the question of what is the expected complexity of geometric structures  
 2 defined over such data. The case where the data is a set of points, and the locations of the points are  
 3 chosen randomly was thoroughly investigated (see [SW93, WW93, HR14] and references therein).  
 4 The problem, when the locations are fixed but the weights associated with the points are chosen  
 5 randomly, is relatively new. Agarwal *et al.* [AHKS14] showed that for a set of disjoint segments in  
 6 the plane, if they are being expanded randomly, then the expected complexity of the union is near  
 7 linear. This result is somewhat surprising as in the worst case the complexity of such a union is  
 8 quadratic.

9 Here we are interested in bounding the expected complexity of weighted generalizations of  
 10 Voronoi diagrams, where the weights (not the site locations) are randomly sampled. Note that the  
 11 result of Agarwal *et al.* [AHKS14] can be interpreted as bounding the expected complexity of level  
 12 sets of the multiplicative weighted Voronoi diagram (of segments). On the other hand, we want to  
 13 bound the entire lower envelope (which implies the same bound on any level set). For the special  
 14 case of multiplicative weighted Voronoi diagrams, a near-linear expected complexity bound was  
 15 provided by Har-Peled and Raichel [HR14]. In this work we consider a much more general class of  
 16 weighted diagrams which allow multiple weights and non-linear distance functions.

## 17 1.1. Our contributions

18 **Conceptual contribution.** We formally define the *candidate diagram* in Section 2.1 — a new  
 19 geometric structure that combines proximity information with utility. For every point  $x$  in the  
 20 plane, the diagram associates a *candidate set*  $C(x)$  of sites that are relevant to  $x$ . That is, all the  
 21 sites that are Pareto optima for  $x$ . Putting it differently, a site is not in  $C(x)$  if it is further away  
 22 from *and* worse in all parameters than some other site. Significantly, unlike the traditional Voronoi  
 23 diagram, the candidate diagram allows the user to change their distance function, as long as the  
 24 function respects the domination relationship. This diagram is a significant extension of the Voronoi  
 25 diagram, and includes other extensions of Voronoi diagrams as special subcases, like multiplicative  
 26 weighted Voronoi diagrams. Not surprisingly, the worst case complexity of this diagram can be  
 27 quite high.

28 **Technical contribution.** We consider the case where each site chooses its  $j$ th attribute from  
 29 some distribution  $\mathcal{D}_j$  independently for each  $j$ . We show that the candidate diagram in expectation  
 30 has near-linear complexity, and that, with high probability, the candidate set has poly-logarithmic  
 31 size for any point in the plane. In the process we derive several results which are interesting in  
 32 their own right.

- 33 (a) We prove that if  $n$  points are sampled from a fixed distribution (see Section 2.2 for assumptions  
 34 on the distribution) over the  $d$ -dimensional hypercube then, with *polynomially* small error  
 35 probability, the number of Pareto optima points is  $O(\log^{d-1} n)$ , which is within a constant  
 36 factor of the expectation (see Lemma 6.4). Previously, this result was only known in a weaker  
 37 form that is insufficient to imply our other results. Specifically, Bentley *et al.* [BKST78] first  
 38 derived the asymptotically tight bound on the expected number of Pareto optima points. Bai  
 39 *et al.* [BDHT05] proved that after normalization the cumulative distribution function of the  
 40 number of Pareto optima is normal, up to an additive error of  $O(1/\text{polylog } n)$ . (See [BR10a,  
 41 BR10b] as well.) In particular, their results (which are quite nice and mathematically involved)  
 42 only imply our statement with poly-logarithmically small error probability. To the best of our  
 43 knowledge this result is new — we emphasize, however, that for our purposes a weaker bound  
 44 of  $O(\log^d n)$  is sufficient, and such a result follows readily from the  $\varepsilon$ -net theorem [HW87]

(naturally, this would add a logarithmic factor to later results).

- (b) *Backward analysis with high probability.* To get this result, we prove a lemma providing high-probability bounds when applying backwards analysis [Sei93] (see Lemma 3.4). Such tail estimates are known in the context of randomized incremental algorithms [CMS93, BCKO08], but our proof is arguably more direct and cleaner, and should be applicable to more cases. (See Section 3).
- (c) *Overlay of the  $k$ th order Voronoi cells in randomized incremental construction.* We prove that the overlay of cells during a randomized incremental construction of the  $k$ th order Voronoi diagram is of complexity  $O(k^4 n \log n)$  (see Lemma 5.8).
- (d) *Complexity of the candidate diagram.* Combining the above results carefully yields a near-linear upper bound on the complexity of the candidate diagram (see Theorem 7.1).

**Outline.** In Section 2 we formally define our problem and introduce some tools that will be used later on. Specifically, after some required preliminaries, we formally introduce the candidate diagram in Section 2.1. The sampling model being used is described in detail in Section 2.2.

Backward analysis with high probability is discussed in Section 3, including Corollary 3.1 which is a sufficient statement for the purposes of this paper. In Section 3.1 we make a short detour and provide a detailed proof of the high-probability backward analysis statement.

To bound the complexity of the candidate diagram (both the size of the planar partition and the total size of the associated candidate sets), in Section 4, the notion of *proxy set* is introduced. Defined formally in Section 4.1, it is (informally) an enlarged candidate set. Section 4.2 bounds the size of the proxy set using backward analysis, both in expectation and with high probability, and Section 4.3 shows that mucking around with the proxy set is useful, by proving that the proxy set contains the candidate set, for any point in the plane.

In Section 5, it is shown that the diagram induced by the proxy sets can be interpreted as the arrangement formed by the overlay of cells during the randomized incremental construction of the  $k$ th order Voronoi diagram. To this end, Section 5.1 defines the  *$k$ th order Voronoi diagram*, interpret as arrangement of planes, and states some basic properties of these entities. For our purposes, we need to bound the size of the conflict lists encountered during the randomized incremental construction, and this is done in Section 5.2 using the Clarkson-Shor technique. In Section 5.3 the  *$k$  environment* of a site is defined, and we related such a notion to the  $k$ th order Voronoi diagram. Next, in Section 5.4 we bound the expected complexity of the proxy diagram.

We bound the expected size of the candidate set for any point in the plane in Section 6. First, in Section 6.1, we analyze the number of staircase points in randomly sampled point sets from the hypercube, and we use this bound, in Section 6.2, to bound the size of the candidate set.

Finally, in Section 7, we put everything together and prove our main result, showing the desired bound on the complexity of the candidate diagram.

## 2. Problem definition and preliminaries

Throughout, we assume the reader is familiar with standard computational geometry terms, such as arrangements [SA95], vertical-decomposition [BCKO08], etc. In the same vein, we assume that the variable  $d$ , the *dimension*, is a small constant and the big- $O$  notation hides constants that are potentially exponential (or worse) in  $d$ .

A quantity is bounded by  $O(f)$  *with high probability* with respect to  $n$ , if for any constant  $\gamma > 0$ , there is another constant  $c$  depending on  $\gamma$  such that the quantity is at most  $c \cdot f$  with probability at least  $1 - n^{-\gamma}$ . In other words, the bound holds for any polynomially small error with the expense

of a multiplicative constant factor on the size of the bound. When there’s no danger of confusion, we sometimes write  $O_{\text{whp}}(f)$  for short.

**Definition 2.1.** Consider two points  $\mathbf{p} = (p_1, \dots, p_d)$  and  $\mathbf{q} = (q_1, \dots, q_d)$  in  $\mathbb{R}^d$ . The point  $\mathbf{p}$  *dominates*  $\mathbf{q}$  (denoted by  $\mathbf{p} \preceq \mathbf{q}$ ) if  $p_i \leq q_i$ , for all  $i$ .

Given a point set  $P \subseteq \mathbb{R}^d$ , there are several terms for the subset of  $P$  that is not dominated, as discussed above, such as *Pareto optima* or *minima*. Here, we use the following term.

**Definition 2.2.** For a point set  $P \subseteq \mathbb{R}^d$ , a point  $\mathbf{p} \in P$  is a *staircase point* of  $P$  if no other point of  $P$  dominates it. The set of all such points, denoted by  $\text{St}(P)$ , is the *staircase* of  $P$ .

Observe that for a nonempty finite point set  $P$ , the staircase  $\text{St}(P)$  is never empty.

## 2.1. Formal definition of the candidate diagram

Let  $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$  be a set of  $n$  distinct *sites* in the plane. For each site  $\mathbf{s}$  in  $S$ , there is an associated list  $\beta = (b_1, \dots, b_d)$  of  $d$  real-valued attributes, each in the interval  $[0, 1]$ . When viewed as a point in the unit hypercube  $[0, 1]^d$ , this list of attributes is the *parametric point* of the site  $\mathbf{s}_i$ . Specifically, a *site* is a point in the plane encoding a facility location, while the term *point* is used to refer to the (parametric) point encoding its attributes in  $\mathbb{R}^d$ .

**Preferences.** Fix a client location  $\mathbf{x}$  in the plane. For each site, there are  $d+1$  associated variables for the client to consider. Specifically, the client distance to the site, and  $d$  additional attributes (e.g., prices of  $d$  different products) associated with the site. Conceptually, the goal of the client is to “pay” as little as possible by choosing the best site (e.g., minimize the overall cost of buying these  $d$  products together from a site, where the price of traveling the distance to the site is also taken into account).

**Definition 2.3.** A client  $\mathbf{x}$  has a *dominating preference* if for any two sites  $\mathbf{s}$  and  $\mathbf{s}'$  in the plane, with parametric points  $\beta$  and  $\beta'$  in  $\mathbb{R}^d$ , respectively, the client would prefer the site  $\mathbf{s}$  over  $\mathbf{s}'$  if  $\|\mathbf{x} - \mathbf{s}\| \leq \|\mathbf{x} - \mathbf{s}'\|$  and  $\beta \preceq \beta'$  (that is,  $\beta$  dominates  $\beta'$ ). We sometimes say site  $\mathbf{s}$  *dominates* site  $\mathbf{s}'$ .

