Asymptotically Optimal Thickness Bounds of Generalized Bar Visibility Graphs

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Abstract

Given a set of disjoint horizontal line segments (call *bars*), the *distance* of two bars is the minimum number of the other bars that a vertical line segment joining the two bars passes through. A graph G is a *bar k-visibility graph* if G can be represented as a set of disjoint bars such that two vertices are adjacent in G if and only if the distance of their associated bars is at most k. A graph G is a *semi bar k-visibility graph* if G can be represented as a set of disjoint bars whose left endpoints have the same x-coordinates such that two vertices are adjacent in G if and only if the distance of their associated bars is at most k. The *thickness* of G is the minimum number of planar subgraphs whose union is G.

Dean *et al.* gave the best previously known upper bound $3k(6k+1)$ on the thickness of bar k-visibility graphs. Hartke *et al.* proved that K_{4k+4} is a bar k-visibility graph, so the upper bound on the thickness of bar k-visibility graphs is at least $\lceil(2k+3)/3\rceil$. Felsner and Massow gave an upper bound on the thickness of semi bar 1-visibility graphs. Felsner and Massow proved that K_{2k+3} is a semi bar k visibility graph, so the upper bound on the thickness of semi bar k visibility graphs is at least $\lceil(2k+5)/6\rceil$. We reduce the upper bound to $3k+3$ on the thickness of bar k-visibility graphs, and give an upper bound $2k$ for semi bar k-visibility graphs.

1 Introduction

All graphs are simple throughout the paper. Consider a set B of disjoint *bars*, that is, horizontal line segments. For any two bars u and v in B, the *vertical distance* $d(u,v)$ in B is the smallest integer k such that there is a vertical line segment whose endpoints are at u and v passing through k other bars. Dean *et al.* [3,4] defined that a graph G is a *bar k-visibility graph* if G can be represented as a set of disjoint bars such that any two vertices are adjacent in G if and only if $d(u,v) \leq k$, where u and v are the associated bars with those vertices. Given a bar k-visibility graph, we called the corresponding representation a *bar k-visibility representation*. The cases with k equals 0 and 1 are illustrated in Figure 1. Bar 0-visibility graphs are also known as the *bar visibility graphs* [2,5]. For $k = \infty$, bar k-visibility graphs are exactly the *interval graphs* (see, for

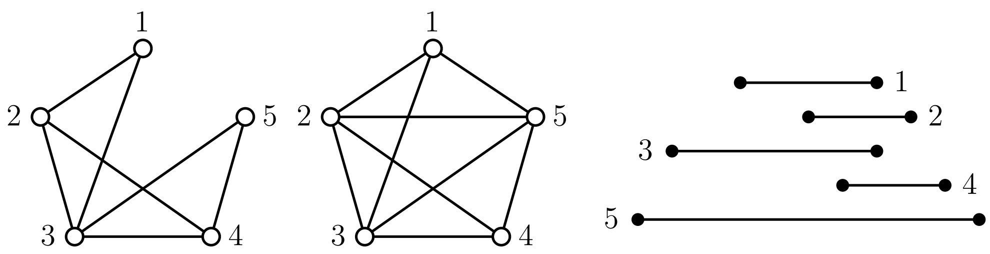

Figure 1: A bar 0-visibility graph, a bar 1-visibility graph, and their common representation.

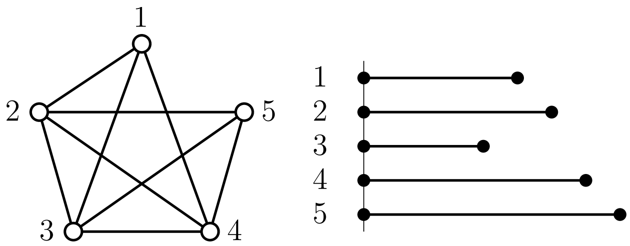

Figure 2: A semi bar 1-visibility graph with its representation.

example, [14]). We denote $\mathcal{B}_k$ as the family of bar k-visibility graphs. Felsner and Massow [6,7] defined that a graph G is a *semi bar k-visibility graph* if G can be represented as a set of disjoint bars whose left endpoints have the same x-coordinates such that any two vertices are adjacent in G if and only if $d(u,v) \leq k$, where u and v are the associated bars with those vertices. The corresponding representation is called a *semi bar k-visibility representation*. The case with $k = 1$ is illustrated in Figure 2. We denote $\mathcal{S}_k$ as the family of semi bar k-visibility graphs. The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G (see, for example, [14]). For any family of graphs $\mathcal{G}$, let $\theta(\mathcal{G}) := \max_{G \in \mathcal{G}} \theta(G)$.