Note that a client having a dominating preference does not identify a specific optimum site for the client, but rather a set of potential optimum sites. Specifically, given a client location  $\mathbf{x}$  in the plane, let its distance to the  $i$ th site be  $\ell_i = \|\mathbf{x} - \mathbf{s}_i\|$ . The set of sites the client might possibly use (assuming the client uses a dominating preference) are the staircase points of the set  $P(\mathbf{x}) = \{(\beta_1, \ell_1), \dots, (\beta_n, \ell_n)\}$  (that is, we are adding the distance to each site as an additional attribute of the site — this attribute depends on the location of  $\mathbf{x}$ ). The set of sites realizing the staircase of  $P(\mathbf{x})$  is the *candidate set*  $C(\mathbf{x})$  of  $\mathbf{x}$ :

$$C(\mathbf{x}) = \left\{ \mathbf{s}_i \in S \mid (\beta_i, \ell_i) \text{ is a staircase point of } P(\mathbf{x}) \text{ in } \mathbb{R}^{d+1} \right\}. \quad (2.1)$$

The *candidate cell* of  $\mathbf{x}$  is the set of all the points in the plane that have the same candidate set associated with them. That is,  $\{\mathbf{p} \in \mathbb{R}^2 \mid C(\mathbf{p}) = C(\mathbf{x})\}$ . The decomposition of the plane into these cells is the *candidate diagram*.

Now, the client  $\mathbf{x}$  has the candidate set  $C(\mathbf{x})$ , and it chooses some site (or potentially several sites) from  $C(\mathbf{x})$  that it might want to use. Note that the client might decide to use different sites for different acquisitions. As an example, consider the case when each site  $\mathbf{s}_i$  is attached with attributes  $\beta_i = (b_{i,1}, b_{i,2})$ . If the client  $\mathbf{x}$  has the preference of choosing the site with smallest value  $b_{i,1}\ell_i$  among

1 all the sites, then this preference is a dominating preference, and therefore the client will choose  
 2 one of the sites from the candidate list  $C(x)$ . (Observe that the preference function corresponds  
 3 to the multiplicative Voronoi diagram with respect to the first coordinate  $b_{i,1}$ .) Similarly, if the  
 4 preference function is to choose the smallest value  $b_{i,1}\ell_i^2 + b_{i,2}$  among all the sites (which again is  
 5 a dominating preference), then this corresponds to a power diagram of the sites.

6 **Complexity of the diagram.** The *complexity* of a planar arrangement is the total number of  
 7 edges, faces, and vertices. A candidate diagram can be interpreted as a planar arrangement, and  
 8 its complexity is defined analogously. The *space complexity* of the candidate diagram is the total  
 9 amount of memory needed to store the diagram explicitly, and is bounded by the complexity of  
 10 the candidate diagram together with the sum of the sizes of candidate sets over all the faces in the  
 11 arrangement of the diagram (which is potentially larger by a factor of  $n$ , the number of sites). Note,  
 12 that the space complexity is a somewhat naïve upper bound, as using persistent data-structures  
 13 might significantly reduce the space needed to store the candidate lists.

14 **Lemma 2.4.** *The complexity of the candidate diagram of  $n$  sites in the plane is  $O(n^4)$ . The space*  
 15 *complexity of the candidate diagram is  $\Omega(n^2)$  in the worst case and  $O(n^5)$  in all cases.*

16 *Proof:* The lower bound is easy, and is left as an exercise to the reader. A naïve upper bound of  
 17  $O(n^5)$  on the space complexity, follows because: (i) all possible pairs of sites induce together  $\binom{n}{2}$   
 18 bisectors, (ii) the complexity of the arrangement of the bisectors is  $O(n^4)$ , and (iii) the candidate  
 19 set of each face in this arrangement might have  $n$  elements inside. ■

20 We leave the problem of closing the gap between the upper and lower bounds of [Lemma 2.4](#) as  
 21 an open problem for further research.

## 22 2.2. Sampling model

23 Fortunately, the situation changes dramatically when randomization is involved. Let  $S$  be a set  
 24 of  $n$  sites in the plane. For each site  $s \in S$ , a parametric point  $\beta = (\beta_1, \dots, \beta_d)$  is sampled  
 25 independently from  $[0, 1]^d$ , with the following constraint: each coordinate  $\beta_i$  is sampled from a  
 26 (continuous) distribution  $\mathcal{D}_i$ , independently for each coordinate. In particular, the sorted order of  
 27 the  $n$  parametric points by a specific coordinate yields a uniform random permutation (for the sake  
 28 of simplicity of exposition we assume that all the values sampled are distinct).

29 Our main result shows that, under the above assumptions, both the complexity and the space  
 30 complexity of the candidate diagram are near linear in expectation — see [Theorem 7.1](#) for the exact  
 31 statement.

## 32 3. Backward analysis with high probability

33 *Randomized incremental construction* is a powerful technique used by geometric algorithms. Here,  
 34 one is given a set of elements  $S$  (e.g., segments in the plane), and one is interested in computing  
 35 some structure induced by these elements (e.g., the vertical decomposition formed by the segments).  
 36 To this end, one computes a random permutation  $\Pi = \langle s_1, \dots, s_n \rangle$  of the elements of  $S$ , and in the  
 37  $i$ th iteration one computes the structure  $V_i$  induced by the  $i$ th prefix  $\Pi_i = \langle s_1, \dots, s_i \rangle$  of  $\Pi$  by  
 38 inserting the  $i$ th element  $s_i$  into  $V_{i-1}$  (e.g., split all the vertical trapezoids of  $V_{i-1}$  that intersect  $s_i$ ,  
 39 and merge together adjacent trapezoids with the same floor and ceiling).



1 In *backward analysis* one is interested in computing the probability that a specific object in  $V_i$   
2 was actually created in the  $i$ th iteration (e.g., a specific vertical trapezoid in the vertical decompo-  
3 sition  $V_i$ ). If the object of interest is defined by at most  $b$  elements of  $\Pi_i$  for some constant  $b$ , then  
4 the desired quantity is the probability that  $\mathbf{s}_i$  is one of these defining elements, which is at most  
5  $b/i$ . In some cases, the sum of these probabilities, over the  $n$  iterations, counts the number of times  
6 certain events happen during the incremental construction. However, this yields only a bound in  
7 expectation. For a high-probability bound, one can not apply this argument directly, as there is a  
8 subtle dependency leakage between the corresponding indicator variables involved between different  
9 iterations. (Without going into details, this is because the defining sets of the objects of interest  
10 can have different sizes, and these sizes depend on which elements were used in the permutation in  
11 earlier iterations.)

12 Let  $\mathsf{P}$  be a set of  $n$  elements. A *property*  $\mathcal{P}$  of  $\mathsf{P}$  is a function that maps any subset  $\mathsf{X}$  of  $\mathsf{P}$  to  
13 a subset  $\mathcal{P}(\mathsf{X})$  of  $\mathsf{X}$ . Intuitively the elements in  $\mathcal{P}(\mathsf{X})$  have some desired property with respect to  
14  $\mathsf{X}$  (for example, let  $\mathsf{X}$  be a set of points in the plane, then  $\mathcal{P}(\mathsf{X})$  may be those points in  $\mathsf{X}$  who lie  
15 on the convex hull of  $\mathsf{X}$ ). The following corollary (implied by [Lemma 3.4](#) below) provides a high-  
16 probability bound for backward analysis, and while the proof is an easy application of the Chernoff  
17 inequality, it nevertheless significantly simplifies some classical results on randomized incremental  
18 construction algorithms.

19 **Corollary 3.1.** *Let  $\mathsf{P}$  be a set of  $n$  elements, let  $c > 1$  and  $k \geq 1$  be prespecified numbers, and let  
20  $\mathcal{P}(\mathsf{X})$  be a property defined over any subset  $\mathsf{X} \subseteq \mathsf{P}$ . Now, consider a uniform random permutation  
21  $\langle \mathbf{p}_1, \dots, \mathbf{p}_n \rangle$  of  $\mathsf{P}$ , and let  $\mathsf{P}_i = \{\mathbf{p}_1, \dots, \mathbf{p}_i\}$ . Furthermore, assume that we have  $|\mathcal{P}(\mathsf{P}_i)| \leq k$  simul-  
22 taneously for all  $i$  with probability at least  $1 - n^{-c}$ . Let  $X_i$  be the indicator variable of the event  
23  $\mathbf{p}_i \in \mathcal{P}(\mathsf{P}_i)$ . Then, for any constant  $\gamma \geq 2e$ , we have*

$$24 \quad \Pr \left[ \sum_{i=1}^n X_i > \gamma \cdot (2k \ln n) \right] \leq n^{-\gamma k} + n^{-c}.$$

25 *(If for all  $\mathsf{X} \subseteq \mathsf{P}$  we have that  $|\mathcal{P}(\mathsf{X})| \leq k$ , then the additional error term  $n^{-c}$  is not necessary.)*

26 In the remainder of this section we prove [Lemma 3.4](#), from which the above corollary is derived,  
27 and provide some examples of its applications. However, the above corollary statement is all that  
28 is required to prove our main results, and so if desired the reader can skip directly to [Section 4](#).

### 29 3.1. A short detour into backward analysis

30 We need the following easy observation.

31 **Lemma 3.2.** *Let  $E_1, \dots, E_t$  be disjoint events and let  $F$  be another event, such that  $\beta = \Pr[F \mid E_i]$   
32 is the same for all  $i$ . Then  $\Pr[F \mid E_1 \cup \dots \cup E_t] = \beta$ .*

33 *Proof:* We have  $\Pr[F \cap (\cup_i E_i)] = \sum_i \Pr[F \cap E_i] = \sum_i \Pr[F \mid E_i] \Pr[E_i] = \beta \sum_i \Pr[E_i]$ . Hence  
34  $\Pr[F \mid \cup_i E_i] = \Pr[F \cap (\cup_i E_i)] / \Pr[\cup_i E_i] = \beta$ . ■

35 **Lemma 3.3.** *Let  $\mathsf{P}$  be a set of  $n$  elements, and let  $k_1, \dots, k_n$  be  $n$  fixed non-negative integers  
36 depending only on  $\mathsf{P}$ . Let  $\mathcal{P}$  be a property of  $\mathsf{P}$  satisfying the following condition: if  $|\mathsf{X}| = i$   
37 then  $|\mathcal{P}(\mathsf{X})| = k_i$ . Now, consider a uniform random permutation  $\langle \mathbf{p}_1, \dots, \mathbf{p}_n \rangle$  of  $\mathsf{P}$ . For any  $i$ ,  
38 let  $\mathsf{P}_i = \{\mathbf{p}_1, \dots, \mathbf{p}_i\}$  and let  $X_i$  be an indicator variable of the event that  $\mathbf{p}_i \in \mathcal{P}(\mathsf{P}_i)$ . Then, the  
39 variables  $X_i$  are mutually independent, for all  $i$ .*

*Proof:* Let  $E_i$  denote the event that  $\mathbf{p}_i \in \mathcal{P}(\mathbf{P}_i)$ . It suffices to show that the events  $E_1, \dots, E_n$  are mutually independent. The insight is to think about the sampling process of creating the random permutation  $\langle \mathbf{p}_1, \dots, \mathbf{p}_n \rangle$  in a different way. Imagine we randomly pick a permutation of elements in  $\mathbf{P}$ , and set the last element to be  $\mathbf{p}_n$ . Next, pick a random permutation of the remaining elements of  $\mathbf{P} \setminus \{\mathbf{p}_n\}$  and set the last element to be  $\mathbf{p}_{n-1}$ . Repeat this process until the whole permutation is generated. Observe that  $E_j$  is determined before  $E_i$  for any  $j > i$ .

Now, consider arbitrary indices  $1 \leq i_1 < i_2 < \dots < i_\psi \leq n$ . Observe that by our thought experiment, when determining the  $i_1$ th value in the permutation, the suffix  $\langle \mathbf{p}_{i_1+1}, \dots, \mathbf{p}_n \rangle$  is fixed. Moreover, the property defined on the remaining set of elements marks  $k_{i_1}$  elements, and these elements are randomly permuted before determining the  $i_1$ th value. Therefore, for any fixed sequence  $\sigma = \langle \mathbf{p}_{i_1+1}, \dots, \mathbf{p}_n \rangle$ , we have for a random permutation  $\tau$  of  $\mathbf{P}$ , that  $\Pr[E_{i_1} | \sigma] = \Pr[\tau \in E_{i_1} | \tau_{i_1+1}^n = \sigma] = k_{i_1}/i_1$ , where  $\tau_{i_1+1}^n = \langle \tau_{i_1+1}, \dots, \tau_n \rangle$  is the suffix of the last  $n - i_1$  elements of  $\tau$ . This also readily implies that  $\Pr[E_{i_1}] = \sum_{\sigma} \Pr[E_{i_1} | \sigma] \Pr[\sigma] = (k_1/i_1) \sum_{\sigma} \Pr[\sigma] = k_1/i_1$ . Thus, for any  $\sigma$ , we have  $\Pr[E_{i_1} | \sigma] = \Pr[E_{i_1}]$ .

**Informal argument.** We are intuitively done — knowing that  $E_{i_2} \cap \dots \cap E_{i_\psi}$  happens only gives us some partial information about what the suffix of the randomly picked permutation might be. However, the above states that even full knowledge of this suffix does not affect the probability of  $E_{i_1}$  happening, thus implying that  $E_{i_1}$  is independent of the other events.

**Formal argument.** Let  $\hat{\Pi}$  be the set of all permutations of  $\mathbf{P}$  such that  $E_{i_2} \cap \dots \cap E_{i_\psi}$  happens. Observe that whether or not a specific permutation  $\tau$  belongs to  $\hat{\Pi}$  depends only on the value of its suffix  $\tau_{i_1+1}^n$  — indeed, once  $\tau_{i_1+1}^n$  is known, one can determine whether the events  $E_{i_2}, \dots, E_{i_\psi}$  happen for  $\tau$ .

For any index  $i$ , let  $(\mathbf{P})_{n-i}$  be the set of sequences of distinct elements of  $\mathbf{P}$  of length  $n - i$ . For a sequence  $\sigma \in (\mathbf{P})_{n-i_1}$ , let  $\hat{\Pi}[\sigma]$  be the set of all permutations of  $\hat{\Pi}$  with the suffix  $\sigma$  (this set might be empty). For two different suffixes  $\sigma, \sigma' \in (\mathbf{P})_{n-i_1}$ , the corresponding sets of permutations  $\hat{\Pi}[\sigma]$  and  $\hat{\Pi}[\sigma']$  are disjoint. As such, the family  $\{\hat{\Pi}[\sigma] \mid \sigma \in (\mathbf{P})_{n-i_1}\}$  is a partition of  $\hat{\Pi}$  into disjoint sets.

By the above, for any  $\sigma \in (\mathbf{P})_{n-i_1}$ , such that  $\hat{\Pi}[\sigma]$  is not empty, we have that  $\Pr[\tau \in E_{i_1} | \tau \in \hat{\Pi}[\sigma]] = \Pr[E_{i_1} | \sigma] = \Pr[E_{i_1}]$ . By [Lemma 3.2](#), for any arbitrary suffix  $\sigma'$  such that  $\hat{\Pi}[\sigma']$  is not empty, we have

$$\Pr[E_{i_1} | E_{i_2} \cap \dots \cap E_{i_\psi}] = \Pr[E_{i_1} | \cup_{\sigma} \hat{\Pi}[\sigma]] = \Pr[E_{i_1} | \sigma'] = \Pr[E_{i_1}].$$

**Putting things together.** By induction, we now have

$$\begin{aligned} \Pr[E_{i_1} \cap \dots \cap E_{i_\psi}] &= \Pr[E_{i_1} | E_{i_2} \cap \dots \cap E_{i_\psi}] \Pr[E_{i_2} \cap \dots \cap E_{i_\psi}] \\ &= \Pr[E_{i_1}] \Pr[E_{i_2} \cap \dots \cap E_{i_\psi}] = \prod_{j=1}^{\psi} \Pr[E_{i_j}], \end{aligned}$$

which implies that the events are mutually independent.  $\blacksquare$

**Lemma 3.4.** *Let  $\mathbf{P}$  be a set of  $n \geq e^2$  elements, and  $k \geq 1$  be a fixed integer depending only on  $\mathbf{P}$ . Let  $\mathcal{P}$  be a property of  $\mathbf{P}$ . Now, consider a uniform random permutation  $\langle \mathbf{p}_1, \dots, \mathbf{p}_n \rangle$  of  $\mathbf{P}$ . For each  $i$ , denote  $\mathbf{P}_i = \{\mathbf{p}_1, \dots, \mathbf{p}_i\}$  and let  $X_i$  be an indicator variable of the event  $\mathbf{p}_i \in \mathcal{P}(\mathbf{P}_i)$ . Then we have:*

(a) *If  $|\mathcal{P}(\mathbf{X})| = k$  whenever  $|\mathbf{X}| \geq k$ , and  $|\mathcal{P}(\mathbf{X})| = |\mathbf{X}|$  whenever  $|\mathbf{X}| < k$ , then for any  $\gamma \geq 2e$ ,*

$$\Pr\left[\sum_i X_i > \gamma \cdot (2k \ln n)\right] \leq n^{-\gamma k}.$$



- 1 (b) The bound in (a) holds under a weaker condition: For all  $X \subseteq P$  we have  $|\mathcal{P}(X)| \leq k$ .  
 2 (c) An even weaker condition suffices: For a random permutation  $\langle p_1, \dots, p_n \rangle$  of  $P$ , assume  $|\mathcal{P}(P_i)| \leq$   
 3  $k'$  for all  $i$ , with probability  $1 - n^{-c}$ , where  $k' = c' \cdot k$ ,  $c$  is an arbitrary constant, and  $c' > 1$  is  
 4 a constant that depends only on  $c$ . Then for any  $\gamma \geq 2e$ ,

$$5 \quad \Pr \left[ \sum_i X_i > \gamma \cdot (2c'k \ln n) \right] \leq n^{-\gamma k} + n^{-c}.$$

6 *Proof:* (a) Let  $E_i$  be the event that  $p_i \in \mathcal{P}(P_i)$ . By [Lemma 3.3](#) the events  $E_1, \dots, E_n$  are mutually  
 7 independent, and  $\Pr[E_i] = |\mathcal{P}(P_i)|/i = \min(k/i, 1)$ . Thus, we have when  $n \geq e^2$ ,

$$8 \quad \mu = \mathbf{E} \left[ \sum_i X_i \right] \leq k + \sum_{k < i \leq n} \frac{k}{i} \leq k(2 + \ln n) \leq 2k \ln n.$$

10 For any constant  $\delta \geq 2e$ , by Chernoff's inequality, we have  $\Pr \left[ \sum_i X_i > \delta \mu \right] < 2^{-\delta \mu}$ . Therefore by  
 11 setting  $\delta = \gamma(2k \ln n)/\mu$  (which is at least  $2e$  by the assumption that  $\gamma \geq 2e$ ), we have

$$12 \quad \Pr \left[ \sum_i X_i > \gamma(2k \ln n) \right] < 2^{-\gamma(2k \ln n)} < n^{-\gamma k}.$$

14 (b) In order to extend the result using the weaker condition, we augment the given property  
 15  $\mathcal{P}$  to a new property  $\mathcal{P}'$  that holds for *exactly*  $k$  elements. So, fix an arbitrary ordering  $\prec$  on the  
 16 elements of  $P$ . Now given any set  $X$  with  $|X| \geq k$ , if  $|\mathcal{P}(X)| = k$  then let  $\mathcal{P}'(X) = \mathcal{P}(X)$ . Otherwise,  
 17 add the  $k - |\mathcal{P}(X)|$  smallest elements in  $X \setminus \mathcal{P}(X)$  according to  $\prec$  to  $\mathcal{P}(X)$ , and let  $\mathcal{P}'(X)$  be the  
 18 resulting subset of size  $k$ . We also set  $\mathcal{P}'(X) = X$  for all  $X$  with  $|X| < k$ . The new property  
 19  $\mathcal{P}'$  complies with the original condition. For any  $X$ ,  $\mathcal{P}(X) \subseteq \mathcal{P}'(X)$ , which implies that an upper  
 20 bound on the probability that the  $i$ th element is in the property set  $\mathcal{P}'$  is an upper bound on the  
 21 corresponding probability for  $\mathcal{P}$ .