The goal of this paper is to study the thickness of bar k-visibility graphs and semi bar k-visibility graphs. For the special case when $k = 1$, Dean *et al.* [3,4] proved that $\theta(\mathcal{B}_1) \leq 4$, and conjectured that $\theta(\mathcal{B}_1) \leq 2$, which was disproved by Felsner and Massow [6,7]. Felsner and Massow also gave a constructive proof for $\theta(\mathcal{S}_1) = 2$. In this paper, we focus on $\theta(\mathcal{B}_k)$ and $\theta(\mathcal{S}_k)$ for general k. Dean *et al.* [3,4] gave the best previously upper bound $3k(6k+1)$ on $\theta(\mathcal{B}_k)$. We reduce the upper bound to $3k + 3$. It is known that $\theta(\mathcal{B}_k)$ is at least $\lceil (2k+3)/3 \rceil$ as Dean *et al.* proved that complete graph K_{4k+4} is in $\mathcal{B}_k$. Hence our first result is asymptotically optimal. We also give the first upper bound $2k$ on $\theta(\mathcal{S}_k)$. Felsner and Massow [6,7] proved that complete graph K_{2k+3} is in $\mathcal{S}_k$, so $\theta(\mathcal{S}_k)$ is at least $\lceil (2k+5)/6 \rceil$. Hence our second result is asymptotically optimal. Table 1 compares previous work and our results. In summary, we prove the following theorem.

Theorem 1.

1. *If G is a bar k-visibility graph, then $\theta(G) \leq 3k + 3$ for any $k \geq 0$.*
2. *If G is a semi bar k-visibility graph, then $\theta(G) \leq 2k$ for any $k \geq 1$.*

The importance of the problem. Mansfield [9] proved that determining the thickness of a graph is NP-hard. The class of graphs whose thickness is known is few—for example, complete graphs and hypercubes (see [10]). If we know better upper bound on the thickness of the graph, then in VLSI design, we can embed the graph using fewer layers [1]. In the scheduling of multihop radio networks, Ramanathan and Lloyd [12,13] gave an approximation algorithm

	$\theta(\mathcal{B}_k)$		$\theta(\mathcal{S}_k)$	
	$k = 1$	$k \geq 1$	$k = 1$	$k \geq 1$
Dean *et al.* [3,4]	≤ 4	$\leq 3k(6k+1)$		
Hartke *et al.* [8]		$\geq \lceil (2k+3)/3 \rceil$		
Felsner and Massow [6,7]	≥ 3		2	$\geq \lceil (2k+5)/6 \rceil$
Our result		$\leq 3k+3$		$\leq 2k$

Table 1: Previous work and our result.

for the schedule where the number of time slots is bounded by a function of the thickness of a graph.

Related work on the problem. Dean *et al.* [3,4] study bar k-visibility graphs and gave bounds on the maximum number of edges and chromatic number of $\mathcal{B}_k$. Hartke *et al.* [8] improved the result by sharpening the bound on maximum number of edges. Hartke *et al.* also provided some other facts about bar k-visibility graphs. They proved that complete graph K_{4k+4} is indeed the largest complete graph in $\mathcal{B}_k$, as conjectured by Dean *et al.* [3,4]; they constructed some forbidden induced subgraphs of the class $\mathcal{B}_k$; and they discussed *regular* bar k-visibility graphs. Felsner and Massow [6,7] gave bounds on semi bar k-visibility graphs, and gave bounds of chromatic number, clique number, maximum number of edges, and connectivity on $\mathcal{S}_k$. They proved that K_{2k+3} is the largest complete graph that can be in $\mathcal{S}_k$. Also the yproved that the upper bounds on geometric thickness of $\mathcal{S}_1$ is also at most 2. Given a semi bar k-visibility graph and an order of bars corresponding to the nodes, Felsner and Massow gave a method to reconstruct a semi bar k-visibility representation.

2 Bar k-visibility graph

Given a graph G, $V(G)$ is the node set of G and $E(G)$ is the edge set of G. Denote n_G the number of nodes in G and m_G the number of edges in G. Consider graph G in $\mathcal{B}_k$. If R is a bar k-visibility representation of G, we denote G as $G(R)$, and the bar in R which corresponds to vertex x in G by b_x or $b(x)$.

2.1 Weak bar k-visibility graph

A graph G is a *weak bar k-visibility graph* if G is a subgraph of a bar k-visibility graph. The case with $k = 1$ is illustrated in Figure 3. We denote $\mathcal{W}_k$ as the family of weak bar k-visibility graphs.