22 (c) We truncate the given property  $\mathcal{P}$  if needed, so that it complies with (b). Specifically, fix  
 23 an arbitrary ordering  $\prec$  on the elements of  $P$ . Given any set  $X$ , if  $|\mathcal{P}(X)| \leq k'$  then  $\mathcal{P}'(X) = \mathcal{P}(X)$ .  
 24 Otherwise,  $|\mathcal{P}(X)| > k'$ , and set  $\mathcal{P}'(X)$  to be the first  $k'$  of  $\mathcal{P}(X)$  according to  $\prec$ . Clearly, the new  
 25 property  $\mathcal{P}'$  complies with the condition in (b). Let  $E$  denote the event  $\mathcal{P}'(P_i) = \mathcal{P}(P_i)$ , for all  $i$ .  
 26 By assumption, we have  $\Pr[E] \geq 1 - n^{-c}$ . Similarly, let  $F$  be the event that  $\sum_i X_i > \gamma(2k' \ln n)$ .  
 27 We now have that

$$28 \quad \Pr[F] \leq \Pr[F | E] \Pr[E] + \Pr[\overline{E}] < \left(1 - \frac{1}{n^c}\right) n^{-\gamma k'} + n^{-c} \leq n^{-\gamma k} + n^{-c}$$

30 for any  $\gamma \geq 2e$ . ■

31 The result of [Lemma 3.4](#) is known in the context of randomized incremental construction al-  
 32 gorithms (see [[BCKO08](#), §6.4]). However, the known proof is more convoluted — indeed, if the  
 33 property  $\mathcal{P}(X)$  has different sizes for different sets  $X$ , then it is no longer true that variables  $X_i$  in  
 34 the proof of [Lemma 3.4](#) are independent. Thus the padding idea in part (b) of the proof is crucial  
 35 in making the result more widely applicable.

36 **Example.** To see the power of [Lemma 3.4](#) we provide two easy applications — both results are of  
 37 course known, and are included here to make it clearer in what settings [Lemma 3.4](#) can be applied.  
 38 The impatient reader is encouraged to skip this example.

(a) **QuickSort**: We conceptually can think about QuickSort as being a randomized incremental algorithm, building up a list of numbers in the order they are used as pivots. Consider the execution of QuickSort when sorting a set  $P$  of  $n$  numbers. Let  $\langle p_1, \dots, p_n \rangle$  be the random permutation of the numbers picked in sequence by QuickSort. Specifically, in the  $i$ th iteration, it randomly picks a number  $p_i$  that was not handled yet, pivots based on this number, and then recursively handles the subproblems. At the  $i$ th iteration, a set  $P_i = \{p_1, \dots, p_i\}$  of pivots has already been chosen by the algorithm. Consider a specific element  $x \in P$ . For any subset  $X \subseteq P$ , let  $\mathcal{P}(X)$  be the two numbers in  $X$  having  $x$  in between them in the original ordering of  $P$  and are closest to each other. In other words,  $\mathcal{P}(X)$  contains the (at most) two elements that are the endpoints of the interval of  $\mathbb{R} \setminus X$  that contains  $x$ . Let  $X_i$  be the indicator variable of the event  $p_i \in \mathcal{P}(P_i)$  — that is,  $x$  got compared to the  $i$ th pivot when it was inserted. Clearly, the total number of comparisons  $x$  participates in is  $\sum_i X_i$ , and by [Lemma 3.4](#) the number of such comparisons is  $O(\log n)$ , with high probability, implying that QuickSort takes  $O(n \log n)$  time, with high probability.

(b) **Point-location queries in a history dag**: Consider a set of lines in the plane, and build their vertical decomposition using randomized incremental construction. Let  $L_n = \langle \ell_1, \dots, \ell_n \rangle$  be the permutation used by the randomized incremental construction. Given a query point  $p$ , the point-location time is the number of times the vertical trapezoid containing  $p$  changes in the vertical decomposition of  $L_i = \langle \ell_1, \dots, \ell_i \rangle$ , as  $i$  increases. Thus, let  $X_i$  the indicator variable of the event that  $\ell_i$  is one of the (at most) four lines defining the vertical trapezoid containing  $p$  the vertical decomposition of  $L_i$ . Again, [Lemma 3.4](#) implies that the query time is  $O(\log n)$ , with high probability. This result is well known, see [\[CMS93\]](#) and [\[BCKO08, §6.4\]](#), but our proof is arguably more direct and cleaner.

## 4. The proxy set

Providing a reasonable bound on the complexity of the candidate diagram directly seems challenging. Therefore, we instead define for each point  $x$  in the plane a slightly different set, called the *proxy set*. First we prove that the proxy set for each point in the plane has small size (see [Lemma 4.2](#) below). Then we prove that, with high probability, the proxy set of  $x$  contains the candidate set of  $x$  for all points  $x$  in the plane simultaneously (see [Lemma 4.4](#) below).

### 4.1. Definitions

As before, the input is a set of sites  $S$ . For each site  $s \in S$ , we randomly pick a parametric point  $\beta \in [0, 1]^d$  according to the sampling method described in [Section 2.2](#).

**Volume ordering.** Given a point  $p = (p_1, \dots, p_d)$  in  $[0, 1]^d$ , the *point volume*  $\text{pv}(p)$  of point  $p$  is defined to be  $p_1 p_2 \cdots p_d$ . That is, the volume of the hyperrectangle with  $p$  and the origin as a pair of opposite corners. When  $p$  is specifically the associated parametric point of an input site  $s$ , we refer to the point volume of  $p$  as the *parametric volume* of  $s$ . Observe that if point  $p$  dominates another point  $q$  then  $p$  must have smaller point volume (that is,  $p$  lies in the hyperrectangle defined by  $q$ ).

The *volume ordering* of sites in  $S$  is a permutation  $\langle s_1, \dots, s_n \rangle$  ordered by increasing parametric volume of the sites. That is,  $\text{pv}(\beta_1) \leq \text{pv}(\beta_2) \leq \dots \leq \text{pv}(\beta_n)$ , where  $\beta_i$  is the parametric point of  $s_i$ . If  $\beta_i$  dominates  $\beta_j$  then  $s_i$  precedes  $s_j$  in the volume ordering. So if we add the sites in volume ordering, then when we add the  $i$ th site  $s_i$  we can ignore all later sites when determining its region

of influence — that is, the region of points whose candidate set  $s_i$  belongs to — as no later site can dominate  $s_i$ .

**$k$  Nearest neighbors.** For a set of sites  $S$  and a point  $x$  in the plane, let  $d_k(x, S)$  denote the  *$k$ th nearest neighbor distance* to  $x$  in  $S$ . That is, the  $k$ th smallest value in the multiset  $\{\|x - s\| \mid s \in S\}$ . The  *$k$  nearest neighbors* to  $x$  in  $S$  is the set

$$N_k(x, S) = \left\{ s \in S \mid \|x - s\| \leq d_k(x, S) \right\}.$$

**Definition 4.1.** Let  $S$  be a set of sites in the plane, and let  $\langle s_1, \dots, s_n \rangle$  be the volume ordering of  $S$ . Let  $S_i$  denote the underlying set of the  $i$ th prefix  $\langle s_1, \dots, s_i \rangle$  of  $\langle s_1, \dots, s_n \rangle$ . For a parameter  $k$  and a point  $x$  in the plane, the  *$k$ th proxy set* of  $x$  is the set of sites

$$\Phi_k(x, S) = \bigcup_{i=1}^n N_k(x, S_i).$$

In words, site  $s$  is in  $\Phi_k(x, S)$  if it is one of the  $k$  nearest neighbors to point  $x$  in some prefix of the volume ordering  $\langle s_1, \dots, s_n \rangle$ .

## 4.2. Bounding the size of the proxy set

The desired bound now follows by using backward analysis and [Corollary 3.1](#).

**Lemma 4.2.** *Let  $S$  be a set of  $n$  sites in the plane, and let  $k \geq 1$  be a fixed parameter. Then we have  $|\Phi_k(x, S)| = O_{whp}(k \log n)$  simultaneously for all points  $x$  in the plane.*

*Proof:* Fix a point  $x$  in the plane. A site  $s$  gets added to the proxy set  $\Phi_k(x, S)$  if site  $s$  is one of the  $k$  nearest neighbors of  $x$  among the underlying set  $S_i$  of some prefix of the volume ordering of  $S$ . Therefore a direct application of [Corollary 3.1](#) implies (by setting  $\mathcal{P}(S_i)$  to be  $N_k(x, S_i)$ ), with high probability, that  $|\Phi_k(x, S)| = O(k \log n)$ .

Furthermore, this holds for all points in the plane simultaneously. Indeed, consider the arrangement determined by the  $\binom{n}{2}$  bisectors formed by all the pairs of sites in  $S$ . This arrangement is a simple planar map with  $O(n^4)$  vertices and  $O(n^4)$  faces. Observe that within each face the proxy set cannot change since all points in this face have the same ordering of their distances to the sites in  $S$ . Therefore, picking a representative point from each of these  $O(n^4)$  faces, applying the high-probability bound to each of them, and then the union bound implies the claim. ■

## 4.3. The proxy set contains the candidate set

The following corollary is implied by a careful (but straightforward) integration argument.

**Corollary 4.3 (Proof in [Appendix A](#)).** *Let  $F_d(\Delta)$  be the volume of the set of points  $p$  in  $[0, 1]^d$  such that the point volume  $\text{pv}(p)$  is at most  $\Delta$ , where  $\Delta \in (0, 1)$ . That is,*

$$F_d(\Delta) = \text{vol} \left( \left\{ p \in [0, 1]^d \mid \text{pv}(p) \leq \Delta \right\} \right).$$

*Then, we have that  $F_d(\Delta) = \sum_{i=0}^{d-1} \frac{\Delta}{i!} \ln^i \frac{1}{\Delta} = O(\Delta \log^{d-1} n)$ .*

**Lemma 4.4.** *Let  $S$  be a set of  $n$  sites in the plane, and let  $k = \Theta(\log^d n)$  be a fixed parameter. For all points  $x$  in the plane,  $C(x) \subseteq \Phi_k(x, S)$  with high probability.*

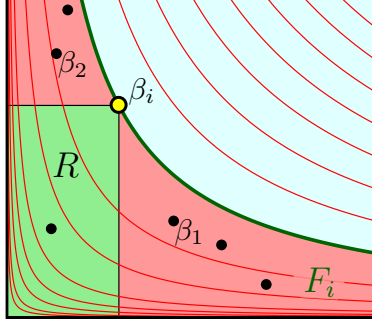


Figure 4.1: The red shaded region  $F_i$  consists of parametric points whose point volume is at most the point volume of  $\beta_i$ . The green shaded region  $R$  consists of the parametric points that dominate  $\beta_i$ . The red curves are contours of the point volume function.

*Proof:* Fix a point  $x$  in the plane, and let  $s_i$  be any site *not* in  $\Phi_k(x, S)$ , and let  $\beta_i$  be the associated parametric point. We claim that, with high probability, the site  $s_i$  is dominated by some other site which is closer to  $x$ , and hence by the definition of dominating preference (Definition 2.3),  $s_i$  cannot be a site used by  $x$  (and thus  $s_i \notin C(x)$ ). Taking the union bound over all sites not in  $\Phi_k(x, S)$  then implies this claim.

By Corollary 4.3, the total measure of the points in  $[0, 1]^d$  with point volume at most  $\Delta = (\log n)/n$  is  $O((\log^d n)/n)$ . As such, by Chernoff's inequality, with high probability, there are  $K = O(\log^d n)$  sites in  $S$  such that their parametric points have point volume smaller than  $\Delta$ . In particular, by choosing  $k$  to be larger than  $K$ , the underlying set  $S_k$  of the  $k$ th prefix of the volume ordering of  $S$  will contain all these small point volume sites, and since  $S_k \subseteq \Phi_k(x, S)$ , so will  $\Phi_k(x, S)$ . Therefore, from this point on, we will assume that  $s_i \notin \Phi_k(x, S)$  and  $\Delta_i = \text{pv}(\beta_i) = \Omega(\log n/n)$ .

Now any site  $s$  with smaller parametric volume than  $s_i$  is in the (unordered) prefix  $S_i$ . In particular, the  $k$  nearest neighbors  $N_k(x, S_i)$  of  $x$  in  $S_i$  all have smaller parametric volume than  $s_i$ . Hence  $\Phi_k(x, S)$  contains  $k$  points all of which have smaller parametric volume than  $s_i$ , and which are closer to  $x$ . Therefore, the claim will be implied if one of these  $k$  points dominates  $s_i$ .

The probability of a site  $s$  (that is closer to  $x$  than  $s_i$ ) with parametric point  $\beta$  to dominate  $s_i$  is the probability that  $\beta \preceq \beta_i$  given that  $\beta \in F_i$ , where  $F_i = \{\beta \in [0, 1]^d \mid \text{pv}(\beta) \leq \Delta_i\}$ . Corollary 4.3 implies that  $\text{vol}(F_i) = F_d(\Delta_i) = O(\Delta_i \log^{d-1} n)$ . The probability that a random parametric point in  $[0, 1]^d$  dominates  $\beta_i$  is exactly  $\Delta_i$ , and as such the desired probability  $\Pr[\beta \preceq \beta_i \mid \beta \in F_i]$  is equal to  $\Delta_i / F_d(\Delta_i)$ , which is  $\Omega(1/\log^{d-1} n)$ . This is depicted in Figure 4.1 — the probability of a random point picked uniformly from the region  $F_i$  under the curve  $y = \Delta_i/x$ , induced by  $s_i$ , to fall in the rectangle  $R$ .

As the parametric point of each one of the  $k$  points in  $N_k(x, S_i)$  has equal probability to be anywhere in  $F$ , this implies the expected number of points in  $N_k(x, S_i)$  which dominate  $s_i$  is  $\Pr[\beta \preceq \beta_i \mid \beta \in F_i] \cdot k = \Omega(\log n)$ . Therefore by making  $k$  sufficiently large, Chernoff's inequality implies the desired result.

It follows that the statement holds, for all points in the plane simultaneously, by following the argument used in the proof of Lemma 4.2. ■

## 5. Bounding the complexity of the $k$ th order proxy diagram

The  *$k$ th proxy cell* of  $x$  is the set of all the points in the plane that have the same  $k$ th proxy set associated with them. Formally, this is the set

$$\{p \in \mathbb{R}^2 \mid \Phi_k(p, S) = \Phi_k(x, S)\}.$$

The decomposition of the plane into these faces is the  *$k$ th order proxy diagram*. In this section, our goal is to prove that the expected total diagram complexity of the  $k$ th order proxy diagram is  $O(k^4 n \log n)$ . To this end, we relate this complexity to the overlay of star-shaped polygons that rise out of the  $k$ th order Voronoi diagram.

### 5.1. Preliminaries

#### 5.1.1. The $k$ th order Voronoi diagram

Let  $S$  be a set of  $n$  sites in the plane. The  *$k$ th order Voronoi diagram* of  $S$  is a partition of the plane into faces such that each cell is the locus of points which have the same set of  $k$  nearest sites in  $S$  (the internal ordering of these  $k$  sites, by distance to the query point, may vary within the cell). It is well known that the worst case complexity of this diagram is  $\Theta(k(n - k))$  (see [AKL13, §6.5]).

#### 5.1.2. Arrangements of planes and lines

One can interpret the  $k$ th order Voronoi diagram in terms of an arrangement of planes in  $\mathbb{R}^3$ . Specifically, “lift” each site to the paraboloid  $(x, y, -(x^2 + y^2))$ . Consider the arrangement of planes  $H$  tangent to the paraboloid at the lifted locations of the sites. A point on the union of these planes is of *level  $k$*  if there are exactly  $k$  planes strictly below it. The  *$k$ -level* is the closure of the set of points of level  $k$ .<sup>3</sup> (For any set of  $n$  hyperplanes in  $\mathbb{R}^d$ , one can define  $k$ -levels of arrangement of hyperplanes analogously.) Consider a point  $x$  in the  $xy$ -plane. The decreasing  $z$ -ordering of the planes vertically below  $x$  is the same as the ordering, by decreasing distance from  $x$ , to the corresponding sites. Hence, let  $E_k(H)$  denote the set of edges in the arrangement  $H$  on the  $k$ -level, where an edge is a maximal portion of the  $k$ -level that lies on the intersection of two planes (induced by two sites). Then the projection of the edges in  $E_{k-1}(H)$  onto the  $xy$ -plane results in the edges of the  $k$ th order Voronoi diagram. When there is no risk of confusion, we also use  $E_k(S)$  to denote the set of edges in  $E_k(H)$ , where  $H$  is obtained by lifting the sites in  $S$  to the paraboloid and taking the tangential planes, as described above.

We also need the notion of  $k$ -levels of arrangement of lines. For set of lines  $L$  in the plane, let  $E_k(L)$  denote the set of edges in the arrangement of  $L$  on the  $k$ -level. We need the following lemma.

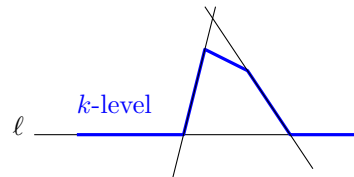
**Lemma 5.1.** *Let  $L$  be a set of lines in general position in the plane, and let  $\ell$  be any line in  $L$ . Then at most  $k + 2$  edges from  $E_k(L)$ , the  $k$ -level of the arrangement of  $L$ , can lie on  $\ell$ .*

*Proof:* This lemma is well known, and its proof is included here for the sake of completeness.

---

<sup>3</sup>The lifting of the sites to the paraboloid  $z = -(x^2 + y^2)$  is done so that the definition of the  $k$ -level coincide with the standard definition.

1 Perform a linear transformation such that  $\ell$  is horizontal and the  
 2  $k$ -level is preserved. As we go from left to right along the now  
 3 horizontal line  $\ell$  (starting from  $-\infty$ ), we may leave and enter the  $k$ -  
 4 level multiple times. However, every time we leave and then return  
 5 to the  $k$ -level we must intersect a negative slope line in between.  
 6 Specifically, both when we leave and return to the  $k$ -level, there must be an intersection with  
 7 another line. If the line intersecting the leaving point has a negative slope then we are done, so  
 8 assume it has positive slope. In this case the level on  $\ell$  decreases as we leave the  $k$ -level, and  
 9 therefore when we return to the  $k$ -level, the point of return must be at an intersection with a  
 10 negative slope line (see figure on the right).



11 So after leaving and returning to the  $k$ -level  $k + 1$  times, there must be at least  $k + 1$  negative  
 12 slope lines below, which implies that the remaining part of  $\ell$  is on level strictly larger than  $k$ . ■

13 **Lemma 5.2.** *Let  $L$  be a set of  $n$  lines in general position in the plane. Fix any arbitrary insertion*  
 14 *ordering of the lines in  $L$ , then the total number of distinct vertices on the  $k$ -level of the arrangement*  
 15 *of  $L$  seen over all iterations of this insertion process is bounded by  $O(nk)$ .*

16 *Proof:* Let  $\ell_i$  be the  $i$ th line inserted, and let  $L_i$  be the set of the first  $i$  lines. Any new vertex  
 17 on the  $k$ th level created by the insertion must lie on  $\ell_i$ . However, by Lemma 5.1 at most  $k + 2$   
 18 edges from  $E_k(L_i)$  can lie on  $\ell_i$ . As each such edge has at most two endpoints, the insertion of  $\ell_i$   
 19 contributes  $O(k)$  vertices to the  $k$ -level. The bound now follows by summing over all  $n$  lines. ■

## 20 5.2. Bounding the size of the below conflict-lists

### 21 5.2.1. The below conflict lists

22 Let  $H$  be a set of  $n$  planes in general position in  $\mathbb{R}^3$ . (For example, in the setting of the  $k$ th order  
 23 Voronoi diagram,  $H$  is the set of planes that are tangent to the paraboloid at the lifted locations of  
 24 the sites.) For any subset  $R \subseteq H$ , let  $V_k(R)$  denote the vertices on the  $k$ -level of the arrangement of  
 25  $R$ . Similarly, let  $V_{\leq k}(R) = \bigcup_{i=0}^k V_i(R)$  be the set of vertices of level at most  $k$  in the arrangement  
 26 of  $R$ , and let  $E_{\leq k}(R)$  be the set of edges of level at most  $k$  in the arrangement of  $R$ . For a vertex  
 27  $v$  in the arrangement of  $R$ , the *below conflict list*  $B(v)$  of  $v$  is the set of planes in  $H$  (not  $R$ ) that  
 28 lie strictly below  $v$ , and let  $b_v = |B(v)|$ . For an edge  $e$  in the arrangement of  $R$ , the *below conflict*  
 29 *list*  $B(e)$  of  $e$  is the set of planes in  $H$  (again, not  $R$ ) which lie below  $e$  (that is, there is at least  
 30 one point on  $e$  that lies above such a plane), and let  $b_e = |B(e)|$ . Our purpose here is to bound the  
 31 quantities  $\mathbf{E}[\sum_{v \in V_{\leq k}(R)} b_v]$  and  $\mathbf{E}[\sum_{e \in E_{\leq k}(R)} b_e]$ .