Lemma 2.1. *If $G \in \mathcal{W}_k$, then there is a graph $H \in \mathcal{B}_k$, such that $n_G = n_H$ and G is a subgraph of H.*

Proof. Suppose that G' is a bar k-visibility graph and G is a subgraph of G'. Let R' be a bar k-visibility representation of G', and $R^* = R' - B$, where B is the set of the associated bars of the vertices in $V(G') - V(G)$. Since for every vertex pair (u, v) where $u \in V(G)$ and $v \in V(G)$, if $d(b_u, b_v) \leq k$ in R', then $d(b_u, b_v) \leq k$ in R^*, we know that for every edge $e \in E(G)$, $e \in E(G(R^*))$. Hence G is a subgraph of $G(R^*)$ and $n_G = n_{G(R^*)}$. □

Lemma 2.2 (Hartke *et al.* [8])**.** *If $G \in \mathcal{B}_k$ and $n_G \geq 2k + 2$, then $m_G \leq (k + 1)(3n_G - 4k - 6)$.*

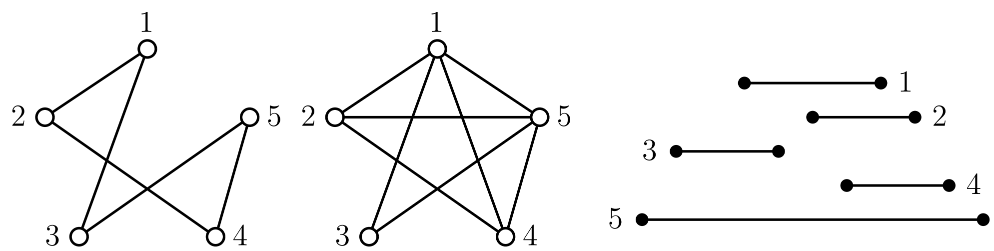

Figure 3: A weak bar 1-visibility graph with its supergraph, and the bar 1-visibility representation of the supergraph.

2.2 Arboricity

The *arboricity* $arb(G)$ of a graph G is the minimum number of forests whose union is G, (see, for example, [14]). We know that

$$\theta(G) \leq arb(G) \tag{1}$$

holds for any graph G, because the thickness of a forest is one.

Lemma 2.3 (Nash-Williams [11]). *For any graph G,*

$$arb(G) = \max\left\{\left\lceil \frac{m_H}{n_H - 1} \right\rceil : H \subseteq G, n_H > 1\right\}.$$

2.3 Proof of Theorem 1.1

Proof. Consider any subgraph H of G. We have the following two cases.

- Case 1: $1 < n_H < 2k + 2$.
  Since the number of edges for every simple graph with n nodes is at most $\binom{n}{2}$, we have

$$\frac{m_H}{n_H - 1} \leq \frac{n_H \cdot (n_H - 1)/2}{n_H - 1} = \frac{n_H}{2} < k + 1.$$

- Case 2: $n_H \geq 2k + 2$.
  By the defintion of $\mathcal{W}_k$ and Lemma 2.1, there exists a graph $H' \in \mathcal{B}_k$, such that $n_H = n_{H'}$ and H is a subgraph of H'. Hence we know $m_H \leq m_{H'}$. By Lemma 2.2, we know $m_{H'} \leq (k+1)(3n_{H'} - 4k - 6)$. Therefore,

$$\begin{aligned}
\frac{m_H}{n_H - 1} &\leq \frac{m_{H'}}{n_H - 1} \\
&\leq \frac{(k+1)(3n_{H'} - 4k - 6)}{n_H - 1} \\
&\leq \frac{(k+1)(3n_H - 4k - 6)}{n_H - 1} \\
&= 3(k+1) - \frac{4k^2 + 7k + 3}{n_H - 1} \\
&\leq 3k + 3.
\end{aligned}$$

It follows from Lemma 2.3, that we know $arb(G) \leq 3k + 3$. By (1), we have $\theta(G) \leq 3k + 3$. □

3 Semi bar k-visibility graph

3.1 Semi bar exactly k-visibility graph

A graph G is a *semi bar exactly k-visibility graph* if G can be represented as a set of disjoint bars whose left endpoints have the same x-coordinates such that any two vertices are adjacent in G if and only if $d(u, v) = k$, where u and v are the associated bars with those vertices. The case with $k = 1$ is illustrated in Figure 4. We denote $\mathcal{SE}_k$ as the family of semi bar exactly k-visibility graphs. The *outdegree* $\deg^+(v)$ of a vertex v is the number of outward directed edges from v (see, for example, [14]).