### 32 5.2.2. The Clarkson-Shor technique

33 In the following, we use the Clarkson-Shor technique [CS89], stated here without proof (see [Har11]  
 34 for details). Specifically, let  $S$  be a set of elements such that any subset  $R \subseteq S$  defines a corresponding  
 35 set of objects  $\mathcal{T}(R)$  (e.g.,  $S$  is a set of planes and any subset  $R \subseteq S$  induces a set of vertices in the  
 36 arrangement of planes  $R$ ). Each potential object,  $\tau$ , has a defining set and a stopping set. The  
 37 *defining set*,  $D(\tau)$ , is a subset of  $S$  that must appear in  $R$  in order for the object to be present in  
 38  $\mathcal{T}(R)$ . We require that the defining set has at most a constant size for every object. The *stopping*  
 39 *set*,  $K(\tau)$ , is a subset of  $S$  such that if any of its member appear in  $R$  then  $\tau$  is not present in  $\mathcal{T}(R)$ .  
 40 We also naturally require that  $K(\tau) \cap D(\tau) = \emptyset$  for all object  $\tau$ . Surprisingly, this already implies  
 41 the following.



**Theorem 5.3 (Bounded Moments [CS89]).** *Using the above notation, let  $\mathcal{S}$  be a set of  $n$  elements, and let  $\mathcal{R}$  be a random sample of size  $r$  from  $\mathcal{S}$ . Let  $f(\cdot)$  be a monotonically increasing function bounded by a polynomial (that is,  $f(n) = n^{O(1)}$ ). We have*

$$\mathbf{E} \left[ \sum_{\tau \in \mathcal{T}(\mathcal{R})} f(|K(\tau)|) \right] = O \left( \mathbf{E} [|\mathcal{T}(\mathcal{R})|] f\left(\frac{n}{r}\right) \right),$$

where the expectation is taken over random sample  $\mathcal{R}$ .

### 5.2.3. Bounding the below conflict-lists

**The technical challenge.** The proof of the next lemma is technically interesting as it does not follow in a straightforward fashion from the Clarkson-Shor technique. Indeed, the below conflict list is *not* the standard conflict list. Specifically, the decision whether a vertex  $v$  in the arrangement of  $\mathcal{R}$  is of level at most  $k$  is a “global” decision of  $\mathcal{R}$ , and as such the defining set of this vertex is neither of constant size, nor unique, as required to use the Clarkson-Shor technique. If this was the only issue, the extension by Agarwal *et al.* [AMS98] could handle this situation. However it is even worse: a plane  $h \in \mathcal{H} \setminus \mathcal{R}$  that is below a vertex  $v \in \mathcal{V}_{\leq k}(\mathcal{R})$  is not necessarily conflicting with  $v$  (that is, in the stopping set of  $v$ ) — as its addition to  $\mathcal{R}$  will not necessarily remove  $v$  from  $\mathcal{V}_{\leq k}(\mathcal{R} \cup \{h\})$ .

**The solution.** Since the standard technique fails in this case, we need to perform our argument somehow indirectly. Specifically, we use a second random sample and then deploy the Clarkson-Shor technique on this smaller sample — this is reminiscent of the proof bounding the size of  $\mathcal{V}_{\leq k}(\mathcal{H})$  by Clarkson-Shor [CS89], and the proof of the exponential decay lemma of Chazelle and Friedman [CF90].

**Lemma 5.4.** *Let  $k$  be a fixed constant, and let  $\mathcal{R}$  be a random sample (without replacement) of size  $r$  from a set of  $\mathcal{H}$  of  $n$  planes in  $\mathbb{R}^3$ , we have*

$$\mathbf{E} \left[ \sum_{v \in \mathcal{V}_{\leq k}(\mathcal{R})} b_v \right] = O(nk^3).$$

*Proof:* For the sake of simplicity of exposition, let us assume that the sampling here is done by picking every element into the random sample  $\mathcal{R}$  with probability  $r/n$ . Doing the computations below using sampling without replacement (so we get the exact size) requires modifying the calculations so that the probabilities are stated using binomial coefficients — this makes the calculation messier, but the results remain the same. See [Sha03] for further discussion of this minor issue.

Fix a random sample  $\mathcal{R}$ . Now sample once again by picking each plane in  $\mathcal{R}$ , with probability  $1/k$ , into a subsample  $\mathcal{R}'$ . Let us consider the probability that a vertex  $v \in \mathcal{V}_{\leq k}(\mathcal{R})$  ends up on the lower envelope of  $\mathcal{R}'$ . A lower bound can be achieved by the standard argument of Clarkson-Shor. Specifically, if a vertex  $v$  is on the lower envelope then its three defining planes must be in  $\mathcal{R}'$ . Moreover, as  $v \in \mathcal{V}_{\leq k}(\mathcal{R})$ , by definition there are at most  $k$  planes below  $v$  that must not be in  $\mathcal{R}'$ . So let  $X_v$  be the indicator variable of whether  $v$  appears on the lower envelope of  $\mathcal{R}'$ . We then have

$$\mathbf{E}_{\mathcal{R}'} [X_v \mid \mathcal{R}] \geq \frac{1}{k^3} (1 - 1/k)^k \geq \frac{1}{e^2 k^3}.$$

Observe that

$$\mathbf{E}_{\mathcal{R}'} \left[ \sum_{v \in \mathcal{V}_0(\mathcal{R}')} b_v \right] = \mathbf{E}_{\mathcal{R}} \left[ \mathbf{E}_{\mathcal{R}'} \left[ \sum_{v \in \mathcal{V}_0(\mathcal{R}')} b_v \mid \mathcal{R} \right] \right] \geq \mathbf{E}_{\mathcal{R}} \left[ \mathbf{E}_{\mathcal{R}'} \left[ \sum_{v \in \mathcal{V}_{\leq k}(\mathcal{R})} X_v b_v \mid \mathcal{R} \right] \right]. \quad (5.1)$$

1 Fixing the value of  $R$ , the lower bound above implies

$$2 \quad \mathbf{E}_{R'} \left[ \sum_{v \in V_{\leq k}(R)} X_v b_v \mid R \right] = \sum_{v \in V_{\leq k}(R)} \mathbf{E}_{R'} [X_v b_v \mid R] = \sum_{v \in V_{\leq k}(R)} b_v \mathbf{E}_{R'} [X_v \mid R] \geq \sum_{v \in V_{\leq k}(R)} \frac{b_v}{e^2 k^3},$$

4 by linearity of expectations and as  $b_v$  is a constant for  $v$ . Plugging this into Eq. (5.1), we have

$$5 \quad \mathbf{E}_{R'} \left[ \sum_{v \in V_0(R')} b_v \right] \geq \mathbf{E}_R \left[ \sum_{v \in V_{\leq k}(R)} \frac{b_v}{e^2 k^3} \right] = \frac{1}{e^2 k^3} \mathbf{E}_R \left[ \sum_{v \in V_{\leq k}(R)} b_v \right]. \quad (5.2)$$

7 Observe that  $R'$  is a random sample of  $R$  which by itself is a random sample of  $H$ . As such, one  
 8 can interpret  $R'$  as a direct random sample of  $H$ . The lower envelope of a set of planes has linear  
 9 complexity, and for a vertex  $v$  on the lower envelope of  $R'$  the set  $B(v)$  is the standard conflict list  
 10 of  $v$ . As such, Theorem 5.3 implies

$$11 \quad \mathbf{E}_{R'} \left[ \sum_{v \in V_0(R')} b_v \right] = O \left( |R'| \cdot \frac{n}{|R'|} \right) = O(n).$$

13 Plugging this into Eq. (5.2) implies the claim. ■

14 **Corollary 5.5.** *Let  $R$  be a random sample (without replacement) of size  $r$  from a set  $H$  of  $n$  planes*  
 15 *in  $\mathbb{R}^3$ . We have that  $\mathbf{E}_R \left[ \sum_{e \in E_{\leq k}(R)} b_e \right] = O(nk^3)$ .*

16 *Proof:* Under general position assumption every vertex in the arrangement of  $H$  is adjacent to 8  
 17 edges. For an edge  $e = uv$ , it is easy to verify that  $B(e) \subseteq B(u) \cup B(v)$ , and as such we charge the  
 18 conflict list of  $e$  to its two endpoints  $u$  and  $v$ , and every vertex get charged  $O(1)$  times. Now, the  
 19 claim follows by Lemma 5.4.

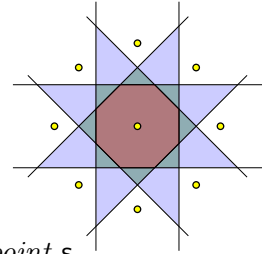
20 This argument fails to capture edges that are rays in the arrangement, but this is easy to  
 21 overcome by clipping the arrangement to a bounding box that contains all the vertices of the  
 22 arrangement. We omit the easy but tedious details. ■

### 23 5.3. Environments and overlays

24 For a site  $s$  in  $S$  and a constant  $k$ , the  *$k$  environment* of  $s$ , denoted by  $\text{env}_k(s, S)$ , is the set of all  
 25 the points in the plane such that  $s$  is one of their  $k$  nearest neighbors in  $S$ :

$$26 \quad \text{env}_k(s, S) = \{x \in \mathbb{R}^2 \mid s \in N_k(x, S)\}.$$

28 See the figure on the right for an example what this environment looks like  
 29 for different values of  $k$ . One can view the  $k$  environment of  $s$  as the union of  
 30 the  $k$ th order Voronoi cells which have  $s$  as one of the  $k$  nearest sites. Observe  
 31 that the overlay of the polygons  $\text{env}_k(s_1, S), \dots, \text{env}_k(s_n, S)$  produces the  $k$ th  
 32 order Voronoi diagram of  $S$ . It is also known that each  $k$  environment of a  
 33 site is a star-shaped polygon (see Aurenhammer and Schwarzkopf [AS92]).