Lemma 3.1. *If $G \in \mathcal{SE}_k$, then there is an orientation of edges of G such that for every vertex v,* $\deg^+(v) \leq 2$.

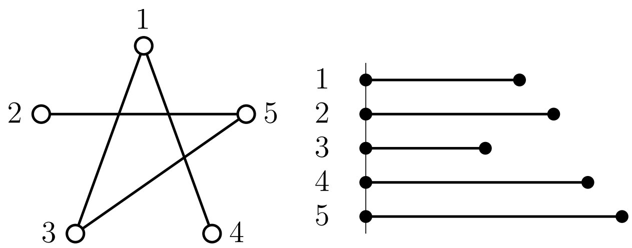

Figure 4: A semi bar exactly 1-visibility graph with its representation.

Proof. We denote the length of bar b by $\ell(b)$. We label the edges of G by $1, 2, \ldots, m_G$, then we orient the edges of G from 1 to m_G according to the following rule: let R be a semi bar exactly k-visibility representation of G. For each $j = 1, \ldots, m_G$, let edge $e_j = (x_j, y_j)$. If $\ell(b(x_j)) \geq \ell(b(y_j))$ in R, then we give e_j the orientation from y_j to x_j, otherwise we give e_j the orientation from x_j to y_j. We name the graph G^*. For any vertex v, suppose that there are more than two bars $b_1, b_2, \ldots, b_q$ which are above b_v, such that for each i with $1 \leq i \leq q$, $d(b_i, b_v) = k$ and the orientation of the edges in G^* corresponding to (b_i, b_v) is pointed out from v. Let two of those bars be b_s and b_t and b_s is above b_t. $\ell(b_s) \geq \ell(b_v)$ and $\ell(b_t) \geq \ell(b_v)$, so every vertical line segment whose endpoints are at b_s and b_v has to pass through b_t. Hence $d(b_t, b_v) \neq d(b_s, b_v)$, which is a contradiction. Therefore, there is at most one bar which is above b_v, such that the orientation of the edge in G^* corresponding to the bar pair is pointed out from v. Similarly, there is at most one bar which is below b_v, such that the orientation of the edge in G^* corresponding to the bar pair is pointed out from v. So, $\deg^+(v) \leq 2$. □

Lemma 3.2. *If G admits an orientation such that $\deg^+(v) \leq d$ for every vertex v, then $\theta(G) \leq d$.*

Proof. By this orientation, we label the outgoing edges of every vertex by $1, 2, \ldots, d$. Let E_i be the set of the edges labeled i, and $G_i = (V(G), E_i)$ for each i with $1 \leq i \leq d$, then we know for any component in G_i for each i with $1 \leq i \leq d$, the number of edges in the component is at most the number of nodes in the component, because G_i has an orientation, such that for every vertex v, $\deg^+(v) \leq 1$. Hence $\theta(G_i) = 1$ for each i with $1 \leq i \leq d$. Since $\bigcup_{i=1}^{d} E_i = E(G)$ and $E_i \cap E_j = \phi$ for any indices i and j with $i \neq j$, we have

$$\theta(G) \leq \sum_{i=1}^{d} \theta(G_i) = \sum_{i=1}^{d} 1 = d. \qquad \square$$

Lemma 3.3 (Felsner and Massow [7])**.** *If $G \in \mathcal{S}_1$, then $\theta(G) \leq 2$.*

3.2 Proof of Theorem 1.2

Proof. Suppose that R is a semi bar k-visibility representation of G. Let

$$E_i = \{(x, y) : d(b_x, b_y) = i\},$$
$$G_i = (V(G), E_i).$$

We have $G_i \in \mathcal{SE}_i$ for each i with $0 \leq i \leq k$, and $\bigcup_{i=0}^{k} E_i = E(G)$. By Lemma 3.1 and Lemma 3.2, we know $\theta(G_i) \leq 2$ for each i with $0 \leq i \leq k$. By the definitions of $\mathcal{S}_k$ and $\mathcal{SE}_k$, we know $G_0 \cup G_1 \in \mathcal{S}_1$. By Lemma 3.3, $\theta(G_0 \cup G_1) \leq 2$. Therefore,

$$\theta(G) \leq \theta(G_0 \cup G_1) + \sum_{i=2}^{k} \theta(G_i) \leq 2 + 2(k-1) = 2k. \qquad \square$$

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