34 **Lemma 5.6.** *The set  $\text{env}_k(s, S)$  is a star-shaped polygon with respect to the point  $s$ .*

35 *Proof:* Consider the set of all  $n - 1$  bisectors determined by  $s$  and any other site in  $S$ . For any  
 36 point  $x$  in the plane,  $p \in \text{env}_k(s, S)$  holds if the segment from  $s$  to  $p$  crosses at most  $k - 1$  of these  
 37 bisectors. The star-shaped property follows as when walking along any ray emanating from  $s$ , the  
 38 number of bisectors crossed is a monotonically increasing function of distance from  $s$ . Moreover,  
 39  $\text{env}_k(s, S)$  is a polygon as its boundary is composed of subsets of straight line bisectors. ■

1        Going back to our original problem. Let  $k$  be a fixed constant, and let  $\langle \mathbf{s}_1, \dots, \mathbf{s}_n \rangle$  be the  
2 volume ordering of  $S$ . As usual, we use  $S_i$  to denote the unordered  $i$ th prefix of  $\langle \mathbf{s}_1, \dots, \mathbf{s}_n \rangle$ . Let  
3  $\text{env}_i = \text{env}_k(\mathbf{s}_i, S_i)$ , the union of all cells in the  $k$ th order Voronoi diagram of  $S_i$  where  $\mathbf{s}_i$  is one of  
4 the  $k$  nearest neighbors.

5        **Observation 5.7.** *The arrangement determined by the overlay of the polygons  $\text{env}_1, \dots, \text{env}_n$  is*  
6 *the  $k$ th order proxy diagram of  $S$ .*

## 7        5.4. Putting it all together

8        The proof of the following lemma is similar in spirit to the argument of Har-Peled and Raichel  
9 [HR14].

10        **Lemma 5.8.** *Let  $S$  be a set of  $n$  sites in the plane, let  $\langle \mathbf{s}_1, \dots, \mathbf{s}_n \rangle$  be the volume ordering of  $S$ , and*  
11 *let  $k$  be a fixed number. The expected complexity of the arrangement determined by the overlay of*  
12 *the polygons  $\text{env}_1, \dots, \text{env}_n$  (and therefore, the expected complexity of the  $k$ th order proxy diagram)*  
13 *is  $O(k^4 n \log n)$ , where  $\text{env}_i = \text{env}_k(\mathbf{s}_i, S_i)$  and  $S_i = \{\mathbf{s}_1, \dots, \mathbf{s}_i\}$  is the underlying set of the  $i$ th prefix*  
14 *of  $\langle \mathbf{s}_1, \dots, \mathbf{s}_n \rangle$ , for each  $i$ .*

15        *Proof:* As the arrangement of the overlay of the polygons  $\text{env}_1, \dots, \text{env}_n$  is a planar map, it suffices  
16 to bound the number of edges in the arrangement. Fix an iteration  $i$ , and observe that  $S_i$  is  
17 fixed once  $i$  is fixed. For an edge  $e \in E_{\leq k}(S_i)$ , let  $X_e$  be the indicator variable of the event that  
18  $e$  was created in the  $i$ th iteration, and furthermore, lies on the boundary of  $\text{env}_i$ . Observe that  
19  $\mathbf{E}[X_e \mid S_i] \leq 4/i$ , as an edge appears for the first time in round  $i$  only if one of its (at most) four  
20 defining sites was the  $i$ th site inserted.

21        For each  $i$ , let  $E(\text{env}_i)$  be the edges in  $E_{\leq k}(S_i)$  that appear on the boundary of  $\text{env}_i$  (for simplicity  
22 we do not distinguish between edges in  $E_{\leq k}(S_i)$  in  $\mathbb{R}^3$  and their projection in the plane). Created  
23 in the  $i$ th iteration, an edge  $e$  in  $E(\text{env}_i)$  is going to be broken into several pieces in the final  
24 arrangement of the overlay. Let  $n_e$  be the number of such pieces that arise from  $e$ .

25        Here we claim that  $n_e \leq c \cdot kb_e$  for some constant  $c$ . Indeed,  $n_e$  counts the number of future  
26 intersections of  $e$  with the edges of  $E(\text{env}_j)$ , for any  $j > i$ . As the edge  $e$  is on the  $k$ -level at the  
27 time of creation, and the edges in  $E(\text{env}_j)$  are on the  $k$ -level when they are being created (in the  
28 future), these edges must lie below  $e$ . Namely, any future intersect on  $e$  are caused by intersections  
29 of (pairs of) planes in  $B(e)$ . So consider the intersection of all planes in  $B(e)$  on the vertical plane  
30 containing  $e$ . (Since  $S_i$  is fixed,  $B(e)$  is also fixed for all  $e \in E_{\leq k}(S_i)$ .) On this vertical plane,  $B(e)$  is  
31 a set of  $b_e$  lines, whose insertion ordering is defined by the suffix of the permutation  $\langle \mathbf{s}_{i+1}, \dots, \mathbf{s}_n \rangle$ .  
32 Now any edge of  $E(\text{env}_j)$ , for some  $j > i$ , that intersects  $e$  must appear as a vertex on the  $k$ -level  
33 at some point during the insertion of these lines. However, by Lemma 5.2, applied to the lines of  
34  $B(e)$  on the vertical plane of  $e$ , under any insertion ordering there are at most  $O(kb_e)$  vertices that  
35 ever appear on the  $k$ -level.

36        Let  $Y_i = \sum_{e \in E(\text{env}_i)} n_e = \sum_{e \in E_{\leq k}(S_i)} n_e X_e$  be the total (forward) complexity contribution to the  
37 final arrangement of edges added in round  $i$ . We thus have

$$\begin{aligned}
\mathbf{E}[Y_i \mid S_i] &= \mathbf{E}\left[\sum_{e \in E_{\leq k}(S_i)} n_e X_e \mid S_i\right] \leq \mathbf{E}\left[\sum_{e \in E_{\leq k}(S_i)} ck b_e X_e \mid S_i\right] \\
&= \sum_{e \in E_{\leq k}(S_i)} ck b_e \mathbf{E}[X_e \mid S_i] \leq \frac{4ck}{i} \cdot \sum_{e \in E_{\leq k}(S_i)} b_e.
\end{aligned}$$

The total complexity of the overlay arrangement of the polygons  $\text{env}_1, \dots, \text{env}_n$  is asymptotically bounded by  $\sum_i Y_i$ , and so by [Corollary 5.5](#) we have

$$\mathbf{E}\left[\sum_i Y_i\right] = \sum_i \mathbf{E}\left[\mathbf{E}\left[Y_i \mid \mathcal{S}_i\right]\right] \leq \sum_i \mathbf{E}\left[\frac{4ck}{i} \cdot \sum_{e \in \mathbf{E}_{\leq k}(\mathcal{S}_i)} \mathbf{b}_e\right] = O\left(\sum_i \frac{nk^4}{i}\right) = O\left(k^4 n \log n\right). \blacksquare$$

## 6. On the expected size of the staircase

### 6.1. Number of staircase points

#### 6.1.1. The two dimensional case

**Corollary 6.1.** *Let  $\mathbf{P}$  be a set of  $n$  points sampled uniformly at random from the unit square  $[0, 1]^2$ . Then the number of staircase points  $\text{St}(\mathbf{P})$  in  $\mathbf{P}$  is  $O_{\text{whp}}(\log n)$ .*

*Proof:* If we order the points in  $\mathbf{P}$  by increasing  $x$ -coordinate, then the staircase points are exactly the points which have the smallest  $y$ -values out of all points in their prefix in this ordering. As the  $x$ -coordinates are sampled uniformly at random, this ordering is a random permutation  $\langle y_1, \dots, y_n \rangle$  of the  $y$ -values  $\mathbf{Y}$ . Let  $X_i$  be the indicator variable of the event that  $y_i$  is the smallest number in  $\mathbf{Y}_i = \{y_1, \dots, y_i\}$  for each  $i$ . By setting property  $\mathcal{P}(\mathbf{Y}_i)$  to be the smallest number in the prefix  $\mathbf{Y}_i$ , we have  $\sum_{i=1}^n X_i = O(\log n)$  with high probability by [Corollary 3.1](#).  $\blacksquare$

#### 6.1.2. Higher dimensions

**Lemma 6.2.** *Fix a dimension  $d \geq 2$ . Let  $m$  and  $n$  be parameters, such that  $m \leq n$ . Let  $\mathbf{Q} = \langle \mathbf{q}_1, \dots, \mathbf{q}_m \rangle$  be an ordered set of  $m$  points picked randomly from  $[0, 1]^d$  as described in [Section 2.2](#). Assume that we have  $|\text{St}(\mathbf{Q}_i)| = O(c_d \log^{d-1} n)$ , with high probability with respect to  $m$  for all  $i$  simultaneously, where  $\mathbf{Q}_i = \{\mathbf{q}_1, \dots, \mathbf{q}_i\}$  is the underlying set of the  $i$ th prefix of  $\mathbf{Q}$ . Then, the set  $\text{St} = \bigcup_{i=1}^m \text{St}(\mathbf{Q}_i)$  has size  $O(c_d \log^d n)$ , with high probability with respect to  $m$ .*

*Proof:* Let  $|\text{St}(\mathbf{Q}_i)| \leq c' \cdot c_d \ln^{d-1} n$  with probability  $1 - m^{-c}$  for large enough constant  $c$  and some constant  $c'$  depending on  $c$ . By setting  $\mathcal{P}(\mathbf{Q}_i) = \text{St}(\mathbf{Q}_i)$ , we have that  $\Pr[|\text{St}| > \gamma(2k \ln m)] \leq m^{-\gamma k} + m^{-c}$  for  $k = c' \cdot c_d \ln^{d-1} n$  and any  $\gamma \geq 2e$ , by [Corollary 3.1](#). Setting  $\gamma = 2e \ln n / \ln m$  implies the claim.  $\blacksquare$

**Lemma 6.3.** *Fix a dimension  $d \geq 2$ . Let  $m, n$  be parameters, such that  $m \leq n$ . Let  $\mathbf{P}$  be a set of  $m$  points picked randomly from  $[0, 1]^d$  as described in [Section 2.2](#). Then,  $|\text{St}(\mathbf{P})| = O(c_d \log^{d-1} n)$  holds, with high probability with respect to  $m$ , for some constant  $c_d$  that depends only on  $d$ .*

*Proof:* The argument follows by induction on dimension. The two-dimensional case follows from [Corollary 6.1](#). Assume we have proven the claim for all dimension smaller than  $d$ .

Now, sort  $\mathbf{P}$  by increasing value of the  $d$ th coordinate, and let  $\mathbf{p}_i = (\mathbf{q}_i, \ell_i)$  be the  $i$ th point in  $\mathbf{P}$  in this order for each  $i$ , where  $\mathbf{q}_i$  is a  $(d-1)$ -dimensional vector and  $\ell_i$  is the value of the  $d$ th coordinate of  $\mathbf{p}_i$ . Observe that the points  $\mathbf{q}_1, \dots, \mathbf{q}_m$  are randomly, uniformly, and independently picked from the hypercube  $[0, 1]^{d-1}$ . Now, if  $\mathbf{p}_i$  is a minima point of  $\mathbf{P}$ , then it is a minima point of  $\{\mathbf{p}_1, \dots, \mathbf{p}_i\}$ . But this implies that  $\mathbf{q}_i$  is a minima point of  $\mathbf{Q}_i = \{\mathbf{q}_1, \dots, \mathbf{q}_i\}$  as well. Namely,  $\mathbf{q}_i \in \text{St} = \bigcup_{i=1}^m \text{St}(\mathbf{Q}_i)$ . This implies that  $|\text{St}(\mathbf{P})| \leq |\text{St}|$ . Now, applying induction hypothesis on each  $\mathbf{Q}_i$  in dimension  $d-1$  we have  $|\text{St}(\mathbf{Q}_i)| = O(c_{d-1} \log^{d-2} n)$  holds for all  $i$ , with high probability with respect to  $m$ . Plugging it into [Lemma 6.2](#) we have  $|\text{St}(\mathbf{P})| \leq |\text{St}| = O(c_{d-1} \log^{d-1} n)$ , with high probability with respect to  $m$ . Choosing a proper constant  $c_d$  now implies the claim.  $\blacksquare$

1 **Lemma 6.4.** Fix a dimension  $d \geq 2$ . Let  $Q = \langle q_1, \dots, q_n \rangle$  be an ordered set of  $n$  points picked  
2 randomly from  $[0, 1]^d$  (as described in [Section 2.2](#)), and  $Q_i = \{q_1, \dots, q_i\}$  is the  $i$ th (unordered)  
3 prefix of  $Q$ . Then, the set  $\bigcup_{i=1}^n \text{St}(Q_i)$  is of size  $O_{\text{whp}}(c_d \log^d n)$ , and the staircase  $\text{St}(P)$  is of size  
4  $O_{\text{whp}}(c_d \log^{d-1} n)$ .

5 *Proof:* By [Lemma 6.2](#), the set  $\bigcup_{i=1}^n \text{St}(Q_i)$  is of size  $O(c_d \log^d n)$ , with high probability. By  
6 [Lemma 6.3](#), the set  $\text{St}(P)$  is of size  $O(c_d \log^{d-1} n)$ , with high probability. ■

7 **Remark 6.5.** In the proof of [Lemma 6.3](#) whether a point is on the staircase (or not) only depends  
8 on the coordinate orderings of the points and not their actual values.

9 The basic recursive argument used in [Lemma 6.3](#) was used by Clarkson [[Cla04](#)] to bound the  
10 expected number of  $k$ -sets for a random point set. Here, using [Corollary 3.1](#) enables us to get a  
11 high-probability bound.

12 Note that the definition of the staircase can be made with respect to any corner of the hyper-  
13 cube (that is, this corner would replace the origin in the definition dominance, point volume, the  
14 exponential grid, etc). Taking the union over all  $2^d$  such staircases gives us the subset of  $P$  on the  
15 orthogonal convex hull of  $P$ . Therefore [Lemma 6.4](#) also bounds the number of input points on the  
16 orthogonal convex hull. As the vertices on the convex hull of  $P$  are a subset of the points in  $P$  on  
17 the orthogonal convex hull, the above also implies the same bound on the number of vertices on  
18 the convex hull.

## 19 6.2. Bounding the size of the candidate set

20 We can now readily bound the size of the candidate set for any point in the plane.

21 **Lemma 6.6.** Let  $S$  be a set of  $n$  sites in the plane, where for each site  $s$  in  $S$ , a parametric point  
22 from a distribution over  $[0, 1]^d$  is sampled (as described in [Section 2.2](#)). Then, the candidate set  
23 has size  $O_{\text{whp}}(\log^d n)$  simultaneously for all points in the plane.

24 *Proof:* Consider the arrangement of bisectors of all pairs of points of  $S$ . This arrangement has  
25 complexity  $O(n^4)$ , and inside each cell the candidate set is the same. Now for any point in a  
26 cell of the arrangement, [Lemma 6.4](#) immediately gives us the stated bound, with high probability.  
27 Therefore picking a representative point from each cell in this arrangement and applying the union  
28 bound imply the claim. ■

## 29 7. The main result

30 We now use the bound on the complexity of the proxy diagram, as well as our knowledge of the  
31 relationship between the candidate set and the proxy set to bound the complexity, as well as the  
32 space complexity, of the candidate diagram.

33 Recall that the *complexity* of a candidate diagram, treated as a planar arrangement, is the total  
34 number of edges, faces, and vertices in the diagram. The *space complexity* of the candidate diagram  
35 is the sum of the sizes of candidate sets over all the faces in the arrangement of the diagram.

36 **Theorem 7.1.** Let  $S$  be a set of  $n$  sites in the plane, where for each site in  $S$  we sample an  
37 associated parametric point in  $[0, 1]^d$ , as described in [Section 2.2](#). Then, the expected complexity of  
38 the candidate diagram is  $O(n \log^{8d+5} n)$ . The expected space complexity of this candidate diagram  
39 is  $O(n \log^{9d+5} n)$ .

1 *Proof:* Fix  $k$  to be sufficiently large such that  $k = \Theta(\log^d n)$ . By [Lemma 5.8](#) the expected complexity  
2 of the proxy diagram is  $O(k^4 n \log n)$ . Triangulating each polygonal cell in the diagram does not  
3 increase its asymptotic complexity. [Lemma 4.2](#) implies that, the proxy set has size  $O_{\text{whp}}(k \log n)$   
4 simultaneously for all the points in the plane. Now, [Lemma 4.4](#) implies that, with high probability,  
5 the proxy set contains the candidate set for any point in the plane.

6 The resulting triangulation has  $O(k^4 n \log n)$  faces, and inside each face all the sites that might  
7 appear in the candidate set are all present in the proxy set of this face. By [Lemma 2.4](#), the complex-  
8 ity of an  $m$ -site candidate diagram is  $O(m^4)$ . Therefore the complexity of the candidate diagram  
9 per face is  $O_{\text{whp}}((k \log n)^4)$  (clipping the candidate diagram of these sites to the containing triangle  
10 does not increase the asymptotic complexity). Multiplying the number of faces,  $O(k^4 n \log n)$ , by  
11 the complexity of the arrangement within each face,  $O((k \log n)^4)$ , yields the desired result.

12 The bound on the space complexity follows readily from the bound on the size of the candidate  
13 set from [Lemma 6.6](#). ■

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## A. An Integral Calculation

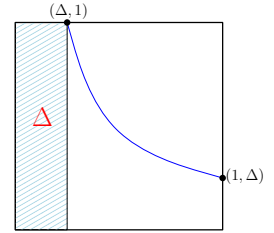
**Lemma A.1.** *Let  $F_d(\Delta)$  be the total measure of the points  $\mathbf{p} = (p_1, \dots, p_d)$  in the hypercube  $[0, 1]^d$ , such that  $\text{pv}(\mathbf{p}) = p_1 p_2 \cdots p_d \leq \Delta$ . That is,  $F_d(\Delta)$  is the measure of all points in hypercube with point volume at most  $\Delta$ . Then*

$$F_d(\Delta) = \sum_{i=0}^{d-1} \frac{\Delta}{i!} \ln^i \frac{1}{\Delta}.$$

*Proof:* The claim follows by tedious but relatively standard calculations. As such, the proof is included for the sake of completeness.

The case for  $d = 1$  is trivial. Consider the  $d = 2$  case. Here the points whose point volume equals  $\Delta$  are defined by the curve  $xy = \Delta$ . This curve intersects the unit square at the point  $(\Delta, 1)$ . As  $F_d(\Delta)$  is the total volume under this curve in the unit square we have that

$$F_2(\Delta) = \Delta + \int_{x=\Delta}^1 \frac{\Delta}{x} dx = \Delta + \Delta \ln \frac{1}{\Delta}.$$



In general, we have

$$\frac{1}{(d-1)!} \int_{x=\Delta}^1 \frac{\Delta}{x} \ln^{d-1} \frac{x}{\Delta} dx = \frac{\Delta}{(d-1)!} \left[ \frac{1}{d} \ln^d \frac{x}{\Delta} \right]_{x=\Delta}^1 = \frac{\Delta}{d!} \ln^d \frac{1}{\Delta}.$$

Now assume inductively that

$$F_{d-1}(\Delta) = \sum_{i=0}^{d-2} \frac{1}{i!} \Delta \ln^i \frac{1}{\Delta},$$

then we have

$$\begin{aligned} F_d(\Delta) &= \Delta + \int_{x_d=\Delta}^1 F_{d-1}\left(\frac{\Delta}{x_d}\right) dx_d = \Delta + \int_{x_d=\Delta}^1 \left( \sum_{i=0}^{d-2} \frac{\Delta}{i! x_d} \ln^i \frac{x_d}{\Delta} \right) dx_d \\ &= \Delta + \sum_{i=0}^{d-2} \frac{1}{i!} \left( \int_{x_d=\Delta}^1 \frac{\Delta}{x_d} \ln^i \frac{x_d}{\Delta} dx_d \right) = \Delta + \sum_{i=1}^{d-1} \frac{\Delta}{i!} \ln^i \frac{1}{\Delta} = \sum_{i=0}^{d-1} \frac{\Delta}{i!} \ln^i \frac{1}{\Delta}. \quad \blacksquare \end{aligned}$$